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The Zeros and Nevanlinna Deficiencies for Some q-Shift Difference Differential Polynomials of Meromorphic Functions

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Abstract The first purpose of this paper is to study the properties on some q-shift difference differential polynomials of meromorphic functions, some theorems about the zeros of some q-shift difference-differential polynomials with more general forms are obtained. The second purpose of this paper is to investigate the properties on the Nevanlinna deficiencies for q-shift difference differential monomials of meromorphic functions, we obtain some relations among $\delta(\infty,f)$, $\delta(\infty,f')$, $\delta(\infty,f(z)^n f(qz+c)^m f'(z))$, $\delta(\infty,f(qz+c)^m f'(z))$ and $\delta(\infty,f(z)^n f(qz+c)^m)$.

Keywords Nevanlinna theory; q-shift difference differential; zero order

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1. Introduction and main results

We assume that the readers shall be familiar with the fundamental theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions which can be found in Hayman [1], Yang [2] and Yi and Yang [3]. For meromorphic function f, let $S(r,f)$ be any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure

$$\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{\mathrm{d}t}{t}<\infty.$$

We also use $S_1(r,f)$ to denote any quantity satisfying $S_1(r,f)=o(T(r,f))$ for all r on a set F of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty}\frac{1}{\log r}\int_{[1,r]\cap F}\frac{1}{t}\mathrm{d}t.$$

Let $\delta(a,f)$ be the Nevanlinna deficiency of a to f, which is defined by

$$\delta(a,f)=\liminf_{r\to+\infty}\frac{m(r,\frac{1}{f-a})}{T(r,f)}=1-\limsup_{r\to+\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

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where $m(r,f)=m(r,\frac{1}{f-a})$ and $N(r,f)=N(r,\frac{1}{f-a})$ if $a=+\infty$, $a\in\widetilde{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$.

If $\delta(a,f)>0$, then the value a is deficient in Nevanlinna's sense. It is clear from Nevanlinna's first fundamental theorem that $0\le\delta(a,f)\le 1$.

In 1959, Hayman [4] studied value distribution of meromorphic function and its derivatives, and obtained the following famous theorem.

Theorem 1.1 ([4]) *Let $f(z)$ be a transcendental entire function. Then for $n\ge 2$, $f(z)^n f'(z)$ assumes all finite values except possibly zero infinitely often.*

For transcendental meromorphic function f, Chen-Fang [5] obtained the following result

Theorem 1.2 ([5, Theorem 1]) *Let $f(z)$ be a transcendental meromorphic function. If $n\ge 1$ is a positive integer, $f(z)^n f'(z)-1$ has infinitely many zeros.*

Recently, many articles have focused on the zeros of $f(z)^n f(z+c)-\alpha$, $f(z)^n f(qz)-\alpha$ or their improvements, α is a nonzero constant, where and in the following, q, c are nonzero complex constants, including [6–15]. The main purpose of these results is to get the sharp number of n to make that the difference polynomials or q-difference polynomials admit infinitely many zeros. For transcendental meromorphic (resp., entire) function f of zero order, Zhang and Korhonen [16] studied the value distribution of q-difference polynomials of meromorphic functions and obtained that if $n\ge 6$ (resp., $n\ge 2$), then $f(z)^n f(qz)$ assumes every nonzero value $a\in\mathbb{C}$ infinitely often [16, Theorem 4.1].

Recently, Liu-Cao [17] considered the value distribution of $f(qz)^n f'(z)-a(z)$ and obtained the following result

Theorem 1.3 ([17, Theorem 4.2]) *Let $f(z)$ be a transcendental entire function with zero-order, $q\in\mathbb{C}\setminus\{0\}$ and $n\ge 9$. Then $f(qz)^n f'(z)-a(z)$ has infinitely many zeros, where and in the following $a(z)$ is a non-zero small function of growth $S(r,f)$.*

In this paper, we further investigate the zeros of several q-shift difference differential polynomials of meromorphic function when $f(qz)$ is replaced by $f(qz+c)$ in Theorem 1.3, and obtain the following results.

Theorem 1.4 *Let $f(z)$ be a transcendental meromorphic function of zero order and $n\ge 11$. Then $f(qz+c)^n+f'(z)+f(z)-a$ has infinitely many zeros.*

Theorem 1.5 *Let $f(z)$ be a transcendental meromorphic function of zero order, and let*

$$F_1(z)=f(z)^n f(qz+c)^m f'(z),$$

where m,n are positive integers and satisfy $m\ge n+10$ or $n\ge m+10$. Then $F_1(z)-a(z)$ has infinitely many zeros.

Let $P_n(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ be a nonzero polynomial, and t_n be the number of the distinct zeros of $P_n(z)$, where $a_0,a_1,\ldots,a_n$ are complex constants. Then we have

Theorem 1.6 *Let $f(z)$ be a transcendental meromorphic function of zero order, and let*

$$F_2(z) = f(z)^m P_n(f(qz+c)) \prod_{j=1}^{k} f^{(j)}(z),$$

where m, n are positive integers and satisfy $m \geq n + t_n + k(k+3) + 4$. Then $F_2(z) - a(z)$ has infinitely many zeros.

Theorem 1.7 *If $f(z)$ is a transcendental meromorphic function of zero order, and let*

$$F_3(z) = P_m(f(z)) f(qz+c)^n \prod_{j=1}^{k} f^{(j)}(z),$$

where m, n are positive integers and satisfy $n \geq m + t_m + k(k+3) + 4$. Then $F_3(z) - a(z)$ has infinitely many zeros.

For meromorphic function $f(z)$, we also deal with the relations of Nevanlinna deficiencies among $f(z)$, $f'(z)$ and

$$F_4(z) = f(qz+c)^m f'(z) \text{ and } F_5(z) = f(z)^n f(qz+c)^m,$$

and obtain the following theorems.

Theorem 1.8 *Let $f(z)$ be a meromorphic function of zero order, and q, c be two nonzero complex constants. If $\delta(\infty, f) > \frac{4m+4}{n+3m+3}$, then we have $\delta(\infty, F_1) > 0$.*

Corollary 1.9 *Let $f(z)$ be a non-constant meromorphic function of zero order. If $\delta(\infty, f) > \frac{4}{m+3}$, then we have $\delta(\infty, F_4) > 0$.*

Corollary 1.10 *Let $f(z)$ be a non-constant meromorphic function of zero order. If $\delta(\infty, f) > \frac{4m}{n+3m}$, then we have $\delta(\infty, F_5) > 0$.*

Theorem 1.11 *Let $f(z)$ be a meromorphic function of zero order satisfying*

$$\limsup_{r \to +\infty} \frac{T(r,f)}{T(r,f')} < +\infty, \tag{1.1}$$

and c be a nonzero complex constant. Then we have

$$\delta(\infty, F_1) \geq \delta(\infty, f'), \quad \delta(\infty, F_4) \geq \delta(\infty, f').$$

Example 1.12 Let $f_1(z) = \frac{z^3+1}{z}$, $q = 2$, $c = 1$ and $n = 3, m = 2$. Then we have

$$f_1'(z) = \frac{2z^3 - 1}{z^2} \text{ and } f_1(2z+1) = \frac{8z^3 + 12z^2 + 6z + 2}{2z+1}.$$

Thus, it yields that

$$F_1(z) = f_1(z)^3 f_1(2z+1)^2 f_1'(z) = \frac{128z^{18} + P_1(z)}{4z^7 + P_2(z)},$$

$$F_4(z) = f_1(2z+1)^2 f_1'(z) = \frac{128z^9 + P_3(z)}{4z^4 + P_4(z)},$$

where $P_1(z), P_2(z), P_3(z), P_4(z)$ are polynomials with $\deg_z P_1(z) \le 17$, $\deg_z P_2(z) \le 4$, $\deg_z P_3(z) \le 8$ and $\deg_z P_4(z) \le 3$. Since

$$\limsup_{r\to+\infty} \frac{T(r,f_1)}{T(r,f_1')} = \limsup_{r\to+\infty} \frac{3\log r}{3\log r} = 1 < +\infty,$$

it thus leads to

$$\delta(\infty, F_1) = \frac{11}{18} > \delta(\infty, f_1') = \frac{1}{3},$$

and

$$\delta(\infty, F_4) = \frac{5}{9} > \delta(\infty, f_1') = \frac{1}{3}.$$

Therefore, this shows that the conclusions of Theorem 1.11 are sharp.

In addition, in view of the above examples, we have that $\delta(\infty, F_1) = \frac{11}{18} < \delta(\infty, f_1) = \frac{2}{3}$ and $\delta(\infty, F_4) = \frac{5}{9} < \delta(\infty, f_1) = \frac{2}{3}$. Hence, a natural question is

Question 1.13 What conditions can guarantee the following inequalities that

$$\delta(\infty, f') \le \delta(\infty, F_1) \le \delta(\infty, f),$$

or

$$\delta(\infty, f') \le \delta(\infty, F_4) \le \delta(\infty, f)?$$

Example 1.14 Let $f_2(z) = z^2$, $q = 2$, $c = 1$. Thus we have that

$$\delta(\infty, f_2') = \delta(\infty, F_1) = \delta(\infty, F_4) = \delta(\infty, f_2) = 1.$$

This shows that the equalities in Theorem 1.11 can be attained.

2. Some lemmas

To prove the above theorems, some lemmas below will be required.

Lemma 2.1 ([3]) *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + \cdots + a_n f^n$, where $a_0, a_1, \ldots, a_n$ are constants and $a_n \ne 0$. Then*

$$T(r, P(f)) = nT(r,f) + S(r,f).$$

By [18] and [19, p.66], we can immediately get the following lemma.

Lemma 2.2 *Let $f(z)$ be a transcendental meromorphic function of zero order and q be a nonzero complex constants. Then*

$$T(r, f(qz+c)) = T(r, f(z)) + S_1(r,f),\ \ N(r, \frac{1}{f(qz)}) = N(r, \frac{1}{f}) + S_1(r,f),$$

$$N(r, f(qz+c)) = N(r,f) + S_1(r,f),\ \ \overline{N}(r, \frac{1}{f(qz+c)}) = \overline{N}(r, \frac{1}{f}) + S_1(r,f),$$

$$\overline{N}(r, f(qz+c)) = \overline{N}(r,f) + S_1(r,f).$$

Lemma 2.3 ([10, Theorem 2.1]) *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m(r, \frac{f(qz+c)}{f(z)}) = S_1(r,f).$$

Lemma 2.4 ([3, p. 37]) *Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then*

$$T(r, f^{(l)}(z)) \leq T(r, f) + l\overline{N}(r, f) + S(r, f), \quad N(r, f^{(l)}(z)) = N(r, f) + l\overline{N}(r, f).$$

Lemma 2.5 *Let f be a transcendental meromorphic function of zero order, $F_5(z)$ be stated as in Theorem 1.5. Then we have*

$$(|m-n|-2)T(r, f) + S_1(r, f) \leq T(r, F_1) \leq (n+m+2)T(r, f) + S_1(r, f). \tag{2.1}$$

Proof If f is a meromorphic function of zero order, from Lemmas 2.1, 2.2 and 2.4, we have

$$\begin{aligned} T(r, f(z)^n f(qz+c)^m f'(z)) &\leq T(r, f(z)^n) + T(r, f(qz+c)^m) + T(r, f'(z)) \\ &\leq (n+m+2)T(r, f) + S_1(r, f). \end{aligned}$$

On the other hand, from Lemmas 2.1–2.4, we have

$$\begin{aligned} &(n+m+1)T(r, f) = T(r, f(qz+c)^{m+n+1}) + S_1(r, f) \\ &\quad = m(r, f(qz+c)^{m+n+1}) + N(r, f(qz+c)^{m+n+1}) + S_1(r, f) \\ &\quad \leq m(r, F_1(z)\frac{f(qz+c)}{f'(z)}\frac{f(qz+c)^n}{f(z)^n}) + N(r, F_1(z)\frac{f(qz+c)}{f'(z)}\frac{f(qz+c)^n}{f(z)^n}) + S_1(r, f) \\ &\quad \leq T(r, F_1) + (2n+3)T(r, f) + S_1(r, f). \end{aligned}$$

Thus, we prove that the first inequality of (2.1) holds when $m > n$.

From Lemmas 2.1, 2.2 and 2.4, we have

$$\begin{aligned} (n+m+1)T(r, f) &= T(r, f(z)^{n+m+1}) + S_1(r, f) \\ &= T(r, \frac{f(z)^{m+1}F_1(z)}{f(qz+c)^m f'(z)}) + S_1(r, f) \\ &\leq T(r, F_1(z)) + T(r, \frac{f(z)}{f'(z)}) + T(r, \frac{f(z)^m}{f(qz+c)^m}) + S_1(r, f) \\ &\leq T(r, F_1(z)) + (2m+3)T(r, f) + S_1(r, f). \end{aligned}$$

Thus, it follows that the second inequality of (2.1) is proved when $n > m$. □

Lemma 2.6 *Let f be a transcendental meromorphic function of zero order, $F_2(z)$ be stated as in Theorem 1.6. Then we have*

$$(m-n-\frac{k(k+3)}{2})T(r, f) \leq T(r, F_2(z)) \leq (m+n+\frac{k(k+3)}{2})T(r, f) + S_1(r, f).$$

Proof Since f is a transcendental meromorphic function of zero order, by Lemmas 2.1, 2.2 and 2.4, we can deduce the second inequality. On the other hand, it follows by Lemmas 2.1, 2.2 and 2.4 that

$$\begin{aligned} (m+k)T(r, f) &= T(r, f^{m+k}) + S_1(r, f) \\ &\leq T(r, \frac{f(z)^k F_2(z)}{P_n(f(qz+c))\prod_{j=1}^k f^{(j)}(z)}) + S_1(r, f) \end{aligned}$$

$$\le T(r,F_2)+T(r,P_n(f(qz+c)))+T(r,\frac{f(z)^k}{\prod_{j=1}^k f^{(j)}(z)})+S_1(r,f)$$
$$\le T(r,F_2)+(n+2k)T(r,f)+\frac{k(k+1)}{2}N(r,f)+S_1(r,f)$$
$$\le T(r,F_2)+\left(n+2k+\frac{k(k+1)}{2}\right)T(r,f)+S_1(r,f),$$

that is

$$T(r,F_2(z))\ge (m-n-\frac{k(k+3)}{2})T(r,f)+S_1(r,f).$$

Thus, this completes the proof of Lemma 2.6. □

Similarly, we get the following lemma.

Lemma 2.7 *Let f be a transcendental meromorphic function of zero order, $F_3(z)$ be stated as in Theorem 1.7. Then we have*

$$(n-m-\frac{k(k+3)}{2})T(r,f)\le T(r,F_3(z))\le (m+n+\frac{k(k+3)}{2})T(r,f)+S_1(r,f).$$

3. Proofs of Theorems

Now, we will show the proofs of our theorems in this section.

Proof of Theorem 1.4 Since $f(z)$ is a transcendental meromorphic function of zero order, we first claim that $f'(z)+f(z)-a\not\equiv 0$. In fact, if $f'(z)+f(z)-a\equiv 0$, that is, $\frac{f'(z)}{f(z)-a}\equiv -1$. By solving the above equation, we have $f(z)=Ae^{-z}+a$, where $A\ne 0$ is a constant. Thus, we have $\rho(f)=1$ which contradicts the fact that $f(z)$ is of zero order. Set

$$F_6(z)=\frac{a-f(z)-f'(z)}{f(qz+c)^n}.$$

Thus, it follows by Lemmas 2.1, 2.2 and 2.4 that

$$nT(r,f)=T(r,f(qz+c)^n)+S_1(r,f)\le T(r,\frac{F_6(z)}{a-f(z)-f'(z)})+S_1(r,f)$$
$$\le T(r,F_6)+T(r,f(z))+T(r,f'(z))+S_1(r,f)$$
$$\le T(r,F_6)+3T(r,f(z))+S_1(r,f),$$

that is,

$$(n-3)T(r,f)\le T(r,F_6)+S_1(r,f). \tag{3.1}$$

On the other hand, we can easily get that

$$T(r,F_6)\le T(r,f(qz+c)^n)+T(r,f)+T(r,f')+S_1(r,f)\le (n+3)T(r,f)+S_1(r,f). \tag{3.2}$$

Noting that $n\ge 4$, and from (3.1) and (3.2), it thus follows that

$$T(r,F_6)=O(T(r,f)).$$

By Lemma 2.2, we have

$$\overline{N}(r,\frac{1}{F_6-1})=\overline{N}(r,\frac{f(qz+c)^n}{f(qz+c)^n+f'(z)+f(z)-a})$$

$$\leq \overline{N}(r, \frac{1}{f(qz+c)^n + f'(z) + f(z) - a}) + \overline{N}(r, f) + S_1(r, f).$$

Assume that $f(qz+c)^n + f'(z) + f(z) - a$ has finitely many zeros, then

$$\overline{N}(r, \frac{1}{F_6 - 1}) \leq \overline{N}(r, f) + S_1(r, f). \tag{3.3}$$

Since

$$\overline{N}(r, \frac{1}{F_6}) \leq \overline{N}(r, f(qz+c)) + \overline{N}(r, \frac{1}{a - f(z) - f'(z)}) + O(1), \tag{3.4}$$

and

$$\overline{N}(r, F_6) \leq \overline{N}(r, \frac{1}{f(qz+c)}) + \overline{N}(r, a - f(z) - f'(z)) + O(1), \tag{3.5}$$

using the second main theorem and Lemmas 2.1, 2.2 and 2.6, from (3.3)–(3.5), we have

$$\begin{aligned} T(r, F_6) &\leq \overline{N}(r, F_6) + \overline{N}(r, \frac{1}{F_6}) + \overline{N}(r, \frac{1}{F_6 - 1}) + S(r, F_6) \\ &\leq T(r, \frac{1}{a - f(z) - f'(z)}) + 4T(r, f) + S_1(r, f) \\ &\leq 7T(r, f) + S_1(r, f). \end{aligned}$$

Thus, it yields by the above inequality that $(n-10)T(r, f) \leq S_1(r, f)$, which is contradiction with $n \geq 11$. Thus, $f(qz+c)^n + f'(z) + f(z) - a$ has infinitely many zeros. □

Proof of Theorem 1.5 (i) Assume that $m \geq n+10$. Since $f(z)$ is a transcendental meromorphic function of zero order, by Lemma 2.5, we have $S(r, f) = S(r, F_1)$. Thus, by using the second fundamental theorem and Lemmas 2.2 and 2.4, again, we have

$$\begin{aligned} &(m-n-2)T(r, f) \leq T(r, F_1) + S_1(r, f) \\ &\quad \leq \overline{N}(r, F_1) + \overline{N}(r, \frac{1}{F_1}) + \overline{N}(r, \frac{1}{F_1(z) - a(z)}) + S(r, F_5) \\ &\quad \leq \overline{N}(r, f(z)) + \overline{N}(r, f(qz+c)) + \overline{N}(r, f'(z)) + \overline{N}(r, \frac{1}{f(z)}) + \overline{N}(r, \frac{1}{f(qz+c)}) + \\ &\qquad \overline{N}(r, \frac{1}{f'}) + \overline{N}(r, \frac{1}{F_1(z) - a(z)}) + S(r, F_1) \\ &\quad \leq 7T(r, f) + \overline{N}(r, \frac{1}{F_1(z) - a(z)}) + S_1(r, f), \end{aligned}$$

that is,

$$\frac{m-n-8}{m+n+2} T(r, F_1) + S(r, F_1) \leq (m-n-8)T(r, f) \leq \overline{N}(r, \frac{1}{F_1(z) - a(z)}) + S_1(r, F_1).$$

In view of $m \geq n+10$, it yields that

$$\delta(a, F_1) \leq 1 - \frac{m-n-8}{m+n+2} < 1.$$

Hence, $F_1(z) - a(z)$ has infinitely many zeros.

(ii) Assume that $n \geq m+10$. Similar to the same discussion as in the proof of Theorem 1.5 (i), it is easy to get that $F_1(z) - a(z)$ has infinitely many zeros. □

Proof of Theorem 1.6 If $f(z)$ is a transcendental meromorphic function of zero order, then by Lemma 2.6, we get $S(r,f)=S(r,F_2)$. Thus, by using the second fundamental theorem and Lemmas 2.2 and 2.6 again, it follows that

$$\begin{aligned}&(m-n-\frac{k(k+3)}{2})T(r,f)\le T(r,F_2)+S_1(r,f)\\&\le \overline{N}(r,F_2)+\overline{N}(r,\frac{1}{F_2})+\overline{N}(r,\frac{1}{F_2(z)-a(z)})+S_1(r,F_2)\\&\le \overline{N}(r,f)+\overline{N}(r,f(qz+c))+\sum_{l=1}^{t_n}\overline{N}(r,\frac{1}{f-\gamma_i})+\overline{N}(r,\frac{1}{f(qz+c)})+\\&\quad\sum_{j=1}^{k}\overline{N}(r,\frac{1}{f^{(j)}})+\overline{N}(r,\frac{1}{F_2(z)-a(z)})+S_1(r,F_2)\\&\le (3+t_n+\frac{k(k+1)}{2}+k)T(r,f)+\overline{N}(r,\frac{1}{F_2(z)-a(z)})+S(r,F_2),\end{aligned}$$

that is,

$$\begin{aligned}&\frac{m-n-t_n-k(k+3)-3}{m+n+\frac{k(k+3)}{2}}T(r,F_2)+S(r,F_2)\\&\le (m-n-t_n-k(k+3)-3)T(r,f)\le \overline{N}(r,\frac{1}{F_2(z)-a(z)})+S_1(r,F_2),\end{aligned}$$

where $\gamma_1,\gamma_2,\ldots,\gamma_t$ are distinct roots of $P_m(t)=0$. Noting that $m\ge n+t_n+k(k+3)+4$, it yields that

$$\delta(a,F_2)\le 1-\frac{m-n-t_n-k(k+3)-3}{m+n+\frac{k(k+3)}{2}}<1.$$

Hence, it follows that $F_2(z)-a(z)$ has infinitely many zeros. □

Proof of Theorem 1.7 Similar to the argument as in the proof of Theorems 1.5 and 1.6, we can prove the conclusions of Theorem 1.7 easily. □

Proof of Theorem 1.8 Set $F_7(z)=f(z)^{n+m+1}$. So, it follows that $N(r,F_7)=(n+m+1)N(r,f)$ and $T(r,F_7)=(n+m+1)T(r,f)+S_1(r,f)$. Thus, it yields that $S_1(r,F_7)=S_1(r,f)$ and $\delta(\infty,F_7)=\delta(\infty,f)$. Besides, we have

$$F_7(z)=F_1(z)(\frac{f(z)}{f(qz+c)})^m\frac{f(z)}{f'(z)}, \tag{3.6}$$

$$\overline{N}(r,f)\le N(r,f)\le \frac{1}{n+m+1}N(r,F_7)\le \frac{1-\delta}{n+m+1}T(r,F_7)+S_1(r,F_7), \tag{3.7}$$

$$\overline{N}(r,\frac{1}{f})\le N(r,\frac{1}{f})\le \frac{1}{n+m+1}N(r,\frac{1}{F_7})\le \frac{1}{n+m+1}T(r,F_7)+O(1). \tag{3.8}$$

Hence, in view of Lemmas 2.1–2.4 and (3.6)–(3.8), it follows that

$$\begin{aligned}T(r,F_7)&=T(r,F_1\frac{f}{f'}(\frac{f(z)}{f(qz+c)})^m)\\&\le T(r,F_1)+mN(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f})+mN(r,f)+\overline{N}(r,f)+S_1(r,f)\\&\le T(r,F_1)+\frac{m+1}{n+m+1}T(r,F_7)+\frac{m+1}{n+m+1}N(r,F_7)+S_1(r,f),\end{aligned}$$

which implies

$$T(r, F_1) \geq \frac{n-(m+1)(1-\delta)}{n+m+1} T(r, F_7) + S_1(r, F_7). \tag{3.9}$$

And by applying Lemmas 2.2–2.4 and (3.6)–(3.8) again, it follows

$$\begin{aligned} N(r, F_1) &\leq N(r, F_7) + mN(r, \frac{f(qz+c)}{f(z)}) + N(r, \frac{f'}{f}) \\ &\leq N(r, F_7) + mN(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f}) + mN(r, f) + \overline{N}(r, f) + S_1(r, f) \\ &\leq \frac{(n+2m+2)(1-\delta)+m+1}{n+m+1} T(r, F_7) + S_1(r, f), \end{aligned}$$

that is,

$$N(r, F_1) \leq \frac{(n+2m+2)(1-\delta)+m+1}{n+m+1} T(r, F_7) + S_1(r, f). \tag{3.10}$$

Thus, from (3.9), (3.10) and $\delta = \delta(\infty, f) = \delta(\infty, F_7) > \frac{4m+4}{n+3m+3}$, it follows that

$$\limsup_{r\to+\infty} \frac{N(r, F_1)}{T(r, F_1)} \leq \frac{(n+2m+2)(1-\delta)+m+1}{n-(m+1)(1-\delta)} < 1,$$

that is,

$$\delta(\infty, F_1) = 1 - \limsup_{r\to+\infty} \frac{N(r, F_1)}{T(r, F_1)} > 0. \quad \square$$

Proofs of Corollaries 1.9 and 1.10 Noting that

$$f(z)^{m+1} = F_4 \frac{f(z)}{f'(z)} \frac{f(z)^m}{f(qz+c)^m}, \quad f(z)^{n+m} = F_5 \frac{f(z)^m}{f(qz+c)^m}$$

thus, by using the similar discussion as in the proof of Theorem 1.8, we can prove Corollaries 1.9 and 1.10 easily. □

Proof of Theorem 1.11 Let

$$f'(z)^{m+1} = F_4(z) \frac{f'(z)^m}{f(z)^m} \frac{f(z)^m}{f(qz+c)^m}.$$

Then we can deduce

$$(m+1)m(r, f') \leq m(r, F_4) + (m+1)m(r, \frac{f'}{f}) + mm(r, \frac{f(z)}{f(qz+c)}) + O(1).$$

In view of Lemma 2.3, it yields that

$$m(r, F_4) \geq (m+1)m(r, f') + S_1(r, f). \tag{3.11}$$

On the other hand, it follows from Lemmas 2.2 and 2.4 that

$$N(r, F_4) \leq (m+1)N(r, f) + \overline{N}(r, f) \leq (m+1)N(r, f'). \tag{3.12}$$

From the assumption of Theorem 1.11, we have

$$\limsup_{r\to+\infty} \frac{S_1(r, f)}{T(r, f')} = \limsup_{r\to+\infty} \frac{S_1(r, f)}{T(r, f)} \frac{T(r, f)}{T(r, f')} = 0. \tag{3.13}$$

Then, it follows from (3.11)–(3.13) that

$$\frac{N(r, F_4)}{T(r, F_4)} \leq \frac{(m+1)N(r, f')}{(m+1)N(r, f') + (m+1)m(r, f') + S_1(r, f)}$$

$$\leq \frac{N(r,f')}{T(r,f')+S_1(r,f)} = \frac{N(r,f')}{(1+o(1))T(r,f')}.$$

Hence we have $\delta(\infty, f') \leq \delta(\infty, F_4)$.

Besides, using the similar method as above and noting that

$$f'(z)^{n+m+1} = F_1 \frac{f'(z)^{n+m}}{f(z)^{n+m}} \frac{f(z)^m}{f(qz+c)^m},$$

we can easily deduce that $\delta(\infty, f') \leq \delta(\infty, F_1)$. □

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