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Derivations and Deformations of Lie-Yamaguti Color Algebras

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Abstract In this paper, we introduce the representation and cohomology theory of Lie-Yamaguti color algebras. Furthermore, we introduce the notions of generalized derivations of Lie-Yamaguti color algebras and present some properties. Finally, we study linear deformations of Lie-Yamaguti color algebras, and introduce the notion of a Nijenhuis operator on a Lie-Yamaguti color algebra, which can generate a trivial deformation.

Keywords Lie-Yamaguti color algebra; representation; cohomology; derivations; deformations

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1. Introduction

Lie triple systems originated from Cartan's study of Riemannian geometry, and Jacobson [1] first introduced Lie triple systems by combining Jordan theory and quantum mechanics. As a generalization of Lie algebra and Lie triple systems, Lie-Yamaguti algebras were introduced by Yamaguti in [2] to give a geometric interpretation in [3]. Lie-Yamaguti algebras were also called "Lie triple algebras" in [4] and the recent terminology is introduced in [5].

In [6], Lin, Chen and Ma studied the formal deformations of Lie-Yamaguti algebras. In [7], Lin, Ma and Chen introduced the quasi-derivations of Lie-Yamaguti algebras. In [8], Zhang and Li developed the deformations and extensions of Lie-Yamaguti algebras. Recently, in [9], Sheng, Zhao and Zhou studied linear deformations of a Lie-Yamaguti algebra and introduced the notions of a Nijenhuis operator, a product structure and a complex structure on a Lie-Yamaguti algebra. Finally, they added a compatibility condition between a product structure and a complex structure to introduce the notion of a complex product structure on a Lie-Yamaguti algebra.

In [10, 11], Zoungrana and Issa introduced the notion of Lie-Yamaguti superalgebras, and gave the killing forms and invariant forms of Lie-Yamaguti superalgebras. Further research on Lie-Yamaguti superalgebras could be found in [12] and references cited therein. In [13], Issa and Zoungrana introduced the concept of Lie-Yamaguti color algebras. They contained usual Lie-Yamaguti algebras and Lie-Yamaguti superalgebras as special cases. Furthermore, they studied the relation between Leibniz color algebras and Lie-Yamaguti color algebras. The purpose of this paper is to study the representations and deformations of Lie-Yamaguti color algebras.

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This paper is organized as follows. In Section 2, we recall the definitions of Lie color algebras and Lie color triple systems. In Section 3, we introduce the representation and cohomology theory of Lie-Yamaguti color algebras. In Section 4, we introduce the notions of generalized derivations of Lie-Yamaguti color algebras and present some properties. In Section 5, we study linear deformations of Lie-Yamaguti color algebras, and introduce the notion of a Nijenhuis operator on a Lie-Yamaguti color algebra, which can generate a trivial deformation.

2. Preliminaries

Throughout this paper, we work on an algebraically closed field $\mathbb{K}$ of characteristic different from 2 and 3, all elements like x, y, z, u, v, w should be homogeneous unless otherwise stated. We recall the definitions of Lie color algebras and Lie color triple systems from [14] and [15].

Definition 2.1 ([14]) *Let Γ be an abelian group. A bicharacter on Γ is a map $\epsilon : \Gamma \times \Gamma \to \mathbb{K}\backslash\{0\}$ satisfying*

(1) $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = 1$,

(2) $\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$,

(3) $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$,

for any $\alpha, \beta, \gamma \in \Gamma$.

Definition 2.2 ([15]) *A Lie color algebra is a triple $(L, [\cdot, \cdot], \epsilon)$ consisting of a Γ-graded space $L = \oplus_{g\in\Gamma} L_g$, a bilinear mapping $[\cdot, \cdot] : L \times L \to L$, and a bicharacter ϵ on Γ satisfying the following conditions,*

(1) $[L_x, L_y] \subseteq L_{x+y}$,

(2) $[x, y] = -\epsilon(x, y)[y, x]$,

(3) $\epsilon(z, x)[x, [y, z]] + \epsilon(x, y)[y, [z, x]] + \epsilon(y, z)[z, [x, y]] = 0$,

for any homogeneous elements $x, y, z \in L$.

Definition 2.3 ([15]) *A Lie color triple system is a triple $(T, [\cdot, \cdot, \cdot], \epsilon)$ consisting of a Γ-graded space $T = \oplus_{g\in\Gamma} T_g$, a ternary operation $[\cdot, \cdot, \cdot] : T \times T \times T \to T$, and a bicharacter ϵ on Γ satisfying the following conditions,*

(1) $[T_x, T_y, T_z] \subseteq T_{x+y+z}$,

(2) $[y, x, z] = -\epsilon(x, y)[x, y, z]$,

(3) $\epsilon(x, z)[x, y, z] + \epsilon(y, x)[y, z, x] + \epsilon(z, y)[z, x, y] = 0$,

(4) $[x, y, [z, u, v]] = [[x, y, z], u, v] + \epsilon(z, x + y)[z, [x, y, u], v] + \epsilon(x + y, z + u)[z, u, [x, y, v]]$,

for any homogeneous elements $x, y, z, u \in T$ and $v \in T$.

3. Representations of Lie-Yamaguti color algebras

We recall the basic definition of Lie-Yamaguti color algebras from [13].

Definition 3.1 ([13]) *A Lie-Yamaguti color algebra (LY color algebra for short) is a quadruple $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \epsilon)$ in which L is a Γ-graded space, $[\cdot, \cdot]$ a binary operation, $\{\cdot, \cdot, \cdot\}$ a ternary* *operation on L and a bicharacter ϵ on Γ such that*

(SHLY1) $[L_x, L_y] \subseteq L_{x+y}$,

(SHLY2) $\{L_x, L_y, L_z\} \subseteq L_{x+y+z}$,

(SHLY3) $[x, y] = -\epsilon(x, y)[y, x]$,

(SHLY4) $\{x, y, z\} = -\epsilon(x, y)\{y, x, z\}$,

(SHLY5) $\epsilon(x, z)([[x, y], z] + \{x, y, z\}) + c.p. = 0$,

(SHLY6) $\epsilon(x, z)\{[x, y], z, u\} + \epsilon(z, y)\{[z, x], y, u\} + \epsilon(y, x)\{[y, z], x, u\} = 0$,

(SHLY7) $\{x, y, [u, v]\} = [\{x, y, u\}, v] + \epsilon(u, x + y)[u, \{x, y, v\}]$,

(SHLY8) $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} + \epsilon(u, x + y)\{u, \{x, y, v\}, w\} + \epsilon(u + v, x + y)\{u, v, \{x, y, w\}\}$,

for any homogeneous elements $x, y, z, u, v, w \in L$ and where c.p. denotes the sum over cyclic permutation of x, y, z, that is

$$\begin{aligned}\epsilon(x, z)([[x, y], z] + \{x, y, z\}) + c.p. =&\epsilon(x, z)([[x, y], z] + \{x, y, z\}) + \epsilon(z, y)([[z, x], y] + \{z, x, y\})+\\ &\epsilon(y, x)([[y, z], x] + \{y, z, x\}).\end{aligned}$$

We denote an LY color algebra by (L, ϵ).

Remark 3.2 (1) If $\Gamma = Z_2$ and $\epsilon(x, y) := (-1)^{|x||y|}$ for all $x, y \in L$. Then the LY color algebra L is just a Lie-Yamaguti superalgebra [10, 11].

(2) If $[x, y] = 0$, for all $x, y \in L$, then $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \epsilon)$ becomes a Lie color triple system $(L, \{\cdot, \cdot, \cdot\}, \epsilon)$.

(3) If $\{x, y, z\} = 0$ for all $x, y, z \in L$, then the LY color algebra $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \epsilon)$ becomes a Lie color algebra $(L, [\cdot, \cdot], \epsilon)$.

A homomorphism between two LY color algebras (L, ϵ) and (L', ϵ) is a linear map $\varphi : L \to L'$ satisfying

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]', \quad \varphi(\{x, y, z\}) = \{\varphi(x), \varphi(y), \varphi(z)\}'.$$

Example 3.3 Let $(L, [\cdot, \cdot], \epsilon)$ be a Lie color algebra. We define $\{\cdot, \cdot, \cdot\} : L \times L \times L \to L$ by

$$\{x, y, z\} := [[x, y], z], \quad \forall\, x, y, z \in L.$$

Then (L, ϵ) becomes a Lie-Yamaguti color algebra naturally.

Example 3.4 Let $\Gamma = \{0, 1\}$. Consider the 5-dimensional Γ-graded vector space $L = L_0 \oplus L_1$, over an arbitrary base filed $\mathbb{K}$ of characteristic different from 2, with basis $\{u_1, u_2, u_3\}$ of L_0 and $\{e_1, e_2\}$ of L_1, and the nonzero products on these elements are induced by the following relations:

$$\begin{aligned}&u_2 * u_1 = -u_3, \quad u_1 * u_2 = u_3, \quad u_1 * u_3 = -2u_1,\\ &u_3 * u_1 = 2u_1, \quad u_3 * u_2 = -2u_2, \quad u_2 * u_3 = 2u_2,\\ &e_1 * u_2 = e_2, \quad e_1 * u_3 = -e_1, \quad e_2 * u_1 = e_1, \quad e_2 * u_3 = e_2.\end{aligned}$$

It is not hard to check that $(L, *)$ is a Leibniz color algebra. By [13], we can define $[\cdot, \cdot]$ and

$\{\cdot,\cdot,\cdot\}$, and the nonzero products on these elements are induced by the following relations:

$$\begin{aligned}
&[u_1,u_2]=2u_3,\quad [u_1,u_3]=-4u_1,\quad [u_2,u_3]=4u_2,\\
&[e_1,u_2]=e_2,\quad [e_1,u_3]=-e_1,\quad [e_2,u_1]=e_1,\quad [e_2,u_3]=e_2,\\
&\{u_1,u_3,u_2\}=2u_3,\quad \{u_2,u_3,u_1\}=2u_3,\quad \{e_1,u_2,u_3\}=-\frac{1}{2}e_2,\\
&\{e_1,u_2,u_1\}=-\frac{1}{2}e_1,\quad \{e_1,u_3,u_2\}=\frac{1}{2}e_2,\quad \{e_1,u_3,u_3\}=-\frac{1}{2}e_1,\\
&\{e_2,u_1,u_2\}=-\frac{1}{2}e_2,\quad \{e_2,u_1,u_3\}=\frac{1}{2}e_1,\quad \{e_2,u_3,u_1\}=-\frac{1}{2}e_1,\\
&\{e_2,u_3,u_3\}=-\frac{1}{2}e_2.
\end{aligned}$$

Then $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\epsilon)$ becomes an LY color algebra.

Definition 3.5 *Let (L,ϵ) be an LY color algebra and V be a Γ-graded vector space. A representation of L on V consists of an even linear map $\rho: L\to \mathrm{End}(V)$ and even bilinear maps $D,\theta: L\times L\to \mathrm{End}(V)$ such that the following conditions are satisfied:*

(SHR1) $D(x,y)-\epsilon(x,y)\theta(y,x)+\theta(x,y)+\rho([x,y])-\rho(x)\rho(y)+\epsilon(x,y)\rho(y)\rho(x)=0$,

(SHR2) $D([x,y],z)+\epsilon(x,y+z)D([y,z],x)+\epsilon(z,x+y)D([z,x],y)=0$,

(SHR3) $\theta([x,y],z)=\epsilon(y,z)\theta(x,z)\rho(y)-\epsilon(x,y+z)\theta(y,z)\rho(x)$,

(SHR4) $D(x,y)\rho(z)=\epsilon(z,x+y)\rho(z)D(x,y)+\rho(\{x,y,z\})$,

(SHR5) $\theta(x,[y,z])=\epsilon(x,y)\rho(y)\theta(x,z)-\epsilon(z,x+y)\rho(z)\theta(x,y)$,

(SHR6) $D(x,y)\theta(u,v)=\epsilon(u+v,x+y)\theta(u,v)D(x,y)+\theta(\{x,y,u\},v)+\epsilon(u,x+y)\theta(u,\{x,y,v\})$,

(SHR7) $\theta(x,\{y,z,u\})=\epsilon(z+u,x+y)\theta(z,u)\theta(x,y)-\epsilon(y,z)\theta(y,u)\theta(x,z)+\epsilon(x,y+z)D(y,z)\theta(x,u)$,

for any homogeneous elements $x,y,z,u,v\in L$. In this case, V is also called an L-module.

Proposition 3.6 *Let (L,ϵ) be an LY color algebra and V be a Γ-graded vector space. Assume that we have a map ρ from L to $\mathrm{End}(V)$ and maps $D,\theta: L\times L\to \mathrm{End}(V)$ satisfying (SHR1)-(SHR7). Then (ρ,D,θ) is a representation of (L,ϵ) on V if and only if $L\oplus V$ is an LY color algebra under the following maps:*

$$\begin{aligned}
&[x+u,y+v]:=[x,y]+\rho(x)(v)-\epsilon(x,y)\rho(y)(u),\\
&\{x+u,y+v,z+w\}:=\{x,y,z\}+D(x,y)(w)-\epsilon(y,z)\theta(x,z)(v)+\epsilon(x,y+z)\theta(y,z)(u),
\end{aligned}$$

for any homogeneous elements $x,y,z\in L$ and $u,v,w\in V$.

Proof It is easy to check that the conditions (SHLY1)–(SHLY4) hold, we only verify that conditions (SHLY5)–(SHLY8) hold for maps defined on $L\oplus V$.

For (SHLY5), we have

$$\begin{aligned}
&\{x+u,y+v,z+w\}+c.p.\\
&\quad=[\{x,y,z\}+D(x,y)(w)-\epsilon(y,z)\theta(x,z)(v)+\epsilon(x,y+z)\theta(y,z)(u)]+c.p.
\end{aligned}$$

and

$$[[x+u,y+v],z+w]+c.p.$$

$$= [[x, y] + \rho(x)(v) - \epsilon(x, y)\rho(y)(u), z + w] + c.p.$$
$$= ([[x, y], z] + \rho([x, y])w - \epsilon(x, z)\rho(z)\rho(x)(v) + \epsilon(x + z, y)\rho(z)\rho(y)(u)) + c.p.$$

Thus by (SHR1), the condition (SHLY5) holds.

For (SHLY6), we have

$$\begin{aligned}
&\epsilon(x, z)(\{[x + u, y + v], z + w, p + t\}) + c.p. \\
&\quad = \epsilon(x, z)(\{[x, y], z, p\} + D([x, y], z)(t) - \theta([x, y], p)(w) + \epsilon(z + p + y, x)\theta(z, p)(\rho(x)(v)) - \\
&\qquad \epsilon(z + p + y, x)\theta(z, p)(\rho(y)(u))) + c.p. \\
&\quad = 0.
\end{aligned}$$

Thus by (SHR2), the condition (SHLY6) holds.

For (SHLY7), we have

$$\begin{aligned}
&\{x + u, y + v, [z + w, p + t]\} \\
&\quad = \{x, y, [z, p]\} + D(x, y)(\rho(z)(t)) - \epsilon(z, p)D(x, y)(\rho(p)(w)) - \\
&\qquad \epsilon(z + p, y)\theta(x, [z, p])(v) + \epsilon(z + p + y, x)\theta(y, [z, p])(u),
\end{aligned}$$

and

$$\begin{aligned}
&[\{x + u, y + v, z + w\}, p + t] + \epsilon(z, x + y)[z + w, \{x + u, y + v, p + t\}] \\
&\quad = [\{x, y, z\}, p] + \rho(\{x, y, z\})(t) - \epsilon(x + y + z, p)\rho(p)(D(x, y)(w)) - \\
&\qquad \epsilon(x + y + z, p)\epsilon(y, z)\rho(p)(\theta(x, z)(v)) + \epsilon(z + y, p + x)\epsilon(p, x)\rho(p)(\theta(y, z)(u)) + \\
&\qquad \epsilon(z, x + y)[z, \{x, y, p\}] + \epsilon(z, x + y)\rho(z)(D(x, y)(t)) - \epsilon(x, z)\epsilon(x + p, y)\rho(z)(\theta(x, p)(v)) + \\
&\qquad \epsilon(z, p)\epsilon(x, z + y + p)\rho(z)(\theta(y, p)(u)) - \epsilon(p, z)\rho(\{x, y, p\})(w).
\end{aligned}$$

Thus by (SHR3), the condition (SHLY7) holds.

Now it suffices to verify (SHLY8). By the definition of the LY color algebra, we have

$$\begin{aligned}
&\{x_1 + u_1, x_2 + u_2, \{y_1 + v_1, y_2 + v_2, y_3 + v_3\}\} \\
&\quad = \{x_1, x_2, \{y_1, y_2, y_3\}\} - \epsilon(x_2, y_1 + y_2 + y_3)\theta(x_1, \{y_1, y_2, y_3\})(u_2) + \\
&\qquad \epsilon(x_1, x_2 + y_1 + y_2 + y_3)\theta(x_2, \{y_1, y_2, y_3\})(u_1) + D(x_1, x_2)(D(y_1, y_2)(v_3)) - \\
&\qquad \epsilon(y_2, y_3)D(x_1, x_2)(\theta(y_1, y_3)(v_2)) + \epsilon(y_1, y_2 + y_3)D(x_1, x_2)(\theta(y_2, y_3)(v_1)), \\
&\{\{x_1 + u_1, x_2 + u_2, y_1 + v_1\}, y_2 + v_2, y_3 + v_3\} \\
&\quad = \{\{x_1, x_2, y_1\}, y_2 + y_3\} + D(\{x_1, x_2, y_1\}, y_2)(v_3) + \epsilon(y_2, y_3)\theta(\{x_1, x_2, y_1\}, y_3)(u_1) + \\
&\qquad \epsilon(y_2 + y_3, x_1 + x_2 + y_1)\theta(y_2, y_3)(D(x_1, x_2)(v_1)) - \epsilon(y_1 + y_2 + y_3, x_2)\epsilon(x_1 + y_1, y_2 + y_3) \\
&\qquad \theta(y_2, y_3)(\theta(x_1, y_1)(u_2)) + \epsilon(y_2 + y_3, x_2 + y_3)\epsilon(x_1, y_2 + x_2)\theta(y_2, y_3)(\theta(x_2, y_1)(u_1)), \\
&\qquad \epsilon(y_1, x_1 + x_2)\{y_1 + v_1, \{x_1 + u_1, x_2 + u_2, y_2 + v_2\}, y_3 + v_3\} \\
&\quad = \epsilon(y_1, x_1 + x_2)\{y_1, \{x_1, x_2, y_2\}, y_3\} + \epsilon(y_1, x_1 + x_2)D(y_1, \{x_1, x_2, y_2\})(v_3) + \\
&\qquad \epsilon(y_1, y_2 + y_3)\theta(\{x_1, x_2, y_2\}, y_3)(v_1) - \epsilon(y_2, y_3)\theta(y_1, y_3)(D(x_1, x_2)(v_3)) + \\
&\qquad \epsilon(y_1 + y_3, x_1 + y_2)\epsilon(x_2, y_1 + y_2 + x_3)\epsilon(y_1, y_2)\theta(y_1, y_3)(\theta(x_1, y_2)(u_2)) -
\end{aligned}$$

$$\begin{aligned}
&\theta(y_1,y_3)(\theta(x_2,y_2)(u_1))+\epsilon(y_1+y_2,x_1+x_2)\{y_1+v_1,y_2+v_2,\{x_1+u_1,x_2+u_2,y_3+v_3\}\}\\
&=\epsilon(y_1+y_2,x_1+x_2)[y_1,y_2,\{x_1,x_2,y_3\}]-\epsilon(y_1,x_1+x_2)\epsilon(y_2,y_3)-\\
&\theta(y_1,\{x_1,x_2,y_3\})(v_2)+\epsilon(y_2,x_1+x_2)\epsilon(y_1,y_3+y_3)\theta(y_2,\{x_1,x_2,y_3\})(v_1)+\\
&\epsilon(y_1+y_2,x_1+x_2)D(y_1,y_2)(D(x_1,x_2)(v_3))-\epsilon(y_1+y_2,x_1+x_3)\epsilon(x_2,y_3)\\
&D(y_1,y_2)(\theta(x_1,y_3)(u_2))+\epsilon(y_1+y_2,x_2+y_3)\epsilon(x_1,x_2+y_1+y_2+y_3)D(y_1,y_2)(\theta(x_2,y_3)(u_1)).
\end{aligned}$$

Thus by (SHR4), the condition (SHLY8) holds. Therefore, we obtain that $L\oplus V$ is an LY color algebra. $\square$

Let V be a representation of LY color algebra (L,ϵ). Let us define the cohomology group of (L,ϵ) with coefficients in V. In order for (δ_I,δ_{II}) to be well-defined. We need that n-linear map $f: L\times L\times\cdots\times L\to V$ satisfies the following condition:

$$f(x_1,\ldots,x_{2i-1},x_{2i},\ldots,x_n)=0,\ \text{ if }\ x_{2i-1}=x_{2i}.$$

The vector space spanned by such linear maps is called an n-cochain of L, which is denoted by $C^n(L,V)$ for $n\geq 1$.

Definition 3.7 *For any $(f,g)\in C^{2n}(L,V)\times C^{2n+1}(L,V)$ the coboundary operator $\delta:(f,g)\to(\delta_I f,\delta_{II}g)$ is a mapping from $C^{2n}(L,V)\times C^{2n+1}(L,V)$ into $C^{2n+2}(L,V)\times C^{2n+3}(L,V)$ defined as follows:*

$$\begin{aligned}
&(\delta_I f)(x_1,x_2,\ldots,x_{2n+2})\\
&\quad=\epsilon(x_{2n+1},x_1+x_2+\cdots+x_{2n})\rho(x_{2n+1})g(x_1,x_2,\ldots,x_{2n},x_{2n+2})-\\
&\quad\epsilon(x_{2n+2},x_1+x_2+\cdots+x_{2n+1})\rho(x_{2n+2})g(x_1,x_2,\ldots,x_{2n},x_{2n+1})-\\
&\quad g(x_1,x_2,\ldots,x_{2n},[x_{2n+1},x_{2n+2}])+\\
&\quad\sum_{k=1}^{n}(-1)^{n+k+1}\epsilon(x_{2k-1}+x_{2k},x_1+x_2+\cdots+x_{2k-2})\times\\
&\quad D(x_{2k-1},x_{2k})f(x_1,\ldots,\widehat{x}_{2k-1},\widehat{x}_{2k},\ldots,x_{2n+2})+\\
&\quad\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k}\epsilon(x_{2k-1}+x_{2k},x_{2k+1}+x_{2k+2}+\cdots+x_{j-1})\times\\
&\quad f(x_1,\ldots,\widehat{x}_{2k-1},\widehat{x}_{2k},\ldots,\{x_{2k-1},x_{2k},x_j\},\ldots,x_{2n+2}),\\
&(\delta_{II}g)(x_1,x_2,\ldots,x_{2n+3})\\
&\quad=\epsilon(x_{2n+2}+x_{2n+3},g+x_1+x_2+\cdots+x_{2n+1})\theta(x_{2n+2},x_{2n+3})g(x_1,\ldots,x_{2n+1})-\\
&\quad\epsilon(x_{2n+1}+x_{2n+3},g+x_1+x_2+\cdots+x_{2n})\epsilon(x_{2n+2},x_{2n+3})\\
&\quad\theta(x_{2n+1},x_{2n+3})g(x_1,\ldots,x_{2n},x_{2n+1})+\\
&\quad\sum_{k=1}^{n+1}(-1)^{n+k+1}\epsilon(x_{2k-1}+x_{2k},g+x_1+x_2+\cdots+x_{2k-2})\times\\
&\quad D(x_{2k-1},x_{2k})g(x_1,\ldots,\widehat{x}_{2k-1},\widehat{x}_{2k},\ldots,x_{2n+3})+
\end{aligned}$$

$$\sum_{k=1}^{n+1}\sum_{j=2k+1}^{2n+3}(-1)^{n+k}\epsilon(x_{2k-1}+x_{2k},x_{2k+1}+x_{2k+2}+\cdots+x_{j-1})\times$$
$$g(x_1,\ldots,\widehat{x}_{2k-1},\widehat{x}_{2k},\ldots,\{x_{2k-1},x_{2k},x_j\},\ldots,x_{2n+3}).$$

Proposition 3.8 *The coboundary operator defined above satisfies $\delta\circ\delta=0$, that is $\delta_I\circ\delta_I=0$ and $\delta_{II}\circ\delta_{II}=0$.*

Proof Similar to [6]. In the case $n=1$, we shall only consider a subspace spanned by the diagonal element $(f,f)\in C^1(T,V)\times C^1(T,V)$ and $\delta(f,f)=(\delta_I f,\delta_{II}f)$ is an element of $C^2(T,V)\times C^3(T,V)$ defined by

$$(\delta_I f)(a,b)=\rho(a)f(b)-\epsilon(a,b)\rho(b)f(a)-f(ab),$$
$$(\delta_{II}f)(a,b,c)=\epsilon(a,b+c)\theta(b,c)f(a)-\epsilon(b,c)\theta(a,c)f(b)+D(a,b)f(c)-f(\{a,b,c\}).$$

Furthermore, for each $(f,g)\in C^2(T,V)\times C^3(T,V)$ another coboundary operation $\delta^*=(\delta_I^*,\delta_{II}^*)$ of $C^2(T,V)\times C^3(T,V)$ into $C^3(T,V)\times C^4(T,V)$ is defined by

$$\begin{aligned}(\delta_I^* f)(a,b,c)=&-\rho(a)f(b,c)-\theta(b,c)\rho(b)f(c,a)-\theta(a+b,c)\rho(c)f(a,b)+f(ab,c)+\\&\theta(a,b+c)f(bc,a)+\theta(a+b,c)f(ca,b)+g(a,b,c)+\\&\theta(a,b+c)g(b,c,a)+\theta(a+b,c)g(c,a,b),\\(\delta_{II}^* f)(a,b,c,d)=&\theta(b+c,d)\theta(a,d)f(b,c)-\theta(a,b+c)\theta(c+a,d)\theta(b,d)f(c,a)+\\&\theta(a+b,c+d)\theta(c,d)f(a,b)+g(ab,c,d)\\&\theta(a,b+c)g(bc,a,d)+\theta(a+b,c)g(ca,b,d).\end{aligned}$$

For each $f\in C^1(T,V)\times C^1(T,V)$, a direct calculation shows that $\delta_I\delta_I f=\delta_I^*\delta_I f=0$ and $\delta_{II}\delta_{II}f=\delta_{II}^*\delta_{II}f=0$. Yamaguti showed that for each $(f,g)\in C^{2p}(T,V)\times C^{2p+1}(T,V)$, $\delta_I\delta_I f=0$ and $\delta_{II}\delta_{II}g=0$. □

The subspace $Z^{2n}(L,V)\times Z^{2n+1}(L,V)$ of $C^{2n}(L,V)\times C^{2n+1}(L,V)$ spanned by all the (f,g) such that $\delta(f,g)=0$ is called the space of cocycles while the space $B^{2n}(L,V)\times B^{2n+1}(L,V)=:\delta(C^{2n-2}(L,V)\times C^{2n-1}(L,V))$ is called the space of coboundaries.

Definition 3.9 *For the case $n\geq 2$, the $(2n,2n+1)$-cohomology group of an LY color algebra L with coefficients in V is defined to be the quotient space:*

$$H^{2n}(L,V)\times H^{2n+1}(L,V):=(Z^{2n}(L,V)\times Z^{2n+1}(L,V))/(B^{2n}(L,V)\times B^{2n+1}(L,V)).$$

In conclusion, we obtain a cochain complex whose cohomology group is called cohomology group of an LY color algebra L with coefficients in V.

4. Derivations on Lie-Yamaguti color algebras

In this section, we give the definition of derivations of LY color algebras, then we study their generalized derivations.

Definition 4.1 ([13]) *Let (L,ϵ) be an LY color algebra. A homogeneous map $D\in\mathrm{End}(L)$ is* *called a derivation of L if, for any $x, y, z \in L$*

$$D([x,y]) = \epsilon(D,x)[x,D(y)] + [D(x),y],$$
$$D(\{x,y,z\}) = \{D(x),y,z\} + \epsilon(D,x)\{x,D(y),z\} + \epsilon(D,x+y)\{x,y,D(z)\}.$$

We denote the set of all homogeneous derivations of L by $\mathrm{Der}(L)$. Obviously, $\mathrm{Der}(L)$ is a subalgebra of $\mathrm{End}(L)$.

Theorem 4.2 $\mathrm{Der}(L)$ *is a Lie color algebra, where the bracket product is defined as follows:*

$$[D,D'] = DD' - \epsilon(D,D')D'D.$$

Proof It suffices to prove $[\mathrm{Der}(L),\mathrm{Der}(L)] \subseteq \mathrm{Der}(L)$ for any homogeneous elements $x, y, z \in L$. Note that

$$\begin{aligned}
[D,D']([x,y]) &= D([x,D'(y)] + \epsilon(D',x)[D'(x),y]) - \epsilon(D,D')D'([x,D(y)] + \epsilon(D,x)[D(x),y]) \\
&= \epsilon(D',x)[D(x),D'(y)] + \epsilon(D+D',x)[x,DD'(y)] + [DD'(x),y] + \\
&\quad \epsilon(D,D'+x)[D'(x),D(y)] - \epsilon(D,D'+x)[D'(x),D(y)] - \\
&\quad \epsilon(D+D',x)\epsilon(D,D')[x,D'D(y)] - \epsilon(D,D')[D'D(x),y] - \epsilon(D',x)[D(x),D'(y)] \\
&= [[D,D'](x),y] + \epsilon(D+D',x)[x,[D,D'](y)].
\end{aligned}$$

Similarly, we can check that

$$\begin{aligned}
&[D,D'](\{x,y,z\}) \\
&\quad = \{[D,D'](x),y,z\} + \epsilon(D+D',x)\{x,[D,D'](y),z\} + \epsilon(D+D',x+y)\{x,y,[D,D'](z)\}.
\end{aligned}$$

It follows that $[D,D'] \in \mathrm{Der}(L)$. $\square$

Definition 4.3 *Let (L,ϵ) be an LY color algebra. $D \in \mathrm{End}_s(L)$ is said to be a homogeneous generalized derivation of L, if there exist three endomorphisms $D', D'', D''' \in \mathrm{End}_s(L)$ such that*

$$[D(x),y] + \epsilon(s,x)[x,D'(y)] = D''([x,y]),$$
$$\{D(x),y,z\} + \epsilon(s,x)\{x,D'(y),z\} + \epsilon(s,x+y)\{x,y,D''(z)\} = D'''(\{x,y,z\}),$$

for all $x, y, z \in L$.

Definition 4.4 *Let (L,ϵ) be an LY color algebra. $D \in \mathrm{End}_s(L)$ is said to be a homogeneous quasi-derivation of L, if there exist endomorphisms $D', D'' \in \mathrm{End}_s(L)$ such that*

$$[D(x),y] + \epsilon(s,x)[x,D(y)] = D'([x,y]),$$
$$\{D(x),y,z\} + \epsilon(s,x)\{x,D(y),z\} + \epsilon(s,x+y)\{x,y,D(z)\} = D''(\{x,y,z\}),$$

for all $x, y, z \in L$.

Let $\mathrm{GDer}(L)$ and $\mathrm{QDer}(L)$ be the sets of homogeneous generalized derivations and of homogeneous quasi-derivations, respectively.

Definition 4.5 *Let (L,ϵ) be an LY color algebra. The centroid of L is the space of linear* *transformations on L given by*

$$\begin{aligned} &C(L)=\{D\in \mathrm{End}(L)|[D(x),y]=\epsilon(D,x)[x,D(y)]=D([x,y]),\\ &\{D(x),y,z\}=\epsilon(D,x)\{x,D(y),z\}=\epsilon(D,x+y)\{x,y,D(z)\}=D(\{x,y,z\})\}. \end{aligned}$$

We denote $C(L)$ and call it the centroid of L.

Definition 4.6 *Let (L,ϵ) be an LY color algebra. The quasi-centroid of L is the space of linear transformations on L given by*

$$\begin{aligned} QC(L)=&\big\{D\in \mathrm{End}(L)|[D(x),y]=\epsilon(D,x)[x,D(y)],\\ &\{D(x),y,z\}=\epsilon(D,x)\{x,D(y),z\}=\epsilon(D,x+y)\{x,y,D(z)\}\big\}, \end{aligned}$$

for all $x,y,z\in L$. We denote $QC(L)$ and call it the quasi-centroid of L.

Remark 4.7 Let (L,ϵ) be an LY color algebra. Then $C(L)\subseteq QC(L)$.

Definition 4.8 *Let (L,ϵ) be an LY color algebra. $D\in\mathrm{End}(L)$ is said to be a central derivation of L if*

$$[D(x),y]=D([x,y])=0,\quad D(\{x,y,z\})=\{D(x),y,z\}=\{x,y,D(z)\}=0,$$

for all $x,y,z\in L$. Denote by $\mathrm{ZDer}(L)$ the set of all central derivations.

Remark 4.9 Let (L,ϵ) be an LY color algebra. Then

$$\mathrm{ZDer}(L)\subseteq \mathrm{Der}(L)\subseteq \mathrm{QDer}(L)\subseteq \mathrm{GDer}(L)\subseteq \mathrm{End}(L).$$

Definition 4.10 *Let (L,ϵ) be an LY color algebra. If $Z(L)=\{x\in L|\ [x,y]=\{x,y,z\}=0,$ $\forall\, y,z\in L\}$, then $Z(L)$ is called the center of L.*

Proposition 4.11 *Let (L,ϵ) be an LY color algebra. Then the following statements hold:*

(1) $\mathrm{GDer}(L),\mathrm{QDer}(L)$ *and* $C(L)$ *are subalgebras of* $\mathrm{End}(L)$.

(2) $\mathrm{ZDer}(L)$ *is an ideal of* $\mathrm{Der}(L)$.

Proof (1) We only prove that $\mathrm{GDer}(L)$ is a subalgebra of $\mathrm{End}(L)$, and similarly for cases of $\mathrm{QDer}(L)$ and $C(L)$. For any homogeneous map $D_1,D_2\in\mathrm{GDer}(L)$ and $x,y,z\in L$, we have

$$\begin{aligned} &\{D_1D_2(x),y,z\}=D_1'''\{D_2(x),y,z\}-\epsilon(D_1,D_2+x)\{D_2(x),D_1'(y),z\}-\\ &\quad\epsilon(D_1,D_2+x+y)\{D_2(x),y,D_1''(z)\}\\ &\ =D_1'''\{D_2'''(\{x,y,z\})-\epsilon(D_2,x)\{x,D_2'(y),z\}-\epsilon(D_2,x+y)\{x,y,D_2''(z)\}\}-\\ &\quad\epsilon(D_1,D_2+x)\{D_2(x),D_1'(y),z\}-\epsilon(D_1,D_2+x+y)\{D_2(x),y,D_1''(z)\}\\ &\ =D_1'''D_2'''(\{x,y,z\})-\epsilon(D_2,x)D_1'''\{x,D_2'(y),z\}-\epsilon(D_2,x+y)D_1'''\{x,y,D_2''(z)\}-\\ &\quad\epsilon(D_1,D_2+x)\{D_2(x),D_1'(y),z\}-\epsilon(D_1,D_2+x+y)\{D_2(x),y,D_1''(z)\}\\ &\ =D_1'''D_2'''(\{x,y,z\})-\epsilon(D_2,x)\{D_1(x),D_2'(y),z\}-\epsilon(D_1+D_2,x)\{x,D_1'D_2'(y),z\}-\\ &\quad\epsilon(D_2,x)\epsilon(D_1,x+y+D_2)\{x,D_2'(y),D_1''(z)\}-\epsilon(D_2,x+y)\{D_1(x),y,D_2''(z)\}-\\ &\quad\epsilon(D_2,x+y)\epsilon(D_1,x)\{x,D_1'(y),D_2''(z)\}-\epsilon(D_1+D_2,x+y)\{x,y,D_1''D_2''(z)\}- \end{aligned}$$

$$\epsilon(D_1, D_2 + x)\{D_2(x), D_1'(y), z\} - \epsilon(D_1, D_2 + x + y)\{D_2(x), y, D_1''(z)\}.$$

Similarly, we have

$$\begin{aligned}
&\{D_2D_1(x), y, z\}\\
&= D_2'''D_1'''(\{x, y, z\}) - \epsilon(D_1, x)\{D_2(x), D_1'(y), z\} - \epsilon(D_1 + D_2, x)\{x, D_2'D_1'(y), z\}-\\
&\quad \epsilon(D_1, x)\epsilon(D_2, x + y + D_1)\{x, D_1'(y), D_2''(z)\} - \epsilon(D_1, x + y)\{D_2(x), y, D_1''(z)\}-\\
&\quad \epsilon(D_1, x + y)\epsilon(D_2, x)\{x, D_2'(y), D_1''(z)\} - \epsilon(D_1 + D_2, x + y)\{x, y, D_2''D_1''(z)\}-\\
&\quad \epsilon(D_2, D_1 + x)\{D_1(x), D_2'(y), z\} - \epsilon(D_2, D_1 + x + y)\{D_1(x), y, D_2''(z)\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\{[D_1, D_2](x), y, z\} = \{D_1D_2(x), y, z\} - \epsilon(D_1, D_2)\{D_2D_1(x), y, z\}\\
&\quad = (D_1'''D_2''' - \epsilon(D_1, D_2)D_2'''D_1''')\{x, y, z\}-\\
&\qquad \epsilon(x, D_1 + D_2)\{x, y, D_1''D_2''(z)\}\{x, (D_1'D_2' - \epsilon(D_1, D_2)D_2'D_1')(y), z\}-\\
&\qquad \epsilon(x + y, D_1 + D_2)\{x, y, (D_1''D_2'' - \epsilon(D_1, D_2)D_2''D_1'')(z)\}\\
&\quad = [D_1''', D_2''']\{x, y, z\} - \epsilon(x, D_1 + D_2)\{x, [D_1', D_2'](y), z\}-\\
&\qquad \epsilon(x + y, D_1 + D_2)\{x, y, [D_1'', D_2''](z)\},
\end{aligned}$$

and it is easy to check that

$$[[D_1, D_2](x), y] = [D_1'', D_2''][x, y] - \epsilon(x, D_1 + D_2)[x, [D_1', D_2'](y)].$$

Obviously, $[D_1', D_2'], [D_1'', D_2'']$ and $[D_1''', D_2''']$ are contained in $\mathrm{End}(L)$, thus $[D_1, D_2] \in \mathrm{GDer}(L) \subseteq \mathrm{End}(L)$, that is, $\mathrm{GDer}(L)$ is a subalgebra of $\mathrm{End}(L)$.

(2) For any $D_1 \in \mathrm{ZDer}(L), D_2 \in \mathrm{Der}(L)$ and $x, y, z \in L$, we have

$$[D_1, D_2](\{x, y, z\}) = D_1D_2(\{x, y, z\}) - \epsilon(D_1, D_2)D_2D_1(\{x, y, z\}) = 0.$$

Also, we have

$$\begin{aligned}
&\{[D_1, D_2](x), y, z\} = \{D_1D_2(x), y, z\} - \epsilon(D_1, D_2)\{D_2D_1(x), y, z\}\\
&\quad = 0 - \epsilon(D_1, D_2)\{D_2D_1(x), y, z\}\\
&\quad = -\epsilon(D_1, D_2)(D_2(\{D_1(x), y, z\}) - \epsilon(D_2, D_1 + x)\{D_1(x), D_2(y), z\})+\\
&\qquad \epsilon(D_2, x + y)\{D_1(x), y, D_2(z)\} = 0,
\end{aligned}$$

and it is easy to check that

$$[[D_1, D_2](x), y] = 0.$$

It follows that $[D_1, D_2] \in \mathrm{ZDer}(L)$. That is, $\mathrm{ZDer}(L)$ is an ideal of $\mathrm{Der}(L)$. $\square$

Lemma 4.12 *Let (L, ϵ) be an LY color algebra. Then the following statements hold:*

(1) $[\mathrm{Der}(L), C(L)] \subseteq C(L)$.

(2) $[\mathrm{QDer}(L), QC(L)] \subseteq QC(L)$.

(3) $[QC(L), QC(L)] \subseteq \mathrm{QDer}(L)$.

(4) $C(L) \subseteq \mathrm{QDer}(L)$.

(5) $\mathrm{QDer}(L) + QC(L) \subseteq \mathrm{GDer}(L)$.

Proof (1)–(4) are easy to prove and we omit them, we only check (5). In fact. Let $D_1 \in \mathrm{QDer}(L), D_2 \in QC(L)$. Then there exist homogeneous maps $D_1', D_1'' \in \mathrm{End_s}(L)$, for any $x, y, z \in L$, we have

$$\begin{aligned}&[D_1(x), y] + \epsilon(s, x)[x, D_1(y)] = D_1'([x, y]),\\&\{D_1(x), y, z\} + \epsilon(s, x)\{x, D_1(y), z\} + \epsilon(s, x+y)\{x, y, D_1(z)\} = D_1''(\{x, y, z\}).\end{aligned}$$

Thus, for any $x, y, z \in L$, we have

$$\begin{aligned}[(D_1 + D_2)(x), y] =&[D_1(x), y] + [D_2(x), y] = D_1'([x, y]) - \epsilon(s, x)[x, D_1(y)] + \epsilon(s, x)[x, D_2(y)]\\=&D_1'([x, y]) - \epsilon(s, x)[x, (D_1 - D_2)(y)],\end{aligned}$$

and

$$\begin{aligned}&\{(D_1 + D_2)(x), y, z\} = \{D_1(x), y, z\} + \{D_2(x), y, z\}\\&= D_1''(\{x, y, z\}) - \epsilon(s, x)\{x, D_1(y), z\} - \epsilon(s, x+y)\{x, y, D_1(z)\} + \epsilon(s, x)\{x, D_2(y), z\}\\&= D_1''(\{x, y, z\}) - \epsilon(s, x)\{x, (D_1 - D_2)(y), z\} - \epsilon(s, x+y)\{x, y, D_1(z)\}.\end{aligned}$$

Therefore, $D_1 + D_2 \in \mathrm{GDer}(L)$. □

Proposition 4.13 *Let* (L, ϵ) *be an LY color algebra. Then* $QC(L) + [QC(L), QC(L)]$ *is a subalgebra of* $\mathrm{GDer}(L)$.

Proof By Lemma 4.12, (3) and (5), we have

$$QC(L) + [QC(L), QC(L)] \subseteq \mathrm{GDer}(L),$$

and it follows that

$$\begin{aligned}&[QC(L) + [QC(L), QC(L)], QC(L) + [QC(L), QC(L)]]\\&\subseteq [QC(L) + \mathrm{QDer}(L), QC(L) + [QC(L), QC(L)]]\\&\subseteq [QC(L), QC(L)] + [QC(L), [QC(L), QC(L)]] +\\&\quad [\mathrm{QDer}(L), QC(L)][\mathrm{QDer}(L), [QC(L), QC(L)]].\end{aligned}$$

It is easy to verify that $[\mathrm{QDer}(L), [QC(L), QC(L)]] \subseteq [QC(L), QC(L)]$ by the Jacobi identity of Lie algebras. Thus

$$\begin{aligned}&[QC(L) + [QC(L), QC(L)], QC(L) + [QC(L), QC(L)]]\\&\subseteq QC(L) + [QC(L), QC(L)].\end{aligned}$$

The proof is completed. □

Theorem 4.14 *Let* (L, ϵ) *be an LY color algebra. Then* $[C(L), QC(L)] \subseteq \mathrm{End}(L, Z(L))$. *Moreover, if* $Z(L) = \{0\}$, *then* $[C(L), QC(L)] = \{0\}$.

Proof For any $D_1 \in C(L), D_2 \in QC(L)$ and $x, y, z \in L$, then we have

$$\begin{aligned}
[[D_1, D_2](x), y] &= [D_1D_2(x), y] - \epsilon(D_1, D_2)[D_2D_1(x), y] \\
&= D_1([D_2(x), y]) - \epsilon(D_1, D_2)\epsilon(D_2, D_1 + x)[D_1(x), D_2(y)] \\
&= D_1([D_2(x), y]) - D_1([D_2(x), y]) = 0, \\
\{[D_1, D_2](x), y, z\} &= \{D_1D_2(x), y, z\} - \epsilon(D_1, D_2)\{D_2D_1(x), y, z\} \\
&= D_1(\{D_2(x), y, z\}) - \epsilon(D_2, x)\{D_1(x), D_2(y), z\} \\
&= \epsilon(x, D_2)D_1(\{x, D_2(y), z\}) - D_1(\{x, D_2(y), z\}) = 0.
\end{aligned}$$

So $[D_1, D_2](x) \subseteq Z(L)$ and therefore $[C(L), QC(L)] \subseteq \mathrm{End}(L, Z(L))$. Moreover, if $Z(L) = \{0\}$, it is easy to see that $[C(L), QC(L)] = \{0\}$. □

5. Deformations of Lie-Yamaguti color algebras

In this section, we study linear deformations of Lie-Yamaguti color algebras, and introduce the notion of a Nijenhuis operator on a Lie-Yamaguti color algebra, which can generate a trivial deformation.

Definition 5.1 *Let (L, ϵ) be an LY color algebra, $\varphi : \bigoplus^2 L \to L$ and $\omega, \psi : \bigoplus^3 L \to L$ be bilinear and trilinear maps. Consider the following linear operators:*

$$[x, y]_t = [x, y] + t\varphi(x, y), \tag{5.1}$$

$$\{x, y, z\}_t = \{x, y, z\} + t\omega(x, y, z) + t^2\psi(x, y, z), \quad \forall\, x, y, z \in L,\ t \in \mathbb{K}. \tag{5.2}$$

If $(L, [\cdot, \cdot]_t, \{\cdot, \cdot, \cdot\}_t, \epsilon)$ is an LY color algebra, then we say that (φ, ω, ψ) generates a linear deformation of the LY color algebra (L, ϵ).

Now we are ready to give the relation between the deformations and cohomologies of LY color algebras.

Theorem 5.2 *Let (L, ϵ) be an LY color algebra, $\varphi \in \mathrm{Hom}(\wedge^2 L, L)$ and $\omega, \psi \in \mathrm{Hom}(\wedge^2 L \otimes L, L)$. Then (ω, ψ) and (φ, ω, ψ) generate a linear deformation of the LY color algebra (L, ϵ) if and only if the following conditions are satisfied:*

(i) (φ, ω) *is a (2,3)-cocycle;*

(ii) (φ, ψ) *defines an LY color algebra structure on* (L, ϵ)*;*

(iii) the following equations hold:

(a) $\psi([x, y], z, w) + \omega(\varphi(x, y), z, w) + \psi([x, y], z, w) + \epsilon(x, y + z)\omega(\varphi(y, z), x, w) +$
$\epsilon(x + y, z)\psi([z, x], y, w) + \epsilon(x + y, z)\omega(\varphi(z, x), y, w) = 0;$

(b) $\omega(x, y, \varphi(z, w)) + \psi(x, y, [z, w]) = \epsilon(x + y, z)\varphi(z, \omega(x, y, w)) +$
$\epsilon(x + y, z)[z, \psi(x, y, w)] + \varphi(\omega(x, y, z), w) + [\psi(x, y, z), w];$

(c) $\psi(x_1, x_2, \{y_1, y_2, y_3\}) + \omega(x_1, x_2, \omega(y_1, y_2, y_3)) + \{x_1, x_2, \psi(y_1, y_2, y_3)\}$
$= \psi(\{x_1, x_2, y_1\}, y_2, y_3) + \omega(\omega(x_1, x_2, y_1), y_2, y_3) + \{\psi(x_1, x_2, y_1), y_2, y_3\} +$

$$\epsilon(x_1+x_2,y_1)\psi(y_1,\{x_1,x_2,y_2\},y_3)+\epsilon(x_1+x_2,y_1)\omega(y_1,\omega(x_1,x_2,y_2),y_3)+$$
$$\epsilon(x_1+x_2,y_1)\{y_1,\psi(x_1,x_2,y_2),y_3\}+\epsilon(x_1+x_2,y_1+y_2)\psi(y_1,y_2,\{x_1,x_2,y_3\})+$$
$$\epsilon(x_1+x_2,y_1+y_2)\omega(y_1,y_2,\omega(x_1,x_2,y_3))+\epsilon(x_1+x_2,y_1+y_2)\{y_1,y_2,\psi(x_1,x_2,y_3)\};$$

(d) $\psi(x_1,x_2,\omega(y_1,y_2,y_3))+\omega(x_1,x_2,\psi(y_1,y_2,y_3))$

$$=\psi(\omega(x_1,x_2,y_1),y_2,y_3)+\omega(\psi(x_1,x_2,y_1),y_2,y_3)+$$
$$\epsilon(x_1+x_2,y_1)\psi(y_1,\omega(x_1,x_2,y_2),y_3)+\epsilon(x_1+x_2,y_1)\omega(y_1,\psi(x_1,x_2,y_2),y_3)+$$
$$\epsilon(x_1+x_2,y_1+y_2)\psi(y_1,y_2,\omega(x_1,x_2,y_3))+$$
$$\epsilon(x_1+x_2,y_1+y_2)\omega(y_1,y_2,\psi(x_1,x_2,y_3)),$$

for any homogeneous elements $x,y,z,x_i,y_j\in L, 1\le i\le 2, 1\le j\le 3$.

Proof Suppose that (φ,ω,ψ) generates a linear deformation $([\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t)$ of an LY color algebra (L,ϵ), then $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\epsilon)$ is an LY color algebra, that is $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\epsilon)$ satisfies (SHLY1)–(SHLY8) of Definition 3.1. By (SHLY5), we have

$$\varphi([x,y],z)+c.p.+[\varphi(x,y),z]+c.p.+\omega(x,y,z)+c.p.=0; \tag{5.3}$$
$$\varphi(\varphi(x,y),z)+c.p.+\psi(x,y,z)+c.p.=0. \tag{5.4}$$

By (SHLY6), we have

$$\omega([x,y],z,w)+\{\varphi(x,y),z,w\}+\epsilon(x,y+z)\omega([y,z],x,w)+\epsilon(x,y+z)\{\varphi(y,z),x,w\}+$$
$$\epsilon(z,y+x)\omega([z,x],y,w)+\epsilon(x+y,z)\{\varphi(z,x),y,w\}=0; \tag{5.5}$$
$$\psi([x,y],z,w)+\omega(\varphi(x,y),z,w)+\epsilon(x,y+z)\psi([y,z],x,w)+\epsilon(x,y+z)\omega(\varphi(y,z),x,w)+$$
$$\epsilon(z,y+x)\psi([z,x],y,w)+\epsilon(x+y,z)\omega(\varphi(z,x),y,w)=0; \tag{5.6}$$
$$\psi(\varphi(x,y),z,w)+\epsilon(x,y+z)\psi(\varphi(y,z),x,w)+\epsilon(x+y,z)\psi(\varphi(z,x),y,w)=0. \tag{5.7}$$

By (SHLY7), calculating the coefficient terms for the corresponding t^i, $i=1,2$, we have

$$\omega(x,y,[z,w])+\{x,y,\varphi(z,w)\}=\epsilon(x+y,z)\varphi(z,\{x,y,w\})+\epsilon(x+y,z)[z,\omega(x,y,w)]+$$
$$\varphi(\{x,y,z\},w)+[\omega(x,y,z),w]; \tag{5.8}$$
$$\omega(x,y,\varphi(z,w))+\psi(x,y,[z,w])=\epsilon(x+y,z)\varphi(z,\omega(x,y,w))+\epsilon(x+y,z)[z,\psi(x,y,w)]+$$
$$\varphi(\omega(x,y,z),w)+[\psi(x,y,z),w]; \tag{5.9}$$
$$\psi(x,y,\varphi(z,w))=\epsilon(x+y,z)\varphi(z,\psi(x,y,w))+\varphi(\psi(x,y,z),w). \tag{5.10}$$

By (SHLY8), calculating the coefficient terms for the corresponding t^i, $i=1,2,3,4$, we have

$$\omega(x_1,x_2,\{y_1,y_2,y_3\})+\{x_1,x_2,\omega(y_1,y_2,y_3)\}$$
$$=\omega(\{x_1,x_2,y_1\},y_2,y_3)+\{\omega(x_1,x_2,y_1),y_2,y_3\}+\epsilon(x_1+x_2,y_1)\omega(y_1,\{x_1,x_2,y_2\},y_3)+$$
$$\epsilon(x_1+x_2,y_1)\{y_1,\omega(x_1,x_2,y_2),y_3\}+\epsilon(x_1+x_2,y_1+y_2)\omega(y_1,y_2,\{x_1,x_2,y_3\})+$$
$$\epsilon(x_1+x_2,y_1+y_2)\{y_1,y_2,\omega(x_1,x_2,y_3)\}; \tag{5.11}$$
$$\psi(x_1,x_2,\{y_1,y_2,y_3\})+\omega(x_1,x_2,\omega(y_1,y_2,y_3))+\{x_1,x_2,\psi(y_1,y_2,y_3)\}$$

$$\begin{aligned}
&= \psi(\{x_1,x_2,y_1\},y_2,y_3)+\omega(\omega(x_1,x_2,y_1),y_2,y_3)+\{\psi(x_1,x_2,y_1),y_2,y_3\}+\\
&\quad \epsilon(x_1+x_2,y_1)\psi(y_1,\{x_1,x_2,y_2\},y_3)+\epsilon(x_1+x_2,y_1)\omega(y_1,\omega(x_1,x_2,y_2),y_3)+\\
&\quad \epsilon(x_1+x_2,y_1)\{y_1,\psi(x_1,x_2,y_2),y_3\}+\epsilon(x_1+x_2,y_1+y_2)\psi(y_1,y_2,\{x_1,x_2,y_3\})+\\
&\quad \epsilon(x_1+x_2,y_1+y_2)\omega(y_1,y_2,\omega(x_1,x_2,y_3))+\epsilon(x_1+x_2,y_1+y_2)\{y_1,y_2,\psi\{x_1,x_2,y_3)\}; \quad (5.12)
\end{aligned}$$

$$\begin{aligned}
&\psi(x_1,x_2,\omega(y_1,y_2,y_3))+\omega(x_1,x_2,\psi(y_1,y_2,y_3))\\
&= \psi(\omega(x_1,x_2,y_1),y_2,y_3)+\omega(\psi(x_1,x_2,y_1),y_2,y_3)+\epsilon(x_1+x_2,y_1)\psi(y_1,\omega(x_1,x_2,y_2),y_3)+\\
&\quad \epsilon(x_1+x_2,y_1)\omega(y_1,\psi(x_1,x_2,y_2),y_3)+\epsilon(x_1+x_2,y_1+y_2)\psi(y_1,y_2,\omega(x_1,x_2,y_3))+\\
&\quad \epsilon(x_1+x_2,y_1+y_2)\omega(y_1,y_2,\psi(x_1,x_2,y_3)); \quad (5.13)
\end{aligned}$$

$$\begin{aligned}
&\psi(x_1,x_2,\psi(y_1,y_2,y_3))\\
&= \psi(\psi(x_1,x_2,y_1),y_2,y_3)+\epsilon(x_1+x_2,y_1)\psi(y_1,\psi(x_1,x_2,y_2),y_3)+\\
&\quad \epsilon(x_1+x_2,y_1+y_2)\psi(y_1,y_2,\psi(x_1,x_2,y_3)). \quad (5.14)
\end{aligned}$$

By (5.3), (5.5), (5.8) and (5.11), Condition (i) is satisfied. By (5.4), (5.7), (5.10) and (5.14), condition (ii) is satisfied. By (5.6), (5.9), (5.12) and (5.13), Condition (iii) is satisfied. □

Definition 5.3 *Let* $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\epsilon)$ *and* $(L,[\cdot,\cdot]'_t,\{\cdot,\cdot,\cdot\}'_t,\epsilon)$ *be two linear deformations of an LY color algebra* (L,ϵ) *generated by* (φ,ω,ψ) *and* (φ',ω',ψ')*, respectively.*

(i) They are called equivalent if there exists a linear map $N\in \mathrm{End}(L)$ *such that* $T_t=\mathrm{Id}+tN:(L,[\cdot,\cdot]'_t,\{\cdot,\cdot,\cdot\}'_t,\epsilon)\to(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\epsilon)$ *is a homomorphism.*

(ii) A linear deformation $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\epsilon)$ *of an LY color algebra* (L,ϵ) *is said to be trivial if it is equivalent to* (L,ϵ).

If two linear deformations are equivalent, we have

$$(\mathrm{Id}+tN)[x,y]'_t=[x+tNx,y+tNy]_t, \quad (5.15)$$

$$(\mathrm{Id}+tN)\{x,y,z\}'_t=\{x+tNx,y+tNy,z+tNz\}_t,\quad \forall\, x,y,z\in L. \quad (5.16)$$

Calculating above two equations, we have

$$\varphi'(x,y)-\varphi(x,y)=[Nx,y]+[x,Ny]-N[x,y], \quad (5.17)$$

$$\omega'(x,y,z)-\omega(x,y,z)=\{Nx,y,z\}+\{x,Ny,z\}+\{x,y,Nz\}-N\{x,y,z\}. \quad (5.18)$$

Theorem 5.4 *Let* $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\epsilon)$ *and* $(L,[\cdot,\cdot]'_t,\{\cdot,\cdot,\cdot\}'_t,\epsilon)$ *be two equivalent deformations of an LY color algebra* (L,ϵ) *generated by* (φ,ω,ψ) *and* (φ',ω',ψ')*, respectively. Then* (φ',ω') *and* (φ,ω) *are in the same cohomology class in the cohomology group* $H^2(L,L)\times H^3(L,L)$.

Next we introduce the notion of a Nijenhuis operator on an LY color algebra by considering trivial deformations.

Let $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\epsilon)$ be a trivial deformation of an LY color algebra (L,ϵ) generated by (φ,ω,ψ). Then there exists a linear map $N\in\mathrm{End}(L)$ such that $T_t=\mathrm{Id}+tN$ satisfies:

$$T_t[x,y]_t=[T_t(x),T_t(y)], \quad (5.19)$$

$$T_t\{x,y,z\}_t=\{T_t(x),T_t(y),T_t(z)\},\quad \forall x,y,z\in L. \quad (5.20)$$

Calculating above two equations, we obtain

$$\varphi(x,y) = [x, Ny] + [Nx, y] - N[x, y]; \tag{5.21}$$

$$[Nx, Ny] = N\varphi(x, y), \tag{5.22}$$

and

$$\omega(x, y, z) = \{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\} - N\{x, y, z\}, \tag{5.23}$$

$$\psi(x, y, z) = \{Nx, Ny, z\} + \{Nx, y, Nz\} + \{x, Ny, Nz\} - N\omega(x, y, z), \tag{5.24}$$

$$\{Nx, Ny, Nz\} = N\psi(x, y, z). \tag{5.25}$$

Definition 5.5 *Let (L, ϵ) be an LY color algebra. A linear map $N : L \to L$ is called a Nijenhuis operator if for any homogeneous elements $x, y, z \in L$, the following conditions are satisfied:*

$$\begin{aligned}[Nx, Ny] =& N([Nx, y] + [x, Ny] - N[x, y]),\\ \{Nx, Ny, Nz\} =& N(\{Nx, Ny, z\} + \{Nx, y, Nz\} + \{x, Ny, Nz\}) -\\ & N^2(\{x, y, z\} + \{x, Ny, z\} + \{x, y, Nz\}) + N^3\{x, y, z\}.\end{aligned}$$

It is obvious that a trivial deformation of an LY color algebra gives rise to a Nijenhuis operator. In the sequel, we show that the converse is also true.

Lemma 5.6 *Let (L, ϵ) be an LY color algebra and L' a vector space endowed with a binary bracket $[\cdot, \cdot]'$ and a ternary bracket $\{\cdot, \cdot, \cdot\}'$. If there exists an isomorphism between vector spaces $f : L' \to L$ such that*

$$\begin{aligned}&f([x, y]') = [f(x), f(y)],\\ &f(\{x, y, z\}') = \{f(x), f(y), f(z)\}, \quad \forall\, x, y, z \in L'.\end{aligned}$$

Then $(L', [\cdot, \cdot]', \{\cdot, \cdot, \cdot\}', \epsilon)$ is an LY color algebra.

Proof Straightforward. □

Theorem 5.7 *Let $N : L \to L$ be a Nijenhuis operator on an LY color algebra (L, ϵ). Then we have a deformation*

$$\begin{aligned}&\varphi(x, y) = [x, Ny] + [Nx, y] - [Nx, Ny],\\ &\omega(x, y, z) = \{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\} - N\{x, y, z\},\\ &\psi(x, y, z) = \{Nx, Ny, z\} + \{Nx, y, Nz\} + \{x, Ny, Nz\} - N\omega\{x, y, z\},\end{aligned}$$

for any homogeneous elements $x, y, z \in L$. Moreover, this deformation is trivial.

Proof Since N is a Nijenhuis operator, the given maps φ, ψ, ω satisfy $[Nx, Ny] = N\varphi(x, y)$, $\{Nx, Ny, Nz\} = N\psi(x, y, z)$, for any homogeneous elements $x, y, z \in L$. Hence, the given maps φ, ψ, ω satisfy conditions (5.21)–(5.25). Therefore, $T_t = \mathrm{Id} + tN$ is a homomorphism from $(L, [\cdot, \cdot]_t, \{\cdot, \cdot, \cdot\}_t, \epsilon)$ to $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \epsilon)$. For t sufficiently small, T_t is an isomorphism between vector spaces. By Lemma 5.6, we can deduce that $(L, [\cdot, \cdot]_t, \{\cdot, \cdot, \cdot\}_t, \epsilon)$ is an LY color algebra for t sufficiently small. Thus, (φ, ψ, ω) generates a linear deformation. It is obvious that the deformation is trivial. $\square$

Corollary 5.8 *Let $N : L \to L$ be a Nijenhuis operator on an LY color algebra (L, ϵ). Then $(L, \varphi, \psi, \epsilon)$ is an LY color algebra.*

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