Journal of Mathematical Research with Applications
Jan., 2022, Vol. 42, No. 1, pp. 41–56
DOI:10.3770/j.issn:2095-2651.2022.01.005
Http://jmre.dlut.edu.cn

Existence of Solutions for a Nonlocal Problem with Variable Exponent Operator

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Abstract The purpose of this paper is to investigate a nonlocal Dirichlet problem with $(p(x), q(x))$-Laplacian-like operator originated from a capillary phenomena. Using the variational methods and the critical point theory, we establish the existence of infinitely many weak solutions for this problem.

Keywords variational methods; nonlocal problem; critical point theory; $(p(x), q(x))$-Laplacian-like operator; weak solutions

MR(2020) Subject Classification 35D30; 35J60; 35J70

1. Introduction and main results

The nonlinear problems driven by variable exponent operators appear in numerous physical models, such as the model of motion of electrorheological fluids [1, 2]. An other application which uses variable exponent Laplacian operators is related to the modeling of image restoration [3]. The field of differential equations with various types of nonstandard growth conditions has witnessed an explosive growth in recent years. In the monograph [4], Radulescu and Repovs provided a thorough introduction to the theory of nonlinear partial differential equations (PDEs) with a variable exponent. The operator $-\Delta_{p(x)}u := -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, where div is the vectorial divergence. The study of nonlinear differential equations with $p(x)$-Laplacian operator is an interesting topic, some results can be found, in [5–13].

In [14], Rodrigues has obtained the existence of nontrivial solution for the Dirichlet problem involving the $p(x)$-Laplacian-like of the following form

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\big) = \lambda f(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$, $p \in C(\overline{\Omega})$ such that $2 < p(x) < N$ for any $x \in \overline{\Omega}$, and $f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition. Boundary problem (1.1) was originated from capillarity phenomena [15]. In short, the capillary action is due to the pressure of cohesion and adhesion which cause the liquid to work against

Received September 19, 2020; Accepted March 11, 2021
Supported by the Fundamental Research Funds for the Central Universities (Grant No. 31920180041) and the Fund for Talent Introduction of Northwest Minzu University (Grant No. xbmuyjrc201907).
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gravity. The study of capillarity can be applied in many areas, such as biomedical, industrial and pharmaceutical to microfluidic systems and so on.

It is worth mentioning that problem (1.1) can be viewed as a generalization of the following elliptic Dirichlet problem

$$\begin{cases} -\operatorname{div}\big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\big) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Problem (1.2) is the capillary surface equation or the mean curvature equation; when $f(x,u) = u$, it describes the equilibrium shape of a liquid surface with constant surface tension in a uniform gravity field, and this is the shape of a pendent drop [16].

In [17], Zhou and Ge studied a class of nonlinear elliptic problems driven by $p(x)$-Laplacian-like with a nonsmooth locally Lipschitz potential, and they proved existence of three solutions of the problem. In [18], the authors obtained weak solutions for a class of nonlinear elliptic problems for the $p(x)$-Laplacian-like operators under no-flux boundary conditions.

Recently, Vetro in [19] investigated the existence and multiplicity of solutions for the following $p(x)$-Laplacian-like equations

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\big) + |u|^{p(x)-2}u = \lambda g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ such that $1 < p(x) < N$ for any $x \in \overline{\Omega}$, $\lambda > 0$ and $g(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition. When reaction term satisfies a sub-critical growth condition, the author established the existence of at least one nontrivial weak solution and three weak solutions, by using variational methods and critical point theory. In [20], by using variational methods, Afrouzi, Kirane and Shokooh studied a class of $p(x)$-Laplacian-like equations with Neumann boundary condition, and they obtained the existence of infinitely many solutions of this problem depending on two parameters.

A lot of works concerning superlinear elliptic boundary value problem have been written by using this usual Ambrosetti-Rabinowitz type superlinear condition ((AR) for short), that is, there exist $L > 0$ and $\mu > p^+$ such that

$$0 < \mu F(x,u) \le f(x,u)u$$

for all $x \in \Omega$ and $u \in \mathbb{R}$ with $|u| \ge L$, where $F(x,u) := \int_0^u f(x,s)\,\mathrm{d}s$. This kind of technical condition implies that $f(x,u)$ grows at a superlinear rate with respect to $|u|^{p^+-2}u$ at infinity, whose role consists in ensuring the boundedness of the Palais-Smale sequences of the energy functional associated with the problem under consideration. However, there are many functions which are superlinear but not satisfies (AR), and these functions have attracted much interest in recent years, for example, see [21–25]. In [26], the authors studied the existence and multiplicity of solutions to a class of $p(x)$-Laplacian-like equations in $\mathbb{R}^N$ originated from a capillary phenomena, and they introduced a revised Ambrosetti-Rabinowitz type condition, and proved that the problem has a nontrivial solution and infinitely many pairs of radially symmetric solutions, respectively. In [27, 28], the authors investigated the existence of weak solutions for problem (1.1), when the nonlinear term is p^+-superlinear at infinity, some solvability conditions of nontrivial periodic solutions are obtained through the use of mountain pass theorem. A major point in [27, 28] is that they ensure compactness without the well-known Ambrosetti-Rabinowitz type superlinearity condition.

When the nonlinear term satisfies (AR) condition and a sub-critical growth condition, Avci in [29] obtained the existence of weak solutions for an elliptic boundary value problem with $p(x)$-Laplacian-like operator and nonlocal term given by

$$\begin{cases} -M(\psi(u))\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\big) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ such that $1 < p(x) < N$ for any $x \in \overline{\Omega}$, $M(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition, and

$$\psi(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\,\mathrm{d}x.$$

Comparing with problems (1.1), (1.2) or (1.3), one typical feature of the equation in problem (1.4) is the nonlocality. Due to the presence of a nonlocal coefficient $M(\psi(u))$, it is no longer pointwise identity, and it is often called nonlocal problem or Kirchhoff-type equation. Boundary value problems like (1.4) model several ecosystems where u describes a process depending on the average of itself, as for example, population densities [30, 31]. This problem has a physical motivation, in boundary problem (1.4), the nonlocal coefficient is a function depending on the average of the kinetic energy [7].

In this paper, we will use a weaker superlinear assumption instead of condition (AR), an existence theorem is obtained for infinitely many weak solutions of a nonlocal $(p(x), q(x))$-Laplacian-like systems by using the symmetric mountain pass theorem [32].

Consider the following nonlocal $(p(x), q(x))$-Laplacian-like systems

$$\begin{cases} -M(\Psi(u))\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\big) = F_u(x,u,v), & \text{in } \Omega, \\ -M(\Phi(v))\mathrm{div}\big(|\nabla v|^{q(x)-2}\nabla v + \frac{|\nabla v|^{2q(x)-2}\nabla v}{\sqrt{1+|\nabla v|^{2q(x)}}}\big) = F_v(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $p(x), q(x) \in C(\overline{\Omega})$ such that $1 < p(x) < N$, $1 < q(x) < N$ for any $x \in \overline{\Omega}$ and

$$1 < p^- := \inf_\Omega p(x) \le p^+ := \sup_\Omega p(x) < +\infty,$$

$$1 < q^- := \inf_\Omega q(x) \le q^+ := \sup_\Omega q(x) < +\infty,$$

$$\Psi(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\,\mathrm{d}x,$$

$$\Phi(v) = \int_\Omega \frac{|\nabla v|^{q(x)} + \sqrt{1 + |\nabla v|^{2q(x)}}}{q(x)} \, \mathrm{d}x.$$

$F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(., s, t)$ is continuous in $\overline{\Omega}$, for all $(s, t) \in \mathbb{R}^2$ and F is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, and F_u, F_v denote the partial derivatives of F, with respect to u, v, respectively.

In this paper, we suppose that the Kirchhoff function M satisfies the following conditions:

(M_0) $M(t)$: $[0, +\infty) \to (m_0, +\infty)$ is a continuous and increasing function with $m_0 > 0$.

(M_1) There exists constant $\eta \geq 1$, such that

$$\widehat{M}(t) := \int_0^t M(s) \, \mathrm{d}s \geq \frac{1}{\eta} M(t) t$$

for all $t \geq 0$.

Now, we state the assumptions on function F:

Let

$$C_+(\overline{\Omega}) = \{h \mid h \in C(\overline{\Omega}), h(x) > 1, \forall x \in \overline{\Omega}\},$$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \geq N. \end{cases}$$

$$q^*(x) = \begin{cases} \frac{Nq(x)}{N-q(x)}, & q(x) < N, \\ +\infty, & q(x) \geq N. \end{cases}$$

$$\alpha(x), \beta(x) \in C_+(\overline{\Omega}),$$

$$p^+ < \alpha^- := \inf_\Omega \alpha(x) \leq \alpha^+ := \sup_\Omega \alpha(x) < p^*(x),$$

$$q^+ < \beta^- := \inf_\Omega \beta(x) \leq \beta^+ := \sup_\Omega \beta(x) < q^*(x).$$

(F_0) There exists constant $c > 0$, such that

$$|F(x, u, v)| \leq c \, (|u|^{\alpha(x)} + |v|^{\beta(x)})$$

for all $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$.

(F_1) $\lim_{|(u,v)| \to 0} \frac{F(x,u,v)}{|u|^{p^+} + |v|^{q^+}} = 0$, uniformly for $x \in \Omega$.

(F_2) $\lim_{|(u,v)| \to +\infty} \frac{F(x,u,v)}{|u|^{\eta p^+} + |v|^{\eta q^+}} = +\infty$, uniformly for $x \in \Omega$, where $\eta \geq 1$ is given in (M_1).

Let

$$\mathcal{F}(x, u, v) := \frac{1}{\max\{\eta p^+, \eta q^+\}} \left[F_u(x, u, v) u + F_v(x, u, v) v \right] - F(x, u, v).$$

(F_3) There are constants $c_1 > 0$, $L > 0$, such that

$$\mathcal{F}(x, u, v) \geq c_1 \, (|u|^{p^-} + |v|^{q^-})$$

for all $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$ with $|(u, v)| > L$.

(F_4) There are constants $c_2 > 0$, $L > 0$, $\sigma > \max\{\frac{p^*}{(1-\theta)(p^*-p^-)}, \frac{q^*}{(1-\theta)(q^*-q^-)}\}$ and $\theta \in (0, 1)$, such that

$$\Big(\frac{|F(x, u, v)|}{|u|^{p^-}}\Big)^\sigma + \Big(\frac{|F(x, u, v)|}{|v|^{q^-}}\Big)^\sigma \leq c_2 \, \mathcal{F}(x, u, v)$$

for all $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$ with $|(u, v)| > L$.

(F_5) $F(x, -u, -v) = F(x, u, v)$ for all $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$.

The main result in this paper is the following.

Theorem 1.1 *Assume that hypotheses (M_0), (M_1) and (F_0)–(F_5) are fulfilled. Then problem (1.5) has infinitely many weak solutions.*

Remark 1.2 A typical example for the nonlocal coefficient $M(t)$ is given by

$$M(t) = (a + b\gamma t)^{\gamma-1},$$

where $t \geq 0$, $\gamma \geq 0$ and a, b are two positive constants, then M satisfies (M_0) and (M_1). In fact, we have

$$M(t) = (a + b\gamma t)^{\gamma-1} \geq a^{\gamma-1} > 0,$$

for all $t \geq 0$. Therefore, (M_0) holds and $M(t)$: $[0, +\infty) \to (a^{\gamma-1}, +\infty)$ is a continuous and increasing function. Let $\eta = \gamma$. One has

$$\begin{aligned}\widehat{M}(t) &= \int_0^t M(s)\,\mathrm{d}s = \frac{1}{b\gamma^2}(a + b\gamma t)^{\gamma} - \frac{a^{\gamma}}{b\gamma^2} \\ &\geq \frac{1}{\gamma}(a + b\gamma t)^{\gamma-1} t = \frac{1}{\eta} M(t) t,\end{aligned}$$

for all $t \geq 0$. Thus, (M_1) holds.

Remark 1.3 The conditions of Theorem 1.1 are different from the results in [5–29]. If

$$M(t) = 1 + \frac{\cos t}{1 + t^2},$$

then $M(t)$ satisfies (M_0)-(M_1) with $\eta = 1$. In this case, if $p(x) = q(x) = 2$, consider the function

$$F(x, u, v) = F(u, v) = (|u|^2 + |v|^2)\ \ln(1 + |u| + |v|).$$

By some simple computations, we can show that all the conditions of Theorem 1.1 hold, and F is not covered by results in [5–29].

2. Preliminaries

In this section we give some preliminary results which will be used in the sequal.

Definition 2.1 ([5]) *Define the variable exponent Lebesgue space:*

$$L^{p(x)}(\Omega) = \Big\{u : \ u \text{ is a measureable real value function satisfies } \int_{\Omega} |u(x)|^{p(x)}\,\mathrm{d}x < \infty\Big\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_{\Omega} |\frac{u(x)}{\lambda}|^{p(x)}\,\mathrm{d}x \leq 1\Big\}.$$

Proposition 2.2 ([5]) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$. We have*

$$\Big|\int_{\Omega} uv\,\mathrm{d}x\Big| \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}$$

for $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Definition 2.3 ([5]) *The Sobolev space with variable exponent* $W^{1,p(x)}(\Omega)$ *is defined as*

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ \nabla u \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Proposition 2.4 ([10]) *We denote by* $W_0^{1,p(x)}(\Omega)$ *the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p(x)}(\Omega)$. *Then* $W_0^{1,p(x)}(\Omega)$ *endowed with an equivalent norm*

$$\|u\|_{W_0^{1,p(x)}}(\Omega) = |\nabla u|_{p(x)},$$

becomes a reflexive and separable Banach space.

Proposition 2.5 ([6]) *(i) Poincare inequality in* $W_0^{1,p(x)}(\Omega)$ *holds, that is, there exists a positive constant* C *such that* $|u|_{p(x)} \le C\,|\nabla u|_{p(x)}$, $\forall u \in W_0^{1,p(x)}(\Omega)$;

(ii) If $q \in C(\overline{\Omega})$ *and* $1 < q(x) < p^*(x)$ *for any* $x \in \bar{\Omega}$, *then the embedding from* $W_0^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous.*

Proposition 2.6 ([6]) *Set*

$$J(u) = \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x,$$

if $u \in W_0^{1,p(x)}(\Omega)$, *we have*

(i) $\|u\|_{W_0^{1,p(x)}(\Omega)} \le 1 \Rightarrow \|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)} \le J(u) \le \|u\|^{p^-}_{W_0^{1,p(x)}(\Omega)}$;

(ii) $\|u\|_{W_0^{1,p(x)}(\Omega)} \ge 1 \Rightarrow \|u\|^{p^-}_{W_0^{1,p(x)}(\Omega)} \le J(u) \le \|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)}$.

Throughout this paper, X denotes the Cartesian product of two Sobolev spaces with variable exponent $W_0^{1,p(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega)$, i.e.,

$$X := W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega).$$

Then X endowed with an equivalent norm

$$\|(u,v)\| = \|(u,v)\|_X := \max\{\|u\|_{W_0^{1,p(x)}(\Omega)}, \|v\|_{W_0^{1,q(x)}(\Omega)}\},$$

becomes a reflexive and separable Banach space [33], where

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,\mathrm{d}x \le 1\Big\}$$

and

$$\|v\|_{W_0^{1,q(x)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{v(x)}{\lambda}|^{q(x)}\,\mathrm{d}x \le 1\Big\}.$$

We say that $(u,v) \in X$ is a weak solution of problem (1.5), if

$$M(\Psi(u)) \int_\Omega \big(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\big)\nabla u \nabla \zeta\,\mathrm{d}x +$$
$$M(\Phi(v)) \int_\Omega \big(|\nabla v|^{p(x)-2} + \frac{|\nabla v|^{2p(x)-2}}{\sqrt{1+|\nabla v|^{2p(x)}}}\big)\nabla v \nabla \epsilon\,\mathrm{d}x$$

$$= \int_\Omega F_u(x,u,v)\zeta \,\mathrm{d}x + \int_\Omega F_v(x,u,v)\epsilon \,\mathrm{d}x,$$

for all $(\zeta,\epsilon) \in X$.

Define the Euler-Lagrange functional associated $I : X \to \mathbb{R}$ to problem (1.5) given by

$$I(u.v) = \widehat{M}\left(\Psi(u)\right) + \widehat{M}\left(\Phi(v)\right) - \int_\Omega F(x,u,v)\,\mathrm{d}x$$

for all $(u,v) \in X$.

From the assumptions (F_0), it is standard to check that $I \in C^1(X,\mathbb{R})$ whose Gateaux derivative is

$$\begin{aligned}\langle I'(u,v),(\zeta,\epsilon)\rangle =& M(\Psi(u))\int_\Omega \big(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\big)\nabla u\nabla\zeta\,\mathrm{d}x + \\ & M(\Phi(v))\int_\Omega \big(|\nabla v|^{p(x)-2} + \frac{|\nabla v|^{2p(x)-2}}{\sqrt{1+|\nabla v|^{2p(x)}}}\big)\nabla v\nabla\epsilon\,\mathrm{d}x - \\ & \int_\Omega F_u(x,u,v)\zeta\,\mathrm{d}x - \int_\Omega F_v(x,u,v)\epsilon\,\mathrm{d}x,\end{aligned}$$

for all $(\zeta,\epsilon) \in X$. Then, $(u,v) \in X$ is a critical point of I if and only if (u,v) is a weak solution of problem (1.5).

3. Proof of Theorem 1.1

In order to obtain infinitely many solutions of (1.5), we shall use the symmetric mountain pass theorem [32] introduced by Jabri.

Theorem 3.1 ([32]) *Let X be a real infinite-dimensional Banach space and, $I \in C^1(X,\mathbb{R})$ is an even functional and $I(0) = 0$. Assume that I satisfies:*

(i) I satisfies the (C) condition, that is, sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $\|I'(x_n)\|(1 + \|x_n\|) \to 0$ as $n \to \infty$ has a convergent sequence;

(ii) There are constants $\rho, \delta > 0$ such that

$$I\,|_{\partial B_\rho \cap X} \geq \delta > 0$$

where $\partial B_\rho = \{x \in X \mid \|x\| = \rho\}$;

(iii) For all finite-dimensional subspace $\widetilde{X}$ of X, there exists positive constant $r(\widetilde{X})$ such that $I(x) \leq 0$ for $u \in \widetilde{X}\backslash B_r(0)$, where $B_r(0)$ is an open ball in $\widetilde{X}$ of radius r centred at 0.
Then I possesses an unbounded sequence of critical values characterized by a minimax argument.

For the sake of convenience, we denote by c_i $(i = 1,2,3,\ldots)$ various positive constants. At first, we show that the functional I satisfies the compactness condition (C).

Lemma 3.2 *Suppose that (M_0), (M_1), (F_0), (F_3) and (F_4) hold. Then functional I satisfies condition (C).*

Proof Suppose that $\{(u_n,v_n)\} \subset X := W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ ia a Cerami sequence of I, that is, $\{I(u_n,v_n)\}$ is bounded and $(1 + \|(u_n,v_n)\|)\|I'(u_n,v_n)\| \to 0$ as $n \to \infty$. There then exists a constant $c_3 > 0$ such that

$$|I(u_n, v_n)| \le c_3, \quad (1 + \|(u_n, v_n)\|)\|I'(u_n, v_n)\| \le c_3 \tag{3.1}$$

for all n. We claim that the sequence $\{(u_n, v_n)\}$ is bounded in X. Suppose that is not the case. Passing to a subsequence if necessary, we can assume that

$$\|(u_n, v_n)\| := \max\left\{\|u_n\|_{W_0^{1,p(x)}(\Omega)}, \|v_n\|_{W_0^{1,q(x)}(\Omega)}\right\} \to +\infty \tag{3.2}$$

as $n \to +\infty$. From condition (F_0), it follows that

$$|\mathcal{F}(x, u, v)| = \Big| \frac{1}{\max\{\eta p^+, \eta q^+\}}[\, F_u(x, u, v)u + F_v(x, u, v)v\,] - F(x, u, v)\Big| \le c_4 \tag{3.3}$$

for all $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$ with $|(u, v)| \le L$.

Combining (F_3) and (3.3) leads to

$$\begin{aligned}\mathcal{F}(x, u, v) =& \frac{1}{\max\{\eta p^+, \eta q^+\}}[\, F_u(x, u, v)u + F_v(x, u, v)v\,] - F(x, u, v)\\ \ge& c_1\,(|u|^{p^-} + |v|^{q^-}\,) - c_4\end{aligned} \tag{3.4}$$

for all $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$.

It follows from (M_0), (M_1) and (3.4) that

$$\begin{aligned}c_5 \ge& I(u_n, v_n) + \frac{1}{\max\{\eta p^+, \eta q^+\}}\,(1 + \|(u_n, v_n)\|)\|I'(u_n, v_n)\|\\
\ge& I(u_n, v_n) - \frac{1}{\max\{\eta p^+, \eta q^+\}}\,\langle I'(u_n, v_n), (u_n, v_n)\rangle\\
=&\widehat{M}\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)} + \sqrt{1 + |\nabla u_n|^{2p(x)}}}{p(x)}\,\mathrm{d}x\Big) +\\
&\widehat{M}\Big(\int_\Omega \frac{|\nabla v_n|^{q(x)} + \sqrt{1 + |\nabla v_n|^{2q(x)}}}{q(x)}\,\mathrm{d}x\Big) - \int_\Omega F(x, u_n, v_n)\,\mathrm{d}x -\\
&\frac{M(\int_\Omega \frac{|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}}{p(x)}\,\mathrm{d}x)}{\max\{\eta p^+, \eta q^+\}}\int_\Omega \big(|\nabla u_n|^{p(x)} + \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1 + |\nabla u_n|^{2p(x)}}}\big)\,\mathrm{d}x -\\
&\frac{M(\int_\Omega \frac{|\nabla v_n|^{q(x)} + \sqrt{1+|\nabla v_n|^{2q(x)}}}{q(x)}\,\mathrm{d}x)}{\max\{\eta p^+, \eta q^+\}}\int_\Omega \big(|\nabla v_n|^{q(x)} + \frac{|\nabla v_n|^{2q(x)}}{\sqrt{1 + |\nabla v_n|^{2q(x)}}}\big)\,\mathrm{d}x +\\
&\frac{1}{\max\{\eta p^+, \eta q^+\}}\Big[\int_\Omega F_u(x, u_n, v_n)u_n\,\mathrm{d}x + \int_\Omega F_v(x, u_n, v_n)v_n\,\mathrm{d}x\Big]\\
\ge& \frac{1}{\eta p^+}\,M\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)} + \sqrt{1 + |\nabla u_n|^{2p(x)}}}{p(x)}\,\mathrm{d}x\Big)\int_\Omega \big(|\nabla u_n|^{p(x)} + \sqrt{1 + |\nabla u_n|^{2p(x)}}\big)\,\mathrm{d}x +\\
&\frac{1}{\eta q^+}\,M\Big(\int_\Omega \frac{|\nabla v_n|^{q(x)} + \sqrt{1 + |\nabla v_n|^{2q(x)}}}{q(x)}\,\mathrm{d}x\Big)\int_\Omega \big(|\nabla v_n|^{q(x)} + \sqrt{1 + |\nabla v_n|^{2q(x)}}\big)\,\mathrm{d}x -\\
&\frac{M(\int_\Omega \frac{|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}}{p(x)}\,\mathrm{d}x)}{\max\{\eta p^+, \eta q^+\}}\int_\Omega \big(|\nabla u_n|^{p(x)} + \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1 + |\nabla u_n|^{2p(x)}}}\big)\,\mathrm{d}x -\\
&\frac{M(\int_\Omega \frac{|\nabla v_n|^{q(x)} + \sqrt{1+|\nabla v_n|^{2q(x)}}}{q(x)}\,\mathrm{d}x)}{\max\{\eta p^+, \eta q^+\}}\int_\Omega \big(|\nabla v_n|^{q(x)} + \frac{|\nabla v_n|^{2q(x)}}{\sqrt{1 + |\nabla v_n|^{2q(x)}}}\big)\,\mathrm{d}x +\end{aligned}$$

$$
\begin{aligned}
&\int_\Omega \mathcal{F}(x,u_n,v_n)\,\mathrm{d}x\\
\geq & m_0\Big(\frac{1}{\eta p^+}-\frac{1}{\max\{\eta p^+,\eta q^+\}}\Big)\int_\Omega \frac{1}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\,\mathrm{d}x+\\
& m_0\Big(\frac{1}{\eta q^+}-\frac{1}{\max\{\eta p^+,\eta q^+\}}\Big)\int_\Omega \frac{1}{\sqrt{1+|\nabla v_n|^{2q(x)}}}\,\mathrm{d}x+\int_\Omega \mathcal{F}(x,u_n,v_n)\,\mathrm{d}x\\
\geq & \int_\Omega \mathcal{F}(x,u_n,v_n)\,\mathrm{d}x\\
\geq & c_1\Big(\int_\Omega |u_n|^{p^-}\,\mathrm{d}x+\int_\Omega |v_n|^{q^-}\,\mathrm{d}x\Big)-c_4\,|\Omega|.
\end{aligned}
$$

So, we have

$$\int_\Omega \mathcal{F}(x,u_n,v_n)\,\mathrm{d}x\leq c_5 \tag{3.5}$$

and

$$\int_\Omega |u_n|^{p^-}\,\mathrm{d}x+\int_\Omega |v_n|^{q^-}\,\mathrm{d}x\leq c_6. \tag{3.6}$$

Set

$$\omega_n=\frac{u_n}{\|u_n\|_{W_0^{1,p(x)}(\Omega)}},$$

then $\|\omega_n\|_{W_0^{1,p(x)}(\Omega)}=1$. By (3.6), it holds that

$$\int_\Omega |\omega_n|^{p^-}\,\mathrm{d}x=\frac{1}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)}}\int_\Omega |u_n|^{p^-}\,\mathrm{d}x\leq\frac{c_6}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)}}. \tag{3.7}$$

Without loss of generality, we may assume $\|u_n\|_{W_0^{1,p(x)}(\Omega)}\geq\|v_n\|_{W_0^{1,q(x)}(\Omega)}>1$. By (3.2) and (3.7), we have

$$\int_\Omega |\omega_n|^{p^-}\,\mathrm{d}x\to 0,\quad \text{as } n\to\infty. \tag{3.8}$$

By $\sigma>\frac{p^*}{(1-\theta)(p^*-p^-)}$, $\theta\in(0,1)$ and $p^*>p^-$, one then arrives that

$$\sigma>1,\ \frac{\sigma-1}{\theta\sigma}>1,\ \frac{(1-\theta)\sigma p^-}{(1-\theta)\sigma-1}<p^*.$$

According to Proposition 2.5, we deduce that the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{\frac{(1-\theta)\sigma p^-}{(1-\theta)\sigma-1}}(\Omega)$ is compact, then there exists constant $c_7>0$, such that

$$\int_\Omega |\omega_n|^{\frac{(1-\theta)\sigma p^-}{(1-\theta)\sigma-1}}\,\mathrm{d}x\leq c_7\big(\|\omega_n\|_{W_0^{1,p(x)}(\Omega)}\big)^{\frac{(1-\theta)\sigma-1}{(1-\theta)\sigma p^-}}=c_7. \tag{3.9}$$

It follows from the Hölder inequality, (3.9) that

$$
\begin{aligned}
\int_\Omega |\omega_n|^{p^-\sigma'}\,\mathrm{d}x&=\int_\Omega |\omega_n|^{\frac{\sigma p^-}{\sigma-1}}\,\mathrm{d}x\\
&=\int_\Omega |\omega_n|^{\frac{\theta\sigma p^-}{\sigma-1}}|\omega_n|^{\frac{(1-\theta)\sigma p^-}{\sigma-1}}\,\mathrm{d}x\\
&\leq\Big[\int_\Omega\big(|\omega_n|^{\frac{\theta\sigma p^-}{\sigma-1}}\big)^{\frac{\sigma-1}{\theta\sigma}}\mathrm{d}x\Big]^{\frac{\theta\sigma}{\sigma-1}}\Big[\int_\Omega\big(|\omega_n|^{\frac{(1-\theta)\sigma p^-}{\sigma-1}}\big)^{\frac{1}{1-\frac{\theta\sigma}{\sigma-1}}}\mathrm{d}x\Big]^{1-\frac{\theta\sigma}{\sigma-1}}
\end{aligned}
$$

$$= \Big(\int_\Omega |\omega_n|^{p^-} \,\mathrm{d}x \Big)^{\frac{\theta\sigma}{\sigma-1}} \Big(\int_\Omega |\omega_n|^{\frac{(1-\theta)\sigma p^-}{(1-\theta)\sigma-1}} \,\mathrm{d}x \Big)^{1-\frac{\theta\sigma}{\sigma-1}}$$
$$\leq \Big(\int_\Omega |\omega_n|^{p^-} \,\mathrm{d}x \Big)^{\frac{\theta\sigma}{\sigma-1}} (c_7)^{1-\frac{\theta\sigma}{\sigma-1}},$$

where $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. Hence, by (3.8) we have

$$\int_\Omega |\omega_n|^{p^-\sigma'} \,\mathrm{d}x \longrightarrow 0, \quad \text{as} \quad n \to \infty. \tag{3.10}$$

From (3.3), (3.5), (F_0), (F_4), using Hölder inequality, we can prove that

$$\Big| \int_\Omega \frac{F(x,u_n,v_n)}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}} \,\mathrm{d}x \Big|$$
$$\leq \int_{\{x\in\Omega||(u,v)|>L\}} \frac{|F(x,u_n,v_n)|}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}} \,\mathrm{d}x +$$
$$\int_{\{x\in\Omega||(u,v)|\leq L\}} \frac{|F(x,u_n,v_n)|}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}} \,\mathrm{d}x$$
$$\leq \int_{\{x\in\Omega||(u,v)|>L\}} \frac{|F(x,u_n,v_n)|}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)}} \,\mathrm{d}x +$$
$$\int_{\{x\in\Omega||(u,v)|\leq L\}} \frac{|F(x,u_n,v_n)|}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}} \,\mathrm{d}x$$
$$\leq \int_{\{x\in\Omega||(u,v)|>L\}} \frac{|F(x,u_n,v_n)|}{|u_n|^{p^-}} |\omega_n|^{p^-} \,\mathrm{d}x +$$
$$\frac{c_8}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}$$
$$\leq \Big(\int_{\{x\in\Omega||(u,v)|>L\}} (\frac{|F(x,u_n,v_n)|}{|u_n|^{p^-}})^\sigma \,\mathrm{d}x \Big)^{\frac{1}{\sigma}} \Big(\int_{\{x\in\Omega||(u,v)|>L\}} |\omega_n|^{p^-\sigma'} \,\mathrm{d}x \Big)^{\frac{1}{\sigma'}} +$$
$$\frac{c_8}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}$$
$$\leq \Big(c_2 \int_{\{x\in\Omega||(u,v)|>L\}} \mathcal{F}(x,u_n,v_n) \,\mathrm{d}x \Big)^{\frac{1}{\sigma}} \Big(\int_{\{x\in\Omega||(u,v)|>L\}} |\omega_n|^{p^-\sigma'} \,\mathrm{d}x \Big)^{\frac{1}{\sigma'}} +$$
$$\frac{c_8}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}$$
$$\leq \Big(c_2 \int_\Omega \mathcal{F}(x,u_n,v_n) \,\mathrm{d}x - c_2 \int_{\{x\in\Omega||(u,v)|\leq L\}} \mathcal{F}(x,u_n,v_n) \,\mathrm{d}x \Big)^{\frac{1}{\sigma}} \Big(\int_\Omega |\omega_n|^{p^-\sigma'} \,\mathrm{d}x \Big)^{\frac{1}{\sigma'}} +$$
$$\frac{c_8}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}$$
$$\leq (c_2c_5 + c_2c_4|\Omega|)^{\frac{1}{\sigma}} \Big(\int_\Omega |\omega_n|^{p^-\sigma'} \,\mathrm{d}x \Big)^{\frac{1}{\sigma'}} + \frac{c_8}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)} + \|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}$$

for large n. From (3.2) and (3.10) one obtains

$$\Big|\int_\Omega \frac{F(x,u_n,v_n)}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)}+\|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}\,\mathrm{d}x\Big| \longrightarrow 0, \quad \text{as} \quad n\to\infty. \tag{3.11}$$

Without loss of generality, we may assume $\|u_n\|_{W_0^{1,p(x)}(\Omega)} \geq \|v_n\|_{W_0^{1,q(x)}(\Omega)} > 1$. Using Proposition 2.6, condition (M_0), we deduce that

$$\begin{aligned}
c_3 \geq& I(u_n,v_n)\\
=&\widehat{M}\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}+\sqrt{1+|\nabla u_n|^{2p(x)}}}{p(x)}\,\mathrm{d}x\Big)+\widehat{M}\Big(\int_\Omega \frac{|\nabla v_n|^{q(x)}+\sqrt{1+|\nabla v_n|^{2q(x)}}}{q(x)}\,\mathrm{d}x\Big)-\\
&\int_\Omega F(x,u_n,v_n)\,\mathrm{d}x\\
\geq& \int_0^{\int_\Omega \frac{|\nabla u_n|^{p(x)}+\sqrt{1+|\nabla u_n|^{2p(x)}}}{p(x)}\,\mathrm{d}x} M(s)\,\mathrm{d}s+\int_0^{\int_\Omega \frac{|\nabla v_n|^{q(x)}+\sqrt{1+|\nabla v_n|^{2q(x)}}}{q(x)}\,\mathrm{d}x} M(s)\,\mathrm{d}s-\\
&\int_\Omega F(x,u_n,v_n)\,\mathrm{d}x\\
\geq& \frac{m_0}{p^+}\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x+\frac{m_0}{q^+}\int_\Omega |\nabla v_n|^{q(x)}\,\mathrm{d}x-\int_\Omega F(x,u_n,v_n)\,\mathrm{d}x\\
\geq& \frac{m_0}{p^+}\,\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)}+\frac{m_0}{q^+}\,\|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}-\int_\Omega F(x,u_n,v_n)\,\mathrm{d}x.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{c_3}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)}+\|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}\\
&\quad\geq \min\{\frac{m_0}{p^+},\frac{m_0}{q^+}\}-\int_\Omega \frac{F(x,u_n,v_n)}{\|u_n\|^{p^-}_{W_0^{1,p(x)}(\Omega)}+\|v_n\|^{q^-}_{W_0^{1,q(x)}(\Omega)}}\,\mathrm{d}x.
\end{aligned} \tag{3.12}$$

Combining (3.2), (3.11) with (3.12), we can infer that

$$0\geq \min\{\frac{m_0}{p^+},\frac{m_0}{q^+}\}.$$

This contradicts the fact that $m_0>0$, $p^+>1$ and $q^+>1$. Hence $\{(u_n,v_n)\}$ is bounded in X.

Notice that the Banach space $W_0^{1,p(x)}(\Omega)$ is reflexive, and there exists $u\in W_0^{1,p(x)}(\Omega)$ such that the sequence $\{u_n\}$, passing to the subsequence, still denoted by $\{u_n\}$, converges weakly to u in $W_0^{1,p(x)}(\Omega)$ and converges strongly to u in $L^{\alpha(x)}(\Omega)$. Furthermore, by (F_0), applying Hölder inequality yields

$$\begin{aligned}
\Big|\int_\Omega F_u(x,u_n,v_n)(u_n-u)\,\mathrm{d}x\Big| &\leq \int_\Omega |F_u(x,u_n,v_n)||u_n-u|\,\mathrm{d}x\\
&\leq c\int_\Omega |1+|u_n|^{\alpha(x)-1}|\,|u_n-u|\,\mathrm{d}x\\
&\leq 2c\,|1+|u_n|^{\alpha(x)-1}|_{(\alpha'(x))}\,|u_n-u|_{\alpha(x)},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int_\Omega F_u(x,u_n,v_n)(u_n-u)\,\mathrm{d}x=0. \tag{3.13}$$

By (F_0), similar to the proof of (3.13), there exists $v \in W_0^{1,q(x)}(\Omega)$ such that passing to the subsequence, still denoted by $\{v_n\}$, we have

$$\lim_{n\to\infty}\int_\Omega F_v(x,u_n,v_n)(v_n-v)\,\mathrm{d}x=0. \tag{3.14}$$

On the other hand, by (3.1), we have

$$\langle I'(u_n,v_n),(u_n-u,v_n-v)\rangle\to 0,\quad \text{as}\quad n\to\infty.$$

Thus, from (M_0), we get

$$\langle A(u_n),u_n-u\rangle=\int_\Omega\big(|\nabla u_n|^{p(x)-2}\nabla u_n+\frac{|\nabla u_n|^{2p(x)-2}\nabla u_n}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\big)(\nabla u_n-\nabla u)\,\mathrm{d}x\to 0,\quad \text{as}\quad n\to\infty,$$

and

$$\langle B(v_n),v_n-v\rangle=\int_\Omega\big(|\nabla v_n|^{q(x)-2}\nabla v_n+\frac{|\nabla v_n|^{2q(x)-2}\nabla v_n}{\sqrt{1+|\nabla v_n|^{2q(x)}}}\big)(\nabla v_n-\nabla v)\,\mathrm{d}x\to 0,\quad \text{as}\quad n\to\infty.$$

By [14, Proposition 3.1], the functional $A: W_0^{1,p(x)}(\Omega)\to(W_0^{1,p(x)}(\Omega))^*$ is a mapping of type (S_+), i.e., $u_n\rightharpoonup u$ (weakly) in $W_0^{1,p(x)}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle\le 0$, implies $u_n\to u$ (strongly) in $W_0^{1,p(x)}(\Omega)$. So we get $u_n\to u$ (strongly) in $W_0^{1,p(x)}(\Omega)$. In a similar way, we obtain that $v_n\to v$ (strongly) in $W_0^{1,q(x)}(\Omega)$, which implies that $(u_n,v_n)\to(u,v)$ (strongly) in $X:=W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$. This proves that I satisfies the (C) condition in $X:=W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$. $\square$

Lemma 3.3 *If (M_0), (F_0) and (F_1) hold. Then there are constants $\rho,\delta>0$ such that*

$$I\,|_{\partial B_\rho\cap X}\ge\delta>0,$$

where $\partial B_\rho:=\{(u,v)\in X\,|\,\|(u,v)\|=\rho\}$.

Proof Condition (F_0) and (F_1) imply that for a given $\varepsilon>0$, we can find $c_\varepsilon>0$ such that

$$|F(x,u,v)|\le\varepsilon(|u|^{p^+}+|v|^{q^+})+c_\varepsilon(|u|^{\alpha(x)}+|v|^{\beta(x)}), \tag{3.15}$$

for all $x\in\Omega$ and every $(u,v)\in\mathbb{R}^2$.

Since the embeddings $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{p^+}(\Omega)$ and $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$ are continuous and compact, for all $u\in W_0^{1,p(x)}(\Omega)$. Then there exist constants c_9, $c_{10}>0$, such that

$$|u|_{p^+}\le c_9\|u\|_{W_0^{1,p(x)}(\Omega)} \tag{3.16}$$

and

$$|u|_{\alpha(x)}\le c_{10}\|u\|_{W_0^{1,p(x)}(\Omega)}. \tag{3.17}$$

For $\|u\|_{W_0^{1,p(x)}(\Omega)}$ small enough, we deduce that

$$\int_\Omega|u|^{\alpha(x)}\mathrm{d}x\le\max\{\,|u|_{\alpha(x)}^{\alpha^-},\,|u|_{\alpha(x)}^{\alpha^+}\,\}\le c_{11}\,\|u\|_{W_0^{1,p(x)}(\Omega)}^{\alpha^-}. \tag{3.18}$$

Similar to the proof of (3.16) and (3.18), there exist constants c_{12}, $c_{13}>0$ such that

$$|v|_{q^+}\le c_{12}\|v\|_{W_0^{1,q(x)}(\Omega)}, \tag{3.19}$$

and for $\|v\|_{W_0^{1,q(x)}(\Omega)}$ small enough, we have

$$\int_\Omega |v|^{\beta(x)}\mathrm{d}x \le \max\{\, |v|^{\beta^-}_{\beta(x)},\, |v|^{\beta^+}_{\beta(x)}\,\} \le c_{13}\,\|v\|^{\beta^-}_{W_0^{1,q(x)}(\Omega)}. \tag{3.20}$$

Taking into account (3.15), (3.16), (3.18)–(3.20), we obtain that

$$\begin{aligned}
&\Big|\int_\Omega F(x,u,v)\,\mathrm{d}x\Big|\\
&\le \varepsilon\Big[\int_\Omega |u|^{p^+}\,\mathrm{d}x + \int_\Omega |v|^{q^+}\,\mathrm{d}x\Big] + c_\varepsilon\Big[\int_\Omega |u|^{\alpha(x)}\,\mathrm{d}x + \int_\Omega |v|^{\beta(x)}\,\mathrm{d}x\Big]\\
&\le \varepsilon\big[c_9\|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)} + c_{12}\|v\|^{q^+}_{W_0^{1,q(x)}(\Omega)}\big] + c_\varepsilon\big[c_{11}\|u\|^{\alpha^-}_{W_0^{1,p(x)}(\Omega)} + c_{13}\|v\|^{\beta^-}_{W_0^{1,q(x)}(\Omega)}\big].
\end{aligned} \tag{3.21}$$

For any $(u,v)\in X$, $\|v\|_{W_0^{1,q(x)}(\Omega)} \le \|u\|_{W_0^{1,p(x)}(\Omega)} < 1$, by condition (M_0) and Proposition 2.6, one has that

$$\begin{aligned}
&\widehat{M}\Big(\int_\Omega \frac{|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\,\mathrm{d}x\Big) + \widehat{M}\Big(\int_\Omega \frac{|\nabla v|^{q(x)}+\sqrt{1+|\nabla v|^{2q(x)}}}{q(x)}\,\mathrm{d}x\Big)\\
&= \int_0^{\int_\Omega \frac{|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\,\mathrm{d}x} M(s)\,\mathrm{d}s + \int_0^{\int_\Omega \frac{|\nabla v|^{q(x)}+\sqrt{1+|\nabla v|^{2q(x)}}}{q(x)}\,\mathrm{d}x} M(s)\,\mathrm{d}s\\
&\ge \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p^+}\,\mathrm{d}x + \frac{m_0}{q^+}\int_\Omega |\nabla v|^{q^+}\,\mathrm{d}x\\
&\ge \frac{m_0}{p^+}\|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)} + \frac{m_0}{q^+}\|v\|^{q^+}_{W_0^{1,q(x)}(\Omega)}.
\end{aligned} \tag{3.22}$$

Choosing $\varepsilon>0$ such that

$$\varepsilon\, c_9 \le \frac{m_0}{2p^+} \quad\text{and}\quad \varepsilon\, c_{12} \le \frac{m_0}{2q^+}$$

for any $(u,v)\in X$, $\|v\|_{W_0^{1,q(x)}(\Omega)} \le \|u\|_{W_0^{1,p(x)}(\Omega)} < 1$ and $\|u\|_{W_0^{1,p(x)}(\Omega)}$ small enough, by (3.21) and (3.22), we have

$$\begin{aligned}
I(u,v) =&\widehat{M}\Big(\int_\Omega \frac{|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\,\mathrm{d}x\Big) + \widehat{M}\Big(\int_\Omega \frac{|\nabla v|^{q(x)}+\sqrt{1+|\nabla v|^{2q(x)}}}{q(x)}\,\mathrm{d}x\Big) -\\
&\int_\Omega F(x,u,v)\,\mathrm{d}x\\
\ge&\Big(\frac{m_0}{p^+} - \varepsilon\, c_9\Big)\|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)} + \Big(\frac{m_0}{q^+} - \varepsilon\, c_{12}\Big)\|v\|^{q^+}_{W_0^{1,q(x)}(\Omega)} -\\
&c_\varepsilon\big[c_{11}\|u\|^{\alpha^-}_{W_0^{1,p(x)}(\Omega)} + c_{13}\|v\|^{\beta^-}_{W_0^{1,q(x)}(\Omega)}\big]\\
\ge&\Big(\frac{m_0}{2p^+} - c_\varepsilon\, c_{11}\,\|u\|^{\alpha^- - p^+}_{W_0^{1,p(x)}(\Omega)}\Big)\|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)} +\\
&\Big(\frac{m_0}{2q^+} - c_\varepsilon\, c_{13}\,\|v\|^{\beta^- - q^+}_{W_0^{1,q(x)}(\Omega)}\Big)\|u\|^{q^+}_{W_0^{1,q(x)}(\Omega)}\\
\ge&\Big(\frac{m_0}{2p^+} - c_\varepsilon\, c_{11}\,\|u\|^{\alpha^- - p^+}_{W_0^{1,p(x)}(\Omega)}\Big)\|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)}\\
=&\Big(\frac{m_0}{2p^+} - c_\varepsilon\, c_{11}\,\|(u,v)\|^{\alpha^- - p^+}\Big)\|(u,v)\|^{p^+}.
\end{aligned} \tag{3.23}$$

At this stage, we fix ρ as follows:

$$\rho := \left(\frac{m_0}{4c_\varepsilon\, c_{11}\, p^+}\right)^{\frac{1}{\alpha^- - p^+}}.$$

Consequently, if $\|(u,v)\| = \rho$, then

$$I(u,v) \geq \frac{m_0}{4p^+}\,\rho^{p^+} := \delta > 0,$$

which implies the conclusion of Lemma 3.3 holds. □

Lemma 3.4 *If (M_1), (F_0) and (F_2) hold. Then, for all finite-dimensional subspace $\widetilde{X}$ of X, there exists positive constant $r(\widetilde{X})$ such that $I(u,v) \leq 0$ for $(u,v) \in \widetilde{X}\backslash B_r(0)$, where $B_r(0)$ is an open ball in $\widetilde{X}$ of radius r centred at 0.*

Proof Let $t_1 > 0$. By (M_1), we find that

$$\frac{M(t)}{\widehat{M}(t)} \leq \frac{\eta}{t}$$

for every $t \in [t_1, +\infty)$. Integrating this inequality, we obtain

$$\ln\frac{\widehat{M}(t)}{\widehat{M}(t_1)} = \int_{t_1}^{t}\frac{M(s)}{\widehat{M}(s)}\,\mathrm{d}s \leq \int_{t_1}^{t}\frac{\eta}{s}\,\mathrm{d}s = \ln\left(\frac{t}{t_1}\right)^\eta$$

for every $t \in [t_1, +\infty)$. Therefore,

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_1)}{t_1^\eta}t^\eta$$

for every $t \in [t_1, +\infty)$. Thus there exist constants $c_{14} > 0$ and $c_{15} > 0$, such that

$$\widehat{M}(t) \leq c_{14}\, t^\eta + c_{15} \tag{3.24}$$

for all $t > 0$, where $c_{14} := \frac{\widehat{M}(t_1)}{t_1^\eta}$ and $c_{15} := \max_{t\in[0,t_1]}\widehat{M}(t)$.

From (F_0) and (F_2), it follows that, $\forall\vartheta > 0$, there exists constant $c_{16} > 0$, such that

$$F(x,u,v) \geq \vartheta(|u|^{\eta p^+} + |v|^{\eta q^+}) - c_{16} \tag{3.25}$$

for all $x \in \Omega$ and $(u,v) \in \mathbb{R}^2$.

Without loss of generality, we may assume $\|u\|_{W_0^{1,p(x)}(\Omega)} \geq \|v\|_{W_0^{1,q(x)}(\Omega)} > 1$. Let

$$\|(u,v)\| := \max\{\|u\|_{W_0^{1,p(x)}(\Omega)}, \|v\|_{W_0^{1,q(x)}(\Omega)}\} \geq r > 1.$$

By (3.24), (3.25) and Proposition 2.6, for $(u,v) \in \widetilde{X}$, we have

$$\begin{aligned} I(u,v) =&\widehat{M}\Big(\int_\Omega \frac{|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\,\mathrm{d}x\Big) + \widehat{M}\Big(\int_\Omega \frac{|\nabla v|^{q(x)} + \sqrt{1+|\nabla v|^{2q(x)}}}{q(x)}\,\mathrm{d}x\Big) - \\ &\int_\Omega F(x,u,v)\,\mathrm{d}x \\ \leq& c_{14}\Big[\Big(\int_\Omega \frac{|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\,\mathrm{d}x\Big)^\eta + \Big(\int_\Omega \frac{|\nabla v|^{q(x)} + \sqrt{1+|\nabla v|^{2q(x)}}}{q(x)}\,\mathrm{d}x\Big)^\eta\Big] + \\ &2c_{15} - \int_\Omega F(x,u,v)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\le c_{14}(\frac{1}{p^-})^\eta\Big[\int_\Omega(2|\nabla u|^{p(x)}+1)\mathrm{d}x\Big]^\eta+c_{14}(\frac{1}{q^-})^\eta\Big[\int_\Omega(2|\nabla v|^{q(x)}+1)\mathrm{d}x\Big]^\eta+\\
&\quad 2c_{15}-\int_\Omega F(x,u,v)\,\mathrm{d}x\\
&\le c_{14}(\frac{1}{p^-})^\eta(2\|u\|^{p^+}_{W_0^{1,p(x)}(\Omega)}+|\Omega|)^\eta+c_{14}(\frac{1}{q^-})^\eta(2\|v\|^{q^+}_{W_0^{1,q(x)}(\Omega)}+|\Omega|)^\eta+\\
&\quad 2c_{15}-\vartheta\Big[\int_\Omega|u|^{\eta p^+}\mathrm{d}x+\int_\Omega|v|^{\eta q^+}\mathrm{d}x\Big]+c_{16}|\Omega|\\
&\le[c_{14}(\frac{2}{p^-})^\eta\|u\|^{\eta p^+}_{W_0^{1,p(x)}(\Omega)}-\vartheta\int_\Omega|u|^{\eta p^+}\mathrm{d}x]+\\
&\quad [c_{14}(\frac{2}{q^-})^\eta\|v\|^{\eta q^+}_{W_0^{1,q(x)}(\Omega)}-\vartheta\int_\Omega|v|^{\eta q^+}\mathrm{d}x]+c_{17}.
\end{aligned}
\tag{3.26}
$$

Because all norms on the finite dimension space are equivalent, choosing ϑ large enough, (3.26) implies the conclusion of Lemma 3.4 holds. □

Proof of Theorem 1.1 By Lemmas 3.2–3.4, the functional I satisfies all the assumptions of Theorem 3.1. Therefore, I possesses an unbounded sequence of critical values. Thus, problem (1.5) has infinitely many weak solutions. The proof of Theorem 1.1 is completed. □

Acknowledgements We thank the referees for their time and comments.

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