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Laeng-Morpurgo-Type Uncertainty Inequalities for the Laguerre Transform

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Abstract In this work, we prove Clarkson-type and Nash-type inequalities for the Laguerre transform $\mathfrak{F}_L$ on $\mathbb{M} = [0,\infty)\times\mathbb{R}$. By combining these inequalities, we show Laeng-Morpurgo-type uncertainty inequalities. We establish also a local-type uncertainty inequalities for the Laguerre transform $\mathfrak{F}_L$, and we deduce a Heisenberg-Pauli-Weyl-type inequality for this transform.

Keywords Laeng-Morpurgo-type inequality; local-type inequality; Heisenberg-Pauli-Weyl-type inequality

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1. Introduction

Uncertainty principles are mathematical arguments that give limitations on the simultaneous concentration of a function and its Fourier transform. They have implications in quantum physics and signal analysis. They also play an important role in harmonic analysis, many of them have already been studied from several points of view for the Fourier transform, Heisenberg-Pauli-Weyl inequality [1] and local uncertainty [2–4]. Laeng-Morpurgo [5] and Morpurgo [6] obtained Heisenberg inequality involving a combination of L^1 and L^2 norms. Folland and Sitaram [7] proved general forms of the Heisenberg-Pauli-Weyl inequality.

In this paper, we consider the Laguerre transform $\mathfrak{F}_L$ (see [8,9]) defined on $L^1(\mathbb{M}, m_k)$ by

$$\mathfrak{F}_L(f)(\lambda,n) := \int_{\mathbb{M}} \varphi_{-\lambda,n}(x,t) f(x,t) \mathrm{d}m_k(x,t), \quad (\lambda,n)\in\mathbb{V},$$

where $\mathbb{M} := [0,\infty)\times\mathbb{R}$, $\mathbb{V} := \mathbb{R}\times\mathbb{N}$, $\mathrm{d}m_k(x,t) := \frac{x^{2k+1}}{\pi\Gamma(k+1)}\mathrm{d}x\mathrm{d}t$ and

$$\varphi_{\lambda,n}(x,t) = \frac{L_n^{(k)}(|\lambda|x^2)}{L_n^{(k)}(0)} \exp(i\lambda t - |\lambda|\frac{x^2}{2}), \quad (x,t)\in\mathbb{M}.$$

Here $L_n^{(k)}$ is the Laguerre polynomial of degree n and order k.

In this work, we establish Clarkson-type inequality and Nash-type inequality for the Laguerre transform $\mathfrak{F}_L$ on $L^1\cap L^2(\mathbb{M}, m_k)$. Next, building on the techniques of Laeng-Morpurgo [5, 6], we deduce uncertainty inequalities of Heisenberg-type for the Laguerre transform $\mathfrak{F}_L$ on $L^1 \cap$

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$L^2(\mathbb{M}, m_k)$. Finally, due to a local uncertainty inequality for the Laguerre transform $\mathfrak{F}_L$ on $L^2(\mathbb{M}, m_k)$, we show uncertainty inequality of Heisenberg-Pauli-Weyl-type for the transform $\mathfrak{F}_L$ on $L^2(\mathbb{M}, m_k)$.

The analog uncertainty inequalities are also proved, for the Dunkl transform $\mathfrak{F}_k$ on $\mathbb{R}^d$ by Soltani [10–12], and for the Segal-Bargmann transform $\mathfrak{B}_k$ by Soltani [13–15].

This paper is organized as follows. In Section 2, we recall some results about the Laguerre transform $\mathfrak{F}_L$ on $\mathbb{M}$. In Section 3, we prove uncertainty inequalities of Heisenberg-type for the Laguerre transform $\mathfrak{F}_L$ on $L^1 \cap L^2(\mathbb{M}, m_k)$. In Section 4, we show uncertainty inequality of Heisenberg-Pauli-Weyl-type for the transform $\mathfrak{F}_L$ on $L^2(\mathbb{M}, m_k)$.

2. Laguerre transform

We consider the Laguerre operator Δ_L defined on $(0, \infty) \times \mathbb{R}$, by

$$\Delta_L := \frac{\partial^2}{\partial x^2} + \frac{2k+1}{x}\frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2}, \quad k > 0.$$

The Laguerre operator has gained considerable interest in various field of mathematics [9, 16, 17]. It gives rise to generalizations of many two-variable analytic structures like the Laguerre transform $\mathfrak{F}_L$ and the Laguerre-convolution product [8, 9, 16], the dispersion and the Gaussian distributions [8, 9], and the Laguerre and Weierstrass transform [18].

Throughout this subsection, let $k > 0$, $\mathbb{M} := [0, \infty) \times \mathbb{R}$ and $\mathbb{V} := \mathbb{R} \times \mathbb{N}$. For $(x, t) \in \mathbb{M}$ we denote by $|(x, t)| := (x^4 + t^2)^{1/4}$ and for $(\lambda, n) \in \mathbb{V}$, we denote by $|(\lambda, n)| := |\lambda|(n + \frac{k+1}{2})$. We denote by $L^p(\mathbb{M}, m_k)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{M}$, such that

$$\|f\|_{L^p(\mathbb{M},m_k)} := \Big(\int_{\mathbb{K}} |f(x,t)|^p \mathrm{d}m_k(x,t) \Big)^{1/p} < \infty, \quad p \in [1, \infty),$$
$$\|f\|_{L^\infty(\mathbb{M},m_k)} := \operatorname{ess} \sup_{(x,t)\in\mathbb{M}} |f(x,t)| < \infty,$$

where m_k is the measure given by

$$\mathrm{d}m_k(x,t) := \frac{x^{2k+1}}{\pi\Gamma(k+1)} \mathrm{d}x\mathrm{d}t.$$

And by $L^p(\mathbb{V}, v_k)$, $p \in [1, \infty]$, the space of measurable functions g on $\mathbb{V}$, such that

$$\|g\|_{L^p(\mathbb{V},v_k)} := \Big(\int_{\mathbb{V}} |g(\lambda,n)|^p \mathrm{d}v_k(\lambda,n) \Big)^{1/p} < \infty, \quad p \in [1, \infty),$$
$$\|g\|_{L^\infty(\mathbb{V},v_k)} := \operatorname{ess} \sup_{(\lambda,n)\in\mathbb{V}} |g(\lambda,n)| < \infty,$$

where v_k is the positive measure defined on $\mathbb{V}$ by

$$\int_{\mathbb{V}} g(\lambda,n)\mathrm{d}v_k(\lambda,n) = \sum_{n=0}^{\infty} L_n^{(k)}(0) \int_{\mathbb{R}} g(\lambda,n)|\lambda|^{k+1}\mathrm{d}\lambda.$$

Here $L_n^{(k)}$ is the Laguerre polynomial of degree n and order k. Let $r > 0$, the measures m_k and v_k satisfy [19]:

$$m_k(|(x,t)| < r) = \frac{\Gamma(\frac{k+1}{2})}{2\sqrt{\pi}(k+2)\Gamma(k+1)\Gamma(\frac{k}{2}+1)} r^{2(k+2)}, \tag{2.1}$$

$$v_k(|(\lambda,n)|<r)=\frac{2r^{k+2}}{k+2}\sum_{n=0}^{\infty}\frac{L_n^{(k)}(0)}{(n+\frac{k+1}{2})^{k+2}},\tag{2.2}$$

We denote by

$$c_1=[\frac{\Gamma(\frac{k+1}{2})}{2\sqrt{\pi}(k+2)\Gamma(k+1)\Gamma(\frac{k}{2}+1)}]^{1/2},\tag{2.3}$$

and

$$c_2=\frac{2}{k+2}\sum_{n=0}^{\infty}\frac{L_n^{(k)}(0)}{(n+\frac{k+1}{2})^{k+2}}.\tag{2.4}$$

For all $(\lambda,n)\in\mathbb{V}$, the system [9]:

$$\Delta_L u=-2|\lambda|(2n+k+1)u,$$

$$\frac{\partial u}{\partial t}=i\lambda u,\quad u(0,0)=1,\ \frac{\partial u}{\partial x}(0,t)=0,\ \forall\, t\in\mathbb{R},$$

admits a unique solution $\varphi_{\lambda,n}(x,t)$, given by

$$\varphi_{\lambda,n}(x,t)=\frac{L_n^{(k)}(|\lambda|x^2)}{L_n^{(k)}(0)}\exp(i\lambda t-|\lambda|\frac{x^2}{2}),\quad (x,t)\in\mathbb{M}.$$

And for all $(\lambda,n)\in\mathbb{V}$, we have

$$\sup_{(x,t)\in\mathbb{M}}|\varphi_{\lambda,m}(x,t)|=1.$$

The Laguerre transform $\mathfrak{F}_L$ (see [8,9]) is defined on $L^1(\mathbb{M},m_k)$ by

$$\mathfrak{F}_L(f)(\lambda,n):=\int_{\mathbb{M}}\varphi_{-\lambda,n}(x,t)f(x,t)\mathrm{d}m_k(x,t),\quad (\lambda,n)\in\mathbb{V},$$

extends uniquely to an isometric isomorphism on $L^2(\mathbb{M},m_k)$ onto $L^2(\mathbb{V},v_k)$, that is

$$\|\mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)}=\|f\|_{L^2(\mathbb{M},m_k)},\quad f\in L^2(\mathbb{M},m_k).\tag{2.5}$$

Moreover, if $f\in L^1(\mathbb{M},m_k)$, then

$$\|\mathfrak{F}_L(f)\|_{L^\infty(\mathbb{V},v_k)}\le\|f\|_{L^1(\mathbb{M},m_k)}.\tag{2.6}$$

Finally, if $f\in L^1(\mathbb{M},m_k)$ such that $\mathfrak{F}_L(f)\in L^1(\mathbb{V},v_k)$, the inverse Laguerre transform is defined by

$$f(x,t)=\int_{\mathbb{V}}\mathfrak{F}_L(f)(\lambda,n)\varphi_{\lambda,n}(x,t)\mathrm{d}v_k(\lambda,n),\quad \text{a.e. } (x,t)\in\mathbb{M}.$$

Throughout this paper we shall use the notation $\beta=k+2$.

3. Heisenberg-type uncertainty principles

Soltani [12] proved a Laeng-Morpurgo-type uncertainty inequalities for the Dunkl transform $\mathfrak{F}_k$ on $\mathbb{R}^d$. In the following, we will give Laeng-Morpurgo-type uncertainty inequalities for the Laguerre transform $\mathfrak{F}_L$ on $\mathbb{M}$.

Lemma 3.1 (Clarkson-type inequality) *Let $a > 0$. If $f \in L^1 \cap L^2(\mathbb{M}, m_k)$, there exists a constant $A > 0$ such that*

$$\|f\|_{L^1(\mathbb{M},m_k)} \leq A\|f\|_{L^2(\mathbb{M},m_k)}^{\frac{a}{\beta+a}} \| \, |(x,t)|^a f\|_{L^1(\mathbb{M},m_k)}^{\frac{\beta}{\beta+a}}. \tag{3.1}$$

Proof Let $f \in L^1 \cap L^2(\mathbb{M}, m_k)$ and $r, a > 0$. Then

$$\|f\|_{L^1(\mathbb{M},m_k)} = \|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{M},m_k)} + \|(1 - \mathbf{1}_{B_r})f\|_{L^1(\mathbb{M},m_k)}, \tag{3.2}$$

where $\mathbf{1}_{B_r}$ is the characteristic function of the set $B_r := \{(x,t) \in \mathbb{M} : |(x,t)| < r\}$. Firstly,

$$\|(1 - \mathbf{1}_{B_r})f\|_{L^1(\mathbb{M},m_k)} \leq r^{-a} \| \, |(x,t)|^a f\|_{L^1(\mathbb{M},m_k)}. \tag{3.3}$$

By (2.1) and Hölder's inequality, we get

$$\|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{M},m_k)} \leq (m_k(B_r))^{1/2} \|f\|_{L^2(\mathbb{M},m_k)} \leq c_1 r^\beta \|f\|_{L^2(\mathbb{M},m_k)}, \tag{3.4}$$

where c_1 is the constant given by (2.3). Combining the relations (3.2)–(3.4), we obtain

$$\|f\|_{L^1(\mathbb{M},m_k)} \leq c_1 r^\beta \|f\|_{L^2(\mathbb{M},m_k)} + r^{-a} \| \, |(x,t)|^a f\|_{L^1(\mathbb{M},m_k)}.$$

By setting

$$r = \left(\frac{a \| \, |(x,t)|^a f\|_{L^1(\mathbb{M},m_k)}}{\beta c_1 \|f\|_{L^2(\mathbb{M},m_k)}}\right)^{\frac{1}{\beta+a}},$$

we get the desired inequality with

$$A = c_1^{\frac{a}{\beta+a}} \left(\frac{\beta}{a}\right)^{\frac{a}{\beta+a}} \left(1 + \frac{a}{\beta}\right). \quad \square \tag{3.5}$$

Lemma 3.2 (Nash-type inequality) *Let $b > 0$. If $f \in L^1 \cap L^2(\mathbb{M}, m_k)$, there exists a constant $B > 0$ such that*

$$\|f\|_{L^2(\mathbb{M},m_k)} \leq B\|f\|_{L^1(\mathbb{M},m_k)}^{\frac{2b}{\beta+2b}} \| \, |(\lambda,n)|^b \mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)}^{\frac{\beta}{\beta+2b}}. \tag{3.6}$$

Proof Let $f \in L^1 \cap L^2(\mathbb{M}, m_k)$ and $r, b > 0$. Then

$$\|f\|^2_{L^2(\mathbb{M},m_k)} = \|\mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)} = \|\mathbf{1}_{\widetilde{B}_r} \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)} + \|(1 - \mathbf{1}_{\widetilde{B}_r})\mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}, \tag{3.7}$$

where $\mathbf{1}_{\widetilde{B}_r}$ is the characteristic function of the set $\widetilde{B}_r := \{(\lambda,n) \in \mathbb{V} : |(\lambda,n)| < r\}$. Firstly,

$$\|(1 - \mathbf{1}_{\widetilde{B}_r})\mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)} \leq r^{-2b} \| \, |(\lambda,n)|^b \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}. \tag{3.8}$$

By (2.2) and (2.6), we get,

$$\|\mathbf{1}_{\widetilde{B}_r} \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)} \leq v_k(\widetilde{B}_r) \|\mathfrak{F}_L(f)\|^2_{L^\infty(\mathbb{V},v_k)} \leq c_2 r^\beta \|f\|^2_{L^1(\mathbb{M},m_k)}, \tag{3.9}$$

where c_2 is the constant given by (2.4). Combining the relations (3.7)–(3.9), we obtain

$$\|f\|^2_{L^2(\mathbb{M},m_k)} \leq c_2 r^\beta \|f\|^2_{L^1(\mathbb{M},m_k)} + r^{-2b} \| \, |(\lambda,n)|^b \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}.$$

By choosing

$$r = \left(\frac{2b \| \, |(\lambda,n)|^b \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}}{\beta c_2 \|f\|^2_{L^1(\mathbb{M},m_k)}}\right)^{\frac{1}{\beta+2b}},$$

we get the inequality with

$$B = c_2^{\frac{b}{\beta+2b}} \left(\frac{\beta}{2b}\right)^{\frac{b}{\beta+2b}} \left(1 + \frac{2b}{\beta}\right)^{1/2}. \quad \square$$

By combining and multiplying the two relations (3.1) and (3.6) we obtain the following uncertainty inequalities of Laeng-Morpurgo-type [5, 6] for the Laguerre transform $\mathfrak{F}_L$ on $L^1 \cap L^2(\mathbb{M}, m_k)$.

Theorem 3.3 *Let $a, b > 0$. If $f \in L^1 \cap L^2(\mathbb{M}, m_k)$, then*

(i) There exists a constant $C > 0$ such that

$$\|f\|^{\beta+a+2b}_{L^2(\mathbb{M},m_k)} \leq C\| \,|(x,t)|^a f\|^{2b}_{L^1(\mathbb{M},m_k)}\| \,|(\lambda,n)|^b \mathfrak{F}_L(f)\|^{\beta+a}_{L^2(\mathbb{V},v_k)}.$$

(ii) There exists a constant $N > 0$ such that

$$\|f\|^{\beta+a+2b}_{L^1(\mathbb{M},m_k)} \leq N\| \,|(x,t)|^a f\|^{\beta+2b}_{L^1(\mathbb{M},m_k)}\| \,|(\lambda,n)|^b \mathfrak{F}_L(f)\|^{a}_{L^2(\mathbb{V},v_k)}.$$

(iii) There exists a constant $D > 0$ such that

$$\|f\|^{\beta+a}_{L^1(\mathbb{M},m_k)}\|f\|^{\beta+2b}_{L^2(\mathbb{M},m_k)} \leq D\| \,|(x,t)|^a f\|^{\beta+2b}_{L^1(\mathbb{M},m_k)}\| \,|(\lambda,n)|^b \mathfrak{F}_L(f)\|^{\beta+a}_{L^2(\mathbb{V},v_k)}.$$

By application of the two relations (3.1) and (3.6) we deduce also a local-type uncertainty inequalities for the Laguerre transform $\mathfrak{F}_L$ on $L^1 \cap L^2(\mathbb{M}, m_k)$.

Theorem 3.4 *Let E be a measurable subset of $\mathbb{V}$ such that $0 < v_k(E) < \infty$, and let $a, b > 0$. If $f \in L^1 \cap L^2(\mathbb{M}, m_k)$, then*

(i) $\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)} \leq A(v_k(E))^{1/2}\|f\|^{\frac{a}{\beta+a}}_{L^2(\mathbb{M},m_k)}\| \,|(x,t)|^a f\|^{\frac{\beta}{\beta+a}}_{L^1(\mathbb{M},m_k)}$,
where A is the constant given by Lemma 3.1.

(ii) $\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^1(\mathbb{V},v_k)} \leq B(v_k(E))^{1/2}\|f\|^{\frac{2b}{\beta+2b}}_{L^1(\mathbb{M},m_k)}\| \,|(\lambda,n)|^b \mathfrak{F}_L(f)\|^{\frac{\beta}{\beta+2b}}_{L^2(\mathbb{V},v_k)}$,
where B is the constant given by Lemma 3.2.

Proof Let $f \in L^1 \cap L^2(\mathbb{M}, m_k)$ and $a, b > 0$.

(i) From (2.6) we have

$$\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)} \leq (v_k(E))^{1/2}\|\mathfrak{F}_L(f)\|_{L^\infty(\mathbb{V},v_k)} \leq (v_k(E))^{1/2}\|f\|_{L^1(\mathbb{M},m_k)}.$$

The desired result follows from Lemma 3.1.

(ii) From (2.5) we have

$$\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^1(\mathbb{V},v_k)} \leq (v_k(E))^{1/2}\|\mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)} \leq (v_k(E))^{1/2}\|f\|_{L^2(\mathbb{M},m_k)}.$$

The desired result follows from Lemma 3.2. □

4. Heisenberg-Pauli-Weyl uncertainty principle

Soltani [10, 11] proved a Heisenberg-Pauli-Weyl uncertainty principle for the Dunkl transform $\mathfrak{F}_k$ on $\mathbb{R}^d$. In the following, we will give Heisenberg-Pauli-Weyl uncertainty principle for the Laguerre transform $\mathfrak{F}_L$ on $L^2(\mathbb{M}, m_k)$.

Lemma 4.1 (local-type inequality) *Let $a > 0$ and let $f \in L^2(\mathbb{M}, m_k)$. If E is a measurable subset of $\mathbb{V}$ such that $0 < v_k(E) < \infty$, then*

$$\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)} \leq A(v_k(E))^{\frac{a}{2(\beta+a)}}\|f\|^{\frac{a}{\beta+a}}_{L^2(\mathbb{M},m_k)}\| \,|(x,t)|^a f\|^{\frac{\beta}{\beta+a}}_{L^2(\mathbb{M},m_k)}, \tag{4.1}$$

where A is the constant given by Lemma 3.1.

Proof Let $f \in L^2(\mathbb{M}, m_k)$ and $a > 0$. The inequality holds if $\| |(x,t)|^a f\|_{L^2(\mathbb{M},m_k)} = \infty$. Assume that $\| |(x,t)|^a f\|_{L^2(\mathbb{M},m_k)} < \infty$. For all $r > 0$, we have

$$\begin{aligned}\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)} \leq & \|\mathbf{1}_E \mathfrak{F}_L(\mathbf{1}_{B_r} f)\|_{L^2(\mathbb{V},v_k)} + \|\mathbf{1}_E \mathfrak{F}_L((1-\mathbf{1}_{B_r})f)\|_{L^2(\mathbb{V},v_k)} \\ \leq & (v_k(E))^{1/2} \|\mathfrak{F}_L(\mathbf{1}_{B_r} f)\|_{L^\infty(\mathbb{V},v_k)} + \|\mathfrak{F}_L((1-\mathbf{1}_{B_r})f)\|_{L^2(\mathbb{V},v_k)}.\end{aligned}$$

Hence it follows from (2.5) and (2.6) that

$$\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)} \leq (v_k(E))^{1/2} \|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{M},m_k)} + \|(1-\mathbf{1}_{B_r})f\|_{L^2(\mathbb{M},m_k)}. \tag{4.2}$$

On the other hand, by Hölder's inequality, we obtain

$$\|\mathbf{1}_{B_r} f\|_{L^1(\mathbb{M},m_k)} \leq c_1 r^\beta \|f\|_{L^2(\mathbb{M},m_k)}, \tag{4.3}$$

where c_1 is the constant given by (2.3). Moreover,

$$\|(1-\mathbf{1}_{B_r})f\|_{L^2(\mathbb{M},m_k)} \leq r^{-a} \| |(x,t)|^a f\|_{L^2(\mathbb{M},m_k)}. \tag{4.4}$$

Combining the relations (4.2)–(4.4), we deduce that

$$\|\mathbf{1}_E \mathfrak{F}_L(f)\|_{L^2(\mathbb{V},v_k)} \leq (v_k(E))^{1/2} c_1 r^\beta \|f\|_{L^2(\mathbb{M},m_k)} + r^{-a} \| |(x,t)|^a f\|_{L^2(\mathbb{M},m_k)}.$$

By choosing

$$r = \Big(\frac{a\| |(x,t)|^a f\|_{L^2(\mathbb{M},m_k)}}{\beta c_1 \|f\|_{L^2(\mathbb{M},m_k)}}\Big)^{\frac{1}{\beta+a}} (v_k(E))^{-\frac{1}{2(\beta+a)}},$$

we obtain the desired inequality. □

We shall use the local uncertainty principle to obtain the following uncertainty principle of Heisenberg-Pauli-Weyl-type for the Laguerre transform $\mathfrak{F}_L$ on $L^2(\mathbb{M}, m_k)$.

Theorem 4.2 *Let $a, b > 0$. If $f \in L^2(\mathbb{M}, m_k)$, there exists a constant $K > 0$ so that*

$$\|f\|^{a+2b}_{L^2(\mathbb{M},m_k)} \leq K \| |(x,t)|^a f\|^{2b}_{L^2(\mathbb{M},m_k)} \| |(\lambda,n)|^b \mathfrak{F}_L(f)\|^a_{L^2(\mathbb{V},v_k)}.$$

Proof Let $a, b > 0$ and let $r > 0$. Then

$$\|f\|^2_{L^2(\mathbb{M},m_k)} = \|\mathbf{1}_{\widetilde{B}_r} \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)} + \|(1-\mathbf{1}_{\widetilde{B}_r})\mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}. \tag{4.5}$$

Firstly,

$$\|(1-\mathbf{1}_{\widetilde{B}_r})\mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)} \leq r^{-2b} \| |(\lambda,n)|^b \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}. \tag{4.6}$$

From (2.2) and (4.1), we get

$$\|\mathbf{1}_{\widetilde{B}_r} \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)} \leq A^2 (c_2 r^\beta)^{\frac{a}{\beta+a}} \|f\|^{\frac{2a}{\beta+a}}_{L^2(\mathbb{M},m_k)} \| |(x,t)|^a f\|^{\frac{2\beta}{\beta+a}}_{L^2(\mathbb{M},m_k)}, \tag{4.7}$$

where c_2 is the constant given by (2.4). Combining the relations (4.5)–(4.7), we obtain

$$\|f\|^2_{L^2(\mathbb{M},m_k)} \leq A^2 (c_2 r^\beta)^{\frac{a}{\beta+a}} \|f\|^{\frac{2a}{\beta+a}}_{L^2(\mathbb{M},m_k)} \| |(x,t)|^a f\|^{\frac{2\beta}{\beta+a}}_{L^2(\mathbb{M},m_k)} + r^{-2b} \| |(\lambda,n)|^b \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}.$$

By setting

$$r = \Big(\frac{2b(\beta+a)\| |(\lambda,n)|^b \mathfrak{F}_L(f)\|^2_{L^2(\mathbb{V},v_k)}}{a\beta A^2 c_2^{\frac{a}{\beta+a}} \|f\|^{\frac{2a}{\beta+a}}_{L^2(\mathbb{M},m_k)} \| |(x,t)|^a f\|^{\frac{2\beta}{\beta+a}}_{L^2(\mathbb{M},m_k)}}\Big)^{\frac{\beta+a}{a\beta+2b(\beta+a)}},$$

we get the inequality with

$$K = A^{4b(\beta+a)} c_2^{2ab} (\frac{2b(\beta+a)}{a\beta})^{a\beta} (1 + \frac{a\beta}{2b(\beta+a)})^{2a\beta+4b(\beta+a)}. \quad \square$$

Remark 4.3 The local uncertainty principle given by Lemma 4.1 is different from the one given by Rahmouni in [19], who proved the Theorem 4.2 in the particular case $a, b \geq 1$, and whose approach known as Ciatti-Ricci-Sundari method [20], is based on the estimation of the Laguerre-type heat convolution.

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