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Additive Maps Preserving the Truncation of Operators

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Abstract Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. An operator A is called the truncation of B in $\mathcal{B}(\mathcal{H})$ if $A = P_ABP_{A^*}$, where P_A and P_{A^*} denote projections onto the closures of $R(A)$ and $R(A^*)$, respectively. In this paper, we determine the structures of all additive surjective maps on $\mathcal{B}(\mathcal{H})$ preserving the truncation of operators in both directions.

Keywords truncation of operator; operator equation; additive map; preserver

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1. Introduction

The study of linear preserver problem is a hot topic that has been studied by a number of authors [1,2]. It goes back well over a century to the so-called first linear preserver problem, due to Frobenius [Berl. Ber. (1897), 994-1015], that determines linear maps preserving the determinant of matrices. Recently, a survey on preservers of spectra and local spectra is given in [3]. At the same time, additive preserver problems are considered by many authors. It is remarkable that those results on linear preserver problems cannot be generalized to additive preserver problems directly. Therefore, a lot of studies have been done on the subject of additive preserver problems with respect to properties, functions, sets and relations, for example, invertible elements, nilpotent elements, zero products and so on [4–6]. One of those preserver problems is the problem of preserving some relation.

Partial order relation has always been one of the hot subjects considered by numerous authors. Star partial order, minus partial order and diamond partial order as well as their preservers have been considered [7–9]. The diamond partial order $\overset{\diamond}{\leq}$ on $M_n(\mathbb{C})$, the algebra of all $n \times n$-complex matrices, was defined in [7] by Baksalary and Hauke. Let $A, B \in M_n(\mathbb{C})$. We say $A \overset{\diamond}{\leq} B$ if $AA^*A = AB^*A$, $R(A) \subseteq R(B)$ and $R(A^*) \subseteq R(B^*)$, where $R(A)$ denotes the range of A and A^* denotes the adjoint of A. Of course this partial order may be extended to $\mathcal{B}(\mathcal{H})$ if we replace $R(A)$ by $\overline{R(A)}$, the closure of $R(A)$. The following Lemma 2.1 shows that $AA^*A = AB^*A$ in the definition of diamond partial order is actually equivalent to $A = P_ABP_{A^*}$, where P_A denotes the projection on $\overline{R(A)}$. This means that A is a "part" of B, that is, A is truncated from B by P_A and P_{A^*} via the decomposition $\mathcal{H} = \overline{R(A^*)} \oplus \ker(A) = \overline{R(A)} \oplus \ker(A^*)$. We say that A is a

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truncation of B if $A = P_A B P_{A^*}$. This defines a relation on $\mathcal{B}(\mathcal{H})$. In this paper, we characterize additive surjective maps preserving the truncation of operators in both directions.

2. The main result

Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We denote by $\mathcal{F}(\mathcal{H}, \mathcal{K})$ the space of all finite rank operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$ with $\dim \mathcal{H} \geq 2$, we denote by $\mathcal{B}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ the algebra of all bounded linear operators and the ideal of all finite rank operators on $\mathcal{H}$, respectively. For an operator $T \in \mathcal{B}(\mathcal{H})$, write $\ker(T)$ for its kernel. For every nonzero $x, y \in \mathcal{H}$, the symbol $x \otimes y$ stands for the rank-one bounded linear operators defined by $(x \otimes y)z = \langle z, y \rangle x$ for all $z \in \mathcal{H}$, where $\langle z, y \rangle$ is the inner product of z and y. Note that every operator of rank one can be written in this form. The operator $x \otimes y$ is an idempotent if and only if $\langle x, y \rangle = 1$ and $x \otimes y$ is a nilpotent if and only if $\langle x, y \rangle = 0$. Without any confusion, I denotes the identity operator on any Hilbert space.

Lemma 2.1 *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $AA^*A = AB^*A$ if and only if $A = P_A B P_{A^*}$.*

Proof The sufficiency is elementary. We need only to show the necessity. Let $\mathcal{H} = \overline{R(A^*)} \oplus \ker(A) = \overline{R(A)} \oplus \ker(A^*)$ and $A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$, where $A_0 \in \mathcal{B}(\overline{R(A^*)}, \overline{R(A)})$ is an injective operator with dense range. Let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. Since $AA^*A = AB^*A$, we can obtain $A_0 = B_{11}$ by matrix calculation. Thus $B = \begin{pmatrix} A_0 & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. Note that $P_A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = \overline{R(A)} \oplus \ker(A^*)$ and $P_{A^*} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = \overline{R(A^*)} \oplus \ker(A)$. Then $P_A B P_{A^*} = A$. $\Box$

Let φ be an additive map on $\mathcal{B}(\mathcal{H})$. We say that φ preserves the truncation of operators if $\varphi(A)$ is the truncation of $\varphi(B)$ whenever A is that of B for any $A, B \in \mathcal{B}(\mathcal{H})$. If φ preserves the truncation of operators such that A is the truncation of B when $\varphi(A)$ is the truncation of $\varphi(B)$ for any $A, B \in \mathcal{B}(\mathcal{H})$, then we say that φ preserves the truncation of operators in both directions. Since $A = P_A B P_{A^*}$ if and only if $AA^*A = AB^*A$ by Lemma 2.1, φ preserves the truncation of operators in both directions if and only if φ preserves the operator equation $AA^*A = AB^*A$ in both directions, that is,

$$\varphi(A)\varphi(A)^*\varphi(A) = \varphi(A)\varphi(B)^*\varphi(A) \Leftrightarrow AA^*A = AB^*A$$

for all $A, B \in \mathcal{B}(\mathcal{H})$.

We first give a characterization of rank one operators for the proof of our main theorem. Let $A \in \mathcal{B}(\mathcal{H})$ and $A \neq 0$. We define

$$A^\# = \{C \in \mathcal{B}(\mathcal{H}) : AC^*A = 0\}.$$

We say that $A^\#$ is maximal if $A^\# \subseteq B^\#$ for some nonzero $B \in \mathcal{B}(\mathcal{H})$ implies that $A^\# = B^\#$.

Lemma 2.2 *Let $A \in \mathcal{B}(\mathcal{H})$ be a nonzero operator. Then A is of rank one if and only if $A^\#$ is maximal.*

Proof $\Longrightarrow$. Note that $A^\#$ is a closed subspace of $\mathcal{B}(\mathcal{H})$ such that $(\lambda A)^\# = A^\#$ for any nonzero $\lambda \in \mathbb{C}$. Let A be of rank 1. Then we may assume that $A = x \otimes y$ for some unit vectors $x, y \in \mathcal{H}$.

In this case, we have $AC^*A = \langle x, Cy\rangle A$. Thus $C \in A^\#$ if and only if $\langle x, Cy\rangle = 0$. For any $C \in \mathcal{B}(\mathcal{H})$, we have $C - \langle x, Cy\rangle A \in A^\#$. It follows that $C = \langle x, Cy\rangle A + (C - \langle x, Cy\rangle A)$. This means that $A^\#$ is one co-dimensional. Therefore, $A^\#$ is maximal.

$\Longleftarrow$. Assume by way of contradiction that A has rank greater than one. Let $H_1 = \overline{R(A^*)}, H_2 = \ker(A), K_1 = \overline{R(A)}, K_2 = \ker(A^*)$. Then $\mathcal{H} = H_1 \oplus H_2 = K_1 \oplus K_2$ and

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $A_0 \in \mathcal{B}(H_1, K_1)$ is an injective operator with dense range. Let $A_0 = U|A_0|$ be the polar decomposition of A_0, where $|A_0| \in \mathcal{B}(H_1)$ is an injective positive operator, and $U \in \mathcal{B}(H_1, K_1)$ is a unitary operator.

Case 1. $|A_0| = aI$ for some $a > 0$.

Note that $\dim(K_1) = \dim(H_1) \geq 2$. Put $H_1 = M_1 \oplus M_2$, where $\dim(M_i) \geq 1, i = 1, 2$. Let $N_1 = U(M_1)$, $N_2 = U(M_2)$, $U_1 = U|_{M_1}$ and $U_2 = U|_{M_2}$. Then $U : M_1 \oplus M_2 \to N_1 \oplus N_2$ can be expressed as $U = \left(\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right)$. Thus

$$A_0 = \begin{pmatrix} aU_1 & 0 \\ 0 & aU_2 \end{pmatrix}.$$

Consider

$$A_1 = \begin{pmatrix} aU_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Observe that $A^\# \subsetneqq A_1^\#$. This contradicts with the fact that $A^\#$ is maximal.

Case 2. $|A_0| \neq aI$ for all $a > 0$. We may assume that $\|A_0\| = 1$.

Let $|A_0| = \int_{[0,1]} \lambda \mathrm{d}\mathrm{E}_\lambda$ be the spectral decomposition of $|A_0|$. There exists a $c \in (0, 1)$ such that both $M_1 = E[0, c)H_1$ and $M_2 = E[c, 1]H_1$ are nonzero subspaces. Thus

$$|A_0| = \begin{pmatrix} |A_0||_{M_1} & 0 \\ 0 & |A_0||_{M_2} \end{pmatrix}.$$

Put $N_1 = U(M_1)$, $N_2 = U(M_2)$, then we have $N_1 \oplus N_2 = K_1$. Let $U_1 = U|_{M_1}$, $U_2 = U|_{M_2}$. Then $U : M_1 \oplus M_2 \to N_1 \oplus N_2$ can be written in the following matrix form $U = \left(\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right)$. This means that

$$A = \begin{pmatrix} U_1|A_0||_{M_1} & 0 \\ 0 & U_2|A_0||_{M_2} \end{pmatrix}.$$

Put

$$A_1 = \begin{pmatrix} U_1|A_0||_{M_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously, $A^\# \subsetneqq A_1^\#$, a contradiction. Hence, A is a rank one operator. $\square$

Before giving the main result, we recall some notions. Let τ be a ring automorphism of $\mathbb{C}$. A map A on $\mathcal{H}$ is said to be τ-quasilinear if $A(ax + by) = \tau(a)Ax + \tau(b)Ay$ for any $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$. If $\tau(a) = \bar{a}$ for any $a \in \mathbb{C}$, then we say that A is conjugate linear. A conjugate linear map U on $\mathcal{H}$ is said to be anti-unitary if U is bijective such that $\langle Ux, Uy\rangle = \langle y, x\rangle$ for any $x, y \in \mathcal{H}$.

Theorem 2.3 *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be an additive surjective map. Then the following statements are equivalent:*

(1) φ preserves the truncation of operators in both directions;

(2) φ preserves operator equation $AA^*A = AB^*A$ for all $A, B \in \mathcal{B}(\mathcal{H})$ in both directions;

(3) There exist a nonzero scalar $\alpha \in \mathbb{C}$ and operators U and V on $\mathcal{H}$ which are both unitary or both anti-unitary such that $\varphi(T) = \alpha UTV$, $\forall T \in \mathcal{B}(\mathcal{H})$ or $\varphi(T) = \alpha UT^*V$, $\forall T \in \mathcal{B}(\mathcal{H})$.

Proof We note that (1) and (2) are equivalent by Lemma 2.1. The implication from (3) to (1) is elementary. We now assume that φ is an additive surjective map satisfying (2).

We firstly claim that φ is injective. Assume that $\varphi(A) = 0$. By the surjectivity of φ, there exists a nonzero operator $X \in \mathcal{B}(\mathcal{H})$ such that $\varphi(X) = I$. Note that every additive map is $\mathbb{Q}$-linear, hence $\varphi(X + rA) = \varphi(X)$ for every $r \in \mathbb{Q}$. Observe that $II^*I = II^*I$. Thus $\varphi(X+rA)\varphi(X+rA)^*\varphi(X+rA) = \varphi(X)\varphi(X)^*\varphi(X)$. This shows that $(X+rA)(X+rA)^*(X+rA) = XX^*X$. By an elementary calculation, we can get

$$r^3AA^*A + r^2(AA^*X + AX^*A + XA^*A) + r(AX^*X + XA^*X + XX^*A) = 0$$

for all $r \in \mathbb{Q}$. It follows that $AA^*A = 0$ and thus $A = 0$.

Thus φ is bijective and φ^{-1} satisfies the same properties as φ. That is, both φ and φ^{-1} preserve the operator equation $AA^*A = AB^*A$ for all $A, B \in \mathcal{B}(\mathcal{H})$. It is now easy to know that $\varphi(A^\#) = (\varphi(A))^\#$ for any $A \in \mathcal{B}(\mathcal{H})$. It follows that φ preserves rank one operators in both directions by Lemma 2.2.

According to [5, Theorem 3.3], there exist a ring automorphism τ on $\mathbb{C}$ and τ-quasilinear bijections A and C on $\mathcal{H}$ such that

$$\varphi(x \otimes y) = Ax \otimes Cy, \quad \forall x, y \in \mathcal{H} \tag{2.1}$$

or

$$\varphi(x \otimes y) = Ay \otimes Cx, \quad \forall x, y \in \mathcal{H}. \tag{2.2}$$

We assume that φ satisfies (2.1). We will complete the proof by two claims.

Claim 1. Both A and C on $\mathcal{H}$ are multiples of unitary or anti-unitary operators.

Take any unit vectors $x, y \in \mathcal{H}$ such that $x \perp y$. It is easy to check that $x\otimes y,\ y\otimes x \in (x\otimes x)^\#$, hence $\varphi(x \otimes y)$, $\varphi(y \otimes x) \in (\varphi(x \otimes x))^\#$. Thus

$$(Ax \otimes Cx)(Ax \otimes Cy)^*(Ax \otimes Cx) = 0 \text{ and } (Ax \otimes Cx)(Ay \otimes Cx)^*(Ax \otimes Cx) = 0.$$

We have $\langle Ax, Ay\rangle = 0$ and $\langle Cx, Cy\rangle = 0$ by elementary calculation. These imply that both A and C preserve orthogonality of vectors.

On the other hand, it is trivial that $I - x \otimes x \in (x \otimes x)^\#$ for any unit vector $x \in \mathcal{H}$. This implies that $\varphi(I) - \varphi(x \otimes x) \in (\varphi(x \otimes x))^\#$. Thus

$$\|Ax\|^2\|Cx\|^2 = \langle Ax, \varphi(I)Cx\rangle \tag{2.3}$$

for any unit vector $x \in \mathcal{H}$. Again for any pair x, y of unit vectors such that $x \perp y$ one has $I-(x\otimes x+y\otimes y) \in (x\otimes x+y\otimes y)^{\#}$. Then $\varphi(I)-(\varphi(x\otimes x)+\varphi(y\otimes y)) \in (\varphi(x\otimes x)+\varphi(y\otimes y))^{\#}$. Thus

$$\begin{aligned}&(\varphi(x\otimes x)+\varphi(y\otimes y))(\varphi(x\otimes x)+\varphi(y\otimes y))^*(\varphi(x\otimes x)+\varphi(y\otimes y))\\&\quad=(\varphi(x\otimes x)+\varphi(y\otimes y))(\varphi(I))^*(\varphi(x\otimes x)+\varphi(y\otimes y)).\end{aligned}$$

This means that

$$\begin{aligned}&|Ax\|^2\|Cx\|^2 Ax\otimes Cx+\|Ay\|^2\|Cy\|^2 Ay\otimes Cy\\&\quad=(\langle Ax,\varphi(I)Cx\rangle Ax+\langle Ax,\varphi(I)Cy\rangle Ay)\otimes Cx+\\&\qquad(\langle Ay,\varphi(I)Cx\rangle Ax+\langle Ay,\varphi(I)Cy\rangle Ay)\otimes Cy.\end{aligned}$$

Note that $Ax \perp Ay$, hence

$$\langle Ax,\varphi(I)Cy\rangle=\langle Ay,\varphi(I)Cx\rangle=0 \tag{2.4}$$

by (2.3). It follows that $Ay \in \{\varphi(I)Cx\}^{\perp}$ for any orthogonal unit vectors $x, y \in \mathcal{H}$. Since A is τ-quasilinear and $Ax \perp Ay$, we have $A\{x\}^{\perp} \subseteq \{Ax\}^{\perp}$. On the other hand, note that φ^{-1} has the same properties as φ. We then have $A^{-1}(\{Ax\}^{\perp}) \subseteq \{x\}^{\perp}$. Thus $A(\{x\}^{\perp}) = \{Ax\}^{\perp}$.

We therefore see that $A(\{x\}^{\perp}) \subseteq \{\varphi(I)Cx\}^{\perp}$. Note that A is bijective. Hence we have $A(\{x\}^{\perp}) = \{\varphi(I)Cx\}^{\perp}$. Then $Ax \in \{\lambda\varphi(I)Cx : \lambda \in \mathbb{C}\}$ for any unit vector $x \in \mathcal{H}$. This implies that there exists a scalar $\lambda_x \in \mathbb{C}$ such that $Ax = \lambda_x\varphi(I)Cx$. Note that A and C are τ-quasi linear bijections and $\varphi(I) \in \mathcal{B}(\mathcal{H})$ is not of rank one. Thus there exists a scalar $\lambda \in \mathbb{C}$ such that $A = \lambda\varphi(I)C$ by [10, Theorem 2.3]. We note that although [10, Theorem 2.3] dealt with linear operators, we still have the same results for τ-quasilinear operators. In fact, we may prove this elementary fact here. Fixed a nonzero $x \in \mathcal{H}$. It is trivial that $A(ax) = \tau(a)Ax = \tau(a)\lambda_x\varphi(I)Cx = \lambda_x\varphi(I)C(ax)$ for any $a \in \mathbb{C}$. Now take any $y \in \mathcal{H}$ such that x and y are linearly independent. Then $Ax = \lambda_x\varphi(I)Cx$ and $Ay = \lambda_y\varphi(I)Cy$ are also linearly independent. Note that $A(x+y) = \lambda_{x+y}\varphi(I)C(x+y) = \lambda_{x+y}\varphi(I)Cx+\lambda_{x+y}\varphi(I)Cy = Ax+Ay = \lambda_x\varphi(I)Cx+\lambda_y\varphi(I)Cy$. It follows that $\lambda_x = \lambda_y = \lambda_{x+y}$. Put $\lambda = \lambda_x$. Then $A = \lambda\varphi(I)C$.

We now have $\lambda\|\varphi(I)Cx\|^2 = |\lambda|^2\|\varphi(I)Cx\|^2\|Cx\|^2$ for any unit vector $x \in \mathcal{H}$ by (2.3) again. In this case, $\lambda > 0$ and

$$\|Cx\|^2=\lambda^{-1} \tag{2.5}$$

for all unit vector x. In particular, $\forall\alpha \in \mathbb{C}$ with $|\alpha| = 1$, we have $\|C\alpha x\| = |\tau(\alpha)|\|Cx\| = \lambda^{-\frac{1}{2}}$. This means that $|\tau(\alpha)| = 1$. We therefore see τ is continuous and thus $\tau(\alpha) = \alpha$ for all $\alpha \in \mathbb{C}$ or $\tau(\alpha) = \overline{\alpha}$ for all $\alpha \in \mathbb{C}$ by [11, Proposition 1.1]. We now have that $C = cV^*$ for a nonzero constant $c \in \mathbb{C}$ and a unitary or an anti-unitary operator V by (2.5). Thus $A = \lambda\varphi(I)C$ is also a bounded linear or conjugate linear operator. It follows that $A = aU$ for a nonzero constant $a \in \mathbb{C}$ and a unitary or an anti-unitary operator U by (2.4) again. Of course both A and C are simultaneously multiples of unitary or anti-unitary operators.

Claim 2. $\varphi(T) = \alpha UTV$, $\forall T \in \mathcal{B}(\mathcal{H})$.

By Claim 1, there exists a nonzero constant $\alpha \in \mathbb{C}$ such that

$$\varphi(F) = AFC^* = \alpha UFV, \quad \forall F \in \mathcal{F}(\mathcal{H}) \tag{2.6}$$

by (2.1). For any $T \in \mathcal{B}(\mathcal{H})$, let

$$\psi(T) = \alpha^{-1}U^{-1}\varphi(T)V^{-1}. \tag{2.7}$$

Clearly, ψ shares the same properties as φ. Furthermore, $\psi(F) = F$ for all $F \in \mathcal{F}(\mathcal{H})$. Let $T \in \mathcal{B}(\mathcal{H})$ be an infinite rank operator and let $x \in \mathcal{H}$ be a unit vector.

If $T^*x \neq 0$, then $(x \otimes T^*x)(x \otimes T^*x)^*(x \otimes T^*x) = (x \otimes T^*x)T^*(x \otimes T^*x)$. It follows that

$$(x \otimes T^*x)(x \otimes T^*x)^*(x \otimes T^*x) = (x \otimes T^*x)\psi(T)^*(x \otimes T^*x).$$

Then $\langle T^*x, T^*x\rangle = \langle \psi(T)^*x, T^*x\rangle \neq 0$. In particular, $\|T^*x\| \leq \|\psi(T)^*x\|$.

If $T^*x = 0$, then we have $\psi(T)^*x = 0$. In fact, note that ψ^{-1} has the same properties as ψ. If $\psi(T)^*x \neq 0$, then we may have $T^*x = (\psi^{-1}(\psi(T)))^*x \neq 0$. This is a contradiction. It follows that $\ker T^* = \ker \psi(T)^*$. Moreover, we also have $\|\psi(T)^*x\| \leq \|(\psi^{-1}(\psi(T)))^*x\| = \|T^*x\|$. Thus $\|T^*x\| = \|\psi(T)^*x\|$ as well as $\langle T^*x, T^*x\rangle = \langle T^*x, \psi(T)^*x\rangle$ for all $x \in \mathcal{H}$. Since $\mathcal{H}$ is uniformly convex, it follows that $\psi(T)^*x = T^*x$, $\forall x \in \mathcal{H}$, that is, $\psi(T)^* = T^*$. Consequently, by (2.7), we have

$$\varphi(T) = \alpha U\psi(T)V = \alpha UTV, \quad \forall T \in \mathcal{B}(\mathcal{H}).$$

This means that the first form of (3) holds.

If φ satisfies (2.2), we may show that the second form of (3) holds. $\square$

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