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New Results on Sign Patterns that Allow Diagonalizability

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Abstract Characterization of sign patterns that allow diagonalizability has been a long-standing open problem. In this paper, we obtain some sufficient and/or necessary conditions for a sign pattern to allow diagonalizability. Moreover, we determine how many entries need to be changed to obtain a matrix $B' \in Q(A)$ with rank $\mathrm{MR}(A)$ from a matrix $B \in Q(A)$ with rank $\mathrm{mr}(A)$. Finally, we also obtain some results on a sign pattern matrix in Frobenius normal form that allows diagonalizability.

Keywords sign pattern; allowing diagonalizability; maximum cycle length; minimum rank; maximum rank; Frobenius normal form

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1. Introduction and preliminaries

The origins of sign pattern matrices are the need to solve certain problems in economics and other areas based only on the signs of the entries of the matrices. Sign pattern matrices have been heavily studied and have found applications in many other areas [1–3]. In particular, various eigenvalue problems played important roles in both traditional matrix theory and and sign pattern matrix theory [3, 4]. The search for sufficient and necessary conditions characterizing sign patterns that allow diagonalizability has been a long-standing open problem, studied by Eschenbach and Johnson [2], by Shao and Gao [5, 6] and by Feng et al. [7, 8]. In this paper, we further investigate sign patterns that allow diagonalizability.

We now introduce some definitions and notation, most of which can be found in [2, 3, 5]. A sign pattern (matrix) is a matrix whose entries are from the set $\{+, -, 0\}$. The set of all $n\times n$ sign patterns is denoted by Q_n. For an $m\times n$ sign pattern $A = [a_{ij}]$, associated with A is a class of real matrices, called the qualitative class of A, defined by $Q(A) = \{B = [b_{ij}] \in M_{m\times n}(R) | \mathrm{sgn}\, b_{ij} = a_{ij}$ for all i and $j\}$. We may indicate the fact that $B \in Q(A)$ by writing $\mathrm{sgn}(B) = A$.

Let P be a property referring to a real matrix. For a sign pattern A, if there exists a real matrix $B \in Q(A)$ such that B has property P, we say that A allows or admits P; if every $B \in Q(A)$ has property P, we say that A requires P.

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The signed digraph of an $n \times n$ sign pattern $A = [a_{ij}]$, denoted by $D(A)$, is the digraph with vertex set $\{1, 2, \ldots, n\}$, where (i, j) is an arc if only and if $a_{ij} \neq 0$. A formal product of nonzero entries of A of the form

$$\gamma = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1},$$

in which the indices $i_1, \ldots, i_k$ are distinct, is called a simple cycle of length k (or a k-cycle). Each i_m $(m = 1, \ldots, k)$ is called a vertex of γ. A composite cycle of length k is a formal product of simple cycles whose total length is k and whose index sets are mutually disjoint. Naturally, a simple cycle is a composite cycle.

The largest possible length of the composite cycles of A is called the maximum cycle length of A, denoted by $c(A)$. If A has no simple cycle at all, then $c(A) = 0$.

A formal product of nonzero entries of a not necessarily square sign pattern $A = [a_{ij}]$ of the form

$$M = a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_k j_k}$$

with distinct row indices $i_1, \ldots, i_k$ and distinct column indices $j_1, j_2, \ldots, j_k$ is called a matching of size k. We say that the matching M is a principal matching if $\{i_1, \ldots, i_k\} = \{j_1, j_2, \ldots, j_k\}$. We say that the matching M supports a submatrix B if the row index set and column index set of B are equal to those of M. A submatching of M is a matching consisting of some entries in M.

Observe that a composite cycle may be viewed as a principal matching, and vice versa.

The maximum rank of a sing pattern A, denoted $\mathrm{MR}(A)$, is given by

$$\mathrm{MR}(A) = \max\{\operatorname{rank} B | B \in Q(A)\}.$$

Similarly, the minimum rank of A, $\mathrm{mr}(A)$, is given by

$$\mathrm{mr}(A) = \min\{\operatorname{rank} B | B \in Q(A)\}.$$

It is clear that $\mathrm{MR}(A)$ is equal to the maximum possible size of a matching of A. But in general it is very difficult to determine $\mathrm{mr}(A)$. An $n \times n$ sign pattern whose minimum rank equals n is said to be sign nonsingular.

A permutation sign pattern is a square sign pattern matrix with entries 0 and $+$, where the entry $+$ occurs precisely once in each row and in each column. Note that a permutation sign pattern $P \in Q_n$ satisfies $P^T P = P P^T = I_n$, where I_n is the identity sign pattern of order n, namely, I_n is the diagonal sign pattern of order n all of whose diagonal entries are $+$. Two sign patterns $A_1, A_2 \in Q_n$ are said to be permutationally similar if $A_2 = P^T A_1 P$, for some permutation sign pattern P.

A signature sign pattern is a square diagonal sign pattern matrix, each of whose diagonal entries is $+$ or $-$. Two sign patterns $A_1, A_2 \in Q_n$ are said to be signature similar if $A_2 = S A_1 S$, for some signature sign pattern S.

The paper is organized as follows. In Section 2, some necessary and/or sufficient conditions for a sign pattern to allow diagonalizability are obtained. In Section 3, we present some results on how to change entries of a matrix to obtain another matrix with the same sign pattern and a prescribed rank. In Section 4, results on special types of sign pattern matrices that allow diagonalizability are considered. In Section 5, we further obtain some necessary and/or sufficient conditions for a sign pattern matrix in Frobenius normal form to allow diagonalizability.

2. Necessary and/or sufficient conditions for allowing diagonalizability

Firstly, we need the following lemma.

Lemma 2.1 ([5,8]) *Let $A \in Q_n$. If there exists some chordless composite cycle of length k in A with $\mathrm{mr}(A) \le k \le \mathrm{MR}(A)$, then A allows diagonalizability.*

Following [8], we use the following terminology.

Definition 2.2 *A real matrix B is said to be rank-principal if B has a nonsingular $k \times k$ principal submatrix C, where $k = \mathrm{rank}(B)$. Such a principal submatrix C is called a rank-principal certificate of B.*

Definition 2.3 *We say that a composite cycle γ of a square sign pattern A supports a rank-principal certificate for A if there exists a real matrix $B \in Q(A)$ that is rank-principal and the index set of γ is equal to the row index set of a rank-principal certificate of B.*

Remark 2.4 As pointed out in [8], the only possible value of k in Lemma 2.1 is $k = \mathrm{mr}(A)$, and every chordless composite cycle of length $k = \mathrm{mr}(A)$ of A supports a rank-principal certificate for A, which ensures that A allows diagonalizability with rank $\mathrm{mr}(A)$. However, some composite cycles with chords could support a rank-principal certificate, as the following example shows.

For

$$A = \begin{pmatrix} 0 & + & + & 0 & + \\ 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & + & + \\ + & 0 & 0 & 0 & + \\ + & + & + & + & + \end{pmatrix},$$

the cycle $\gamma = a_{12}a_{23}a_{34}a_{41}$ has a chord a_{13}, but γ supports a rank-principal certificate for A. Thus A allows diagonalizability with rank 4.

For a sign pattern whose entries are from the set $\{0, +\}$ (or $\{0, -\}$), how many chords can we add to a cycle which still can support a rank principal certificate? In the above matrix, we can add 6 chords, and change it to the following matrix:

$$A' = \begin{pmatrix} + & + & + & + & + \\ + & 0 & + & + & + \\ + & 0 & 0 & + & + \\ + & 0 & 0 & 0 & + \\ + & + & + & + & + \end{pmatrix}.$$

Moreover, it is easy to see following results.

Theorem 2.5 *Let $A \in Q_n$.*

(1) (see [8]) If there exists a chordless composite cycle of length k that can support a rank-principal certificate for A, then A allows diagonalizability with rank k, and $k = \mathrm{mr}(A)$;

(2) For a sign pattern whose entries are from the set $\{0,+\}$ (or $\{0,-\}$), if a composite cycle of length k can support a rank-principal certificate, then it has at most $\frac{k(k-1)}{2}$ chords.

Theorem 2.6 ([7]) *A sign pattern $A \in Q_n$ allows diagonalizability with rank k if and only if A allows a rank-principal matrix of rank k.*

Lemma 2.7 ([7]) *The set of sign patterns that allow diagonalizability is closed under the following operations:*

(i) negation,

(ii) transposition,

(iii) permutational similarity,

(iv) signature similarity.

It is worth noting that set of sign patterns that allow diagonalizability is also closed under direct sums, and Kronecker products.

Theorem 2.8 ([8]) *Let $A \in Q_n$. Suppose that there exist two composite cycles γ_1 and γ_2 with $\gamma_1 \subset \gamma_2$. If γ_1 can support a rank-principal certificate, then γ_2 can also support a rank-principal certificate.*

Theorem 2.9 ([7]) *A square sign pattern A allows diagonalizability with rank $\mathrm{MR}(A)$ if and only if $c(A) = \mathrm{MR}(A)$.*

3. Entry modifications between matrices achieving extreme ranks

Ranks play an important role in many matrix problems such as diagonalizability. For a sign pattern A, we consider the minimum number of entries which need to be modified to obtain a matrix $B' \in Q(A)$ with rank $\mathrm{MR}(A)$ starting with a matrix $B \in Q(A)$ with rank $\mathrm{mr}(A)$. We observe the following result.

Theorem 3.1 *Let A be a sign pattern. Starting with any matrix $B \in Q(A)$ with $\mathrm{rank}(B) = \mathrm{mr}(A)$, we may change at most $\mathrm{MR}(A)$ entries of B to obtain a matrix $B' \in Q(A)$ with $\mathrm{rank}\,\mathrm{MR}(A)$.*

Proof It is known that A has a maximum matching M of size $\mathrm{MR}(A)$. For any $B \in Q(A)$ with $\mathrm{rank}(B) = \mathrm{mr}(A)$, we can change the entries of B in M to very large values. Then the resulting matrix $B' \in Q(A)$ has $\mathrm{rank}\,\mathrm{MR}(A)$. $\square$

In fact, it suffices to change $\mathrm{MR}(A) - 1$ entries of B in M in the preceding argument, as a dominant nonzero term in the determinant expansion of a submatrix of order $\mathrm{MR}(A)$ may be created this way. This leads to the following result.

Corollary 3.2 *Let A be a sign pattern. It suffices to change* $\mathrm{MR}(A)-1$ *entries of any* $B \in Q(A)$ *with* $\mathrm{rank}(B) = \mathrm{mr}(A)$ *to get a matrix* $B' \in Q(A)$ *with* $\mathrm{rank}\,\mathrm{MR}(A)$.

The next result is on sign patterns whose minimum rank and maximum rank differ by 1.

Theorem 3.3 *Let* $A = [a_{ij}] \in Q_n$ *with* $\mathrm{MR}(A) = \mathrm{mr}(A) + 1$. *Then one can change just one entry of a suitable matrix* $B \in Q(A)$ *with* $\mathrm{rank}\,\mathrm{mr}(A)$ *to get a matrix* $B' \in Q(A)$ *with* $\mathrm{rank}(B') = \mathrm{MR}(A) = n$.

Proof Let $m = \mathrm{MR}(A)$ and let $M = a_{i_1j_1}a_{i_2j_2}\cdots a_{i_mj_m}$ be a matching of size m in A. Let $B_0 \in Q(A)$ be a matrix with $\mathrm{rank}(B_0) = \mathrm{mr}(A) = m-1$. Replace the (i_1, j_1) entry of B_0 by a much larger number of same sign so that the absolute value of the new entry is greater than the sum of the absolute values of the other entries in the same row of B_0. Denote the resulting matrix by B_1. We then replace the (i_2, j_2) entry of B_1 by a much larger number of same sign to get a matrix B_2. Proceeding this way, we get $B_0, B_1, \ldots, B_m \in Q(A)$. Since B_m has m large entries that form a generalized dominant diagonal of a submatrix of order m, we see that $\mathrm{rank}(B_m) = m = \mathrm{MR}(A)$. Let k be the smallest positive integer such that $\mathrm{rank}(B_k) = m$. Then we have rank $(B_{k-1}) = m-1$ and $\mathrm{rank}(B_k) = m$, and these matrices differ in one entry only. This completes the proof. □

We now show a similar result when the gap between the minimum rank and the maximum rank may be larger.

Theorem 3.4 *Let A be a sign pattern with* $\mathrm{MR}(A) = m$ *and* $\mathrm{mr}(A) = r$. *Suppose that there is a matrix* $B \in Q(A)$ *with* $\mathrm{rank}(B) = r$ *such that an* $r \times r$ *nonsingular submatrix C of B is supported by a submatching of a matching M of size m of A. Then one can change only* $m-r$ *entries of* $B \in Q(A)$ *to obtain a matrix* $B' \in Q(A)$ *with* $\mathrm{rank}(B') = \mathrm{MR}(A)$.

Proof Permuting the rows and columns of A if necessary, without loss of generality, we may assume that C is in the upper left corner of B, that is, $B = \left(\begin{smallmatrix} C & D \\ E & F \end{smallmatrix}\right)$. The hypothesis ensures that C contains a matching M_1 of size r and F contains a matching M_2 of size $m-r$, where $M_1M_2 = M$.

Since the rows (respectively, columns) of B containing C are linearly independent and can span the other rows (respectively, columns) of B, we may use suitable elementary row and column operations to reduce B to $\left(\begin{smallmatrix} C & 0 \\ 0 & 0 \end{smallmatrix}\right)$. If the entries of F in the matching M_2 are replaced with twice of the original values to get a matrix F', then when the above-mentioned elementary row and column operations are applied to the matrix $B' = \left(\begin{smallmatrix} C & D \\ E & F' \end{smallmatrix}\right) \in Q(A)$, we get a matrix of the form $\left(\begin{smallmatrix} C & 0 \\ 0 & G \end{smallmatrix}\right)$, where G has exactly $m-r$ nonzero entries, in the same positions as entries in M_2. It follows that $\mathrm{rank}(B') = m$, and B' is obtained from B by modifying exactly $m-r$ entries. □

We suspect that the hypothesis of the preceding theorem is satisfied by every sign pattern matrix, which motivates the following conjecture.

Conjecture 3.5 *For every sign pattern A, it is always possible to change* $\mathrm{MR}(A)-\mathrm{mr}(A)$ *entries of some matrix* $B \in Q(A)$ *with* $\mathrm{rank}(B) = \mathrm{mr}(A)$ *to get a matrix* $B' \in Q(A)$ *with* $\mathrm{rank}\,\mathrm{MR}(A)$.

4. Further results on sign patterns that allow diagonalizability

For a square matrix B, let $z(B)$ and $g(B)$ denote the algebraic and geometric multiplicities of 0 as an eigenvalue of B. For a square sign pattern matrix A, $z(A) = \min\{z(B)|B \in Q(A)\}$ denotes the minimum algebraic multiplicity of 0 as an eigenvalue of a matrix in $Q(A)$, and similarly, $Z(A)$ denotes the maximum algebraic multiplicity of 0 as an eigenvalue of a matrix in $Q(A)$, $g(A)$ denotes the minimum geometric multiplicity of 0 as an eigenvalue of a matrix in $Q(A)$, and $G(A)$ denotes the maximum geometric multiplicity of 0 as an eigenvalue of a matrix in $Q(A)$. The following result can be found in [6].

Theorem 4.1 ([6]) *Let A be an $n \times n$ sign pattern. If $z(A) = g(A)$, then A allows diagonalizability.*

Proof It can be seen that $z(A) = n - c(A)$, and $g(A) = n - \mathrm{MR}(A)$. Hence, $z(A) = g(A)$ ensures that $c(A) = \mathrm{MR}(A)$. Let $k = c(A)$.

Thus by emphasizing the entries on a composite cycle of length k, we get a matrix $B \in Q(A)$ with $\mathrm{rank}(B) = k$ and $z(B) = g(B) = n - k$. Then the characteristics polynomial of B has the form

$$p_B(t) = t^n - E_1 t^{n-1} + E +_2 t^{n-2} + \cdots + (-1)^k E_k\ t^{n-k},$$

where $E_i, i = 1, \ldots, n-k$, is the sum of all i-by-i principal minors of B, and $E_k \neq 0$. Thus, the matrix B is rank-principal. Therefore, A allows diagonalizability by Theorem 2.6. □

As pointed out in the preceding proof, $z(A) = g(A)$ is equivalent to $c(A) = \mathrm{MR}(A)$, which holds if and only if A allows diagonalizability with rank $\mathrm{MR}(A)$ by Theorem 2.9.

We also have the following two results similar to Theorem 4.1.

Theorem 4.2 *Let $A \in Q_n$ be a sign pattern such that $Z(A) = G(A)$. Then A allows diagonalizability with* $\mathrm{rank}\,\mathrm{mr}(A)$.

Proof Take a matrix $B \in Q(A)$ such that $\mathrm{rank}(B) = \mathrm{mr}(A)$. Since $Z(A) = G(A)$, we have

$$z(B) \leq Z(A) = G(A) = n - \mathrm{mr}(A) = n - \mathrm{rank}(B).$$

Hence, $n - z(B) \geq \mathrm{rank}(B)$. But since the rank of any square matrix is clearly always greater than or equal to the number of nonzero eigenvalues of the matrix, we also have the opposite inequality $n - z(B) \leq \mathrm{rank}(B)$. Thus $n - z(B) = \mathrm{rank}(B)$, namely, the number of nonzero eigenvalues of B is equal to rank of B. It follows that B is rank-principal. By Theorem 2.6, A allows diagonalizability with $\mathrm{rank}\,\mathrm{mr}(A)$. □

As a square matrix for which the algebraic multiplicity and geometric multiplicity of the eigenvalue 0 are equal is rank-principal, by Theorem 2.6, we have the following fact.

Corollary 4.3 *A square sign pattern A allows diagonalizability if and only if there exists a real matrix $B \in Q(A)$ for which the algebraic multiplicity and geometric multiplicity of the eigenvalue 0 are equal.*

Observe that up to permutational similarity, every square rank-principal matrix arises as a matrix of the form

$$\begin{pmatrix} C & D \\ E & EC^{-1}D \end{pmatrix},$$

where C is nonsingular, and D, E are of suitable sizes. Thus, up to permutational similarity, the sign patterns that allow diagonalizability are just sign patterns of matrices of this form. However, we are more interested in the combinatorial characterizations of sign patterns that allow diagonalizability.

We now present some special sign patterns that allow diagonalizability.

Theorem 4.4 *Suppose a square sign pattern A has minimum rank $k > 0$ and A has a sign nonsingular $k \times k$ principal submatrix. Then A allows diagonalizability with rank k.*

Proof Every matrix $B \in Q(A)$ with rank k is clearly rank-principal due to the presence of a sign nonsingular $k \times k$ principal submatrix of A. Thus A allows diagonalizability with rank k by Theorem 2.6. □

Next, we give a characterization of the square sign patterns that require a unique rank and allow diagonalizability.

Theorem 4.5 *Let A be a square sign pattern such that $\mathrm{mr}(A) = \mathrm{MR}(A) = k$. Then A allows diagonalizability if and only if $c(A) = k$.*

Proof Both the necessity and the sufficiency follow from Theorem 2.9. □

Upper triangular sign patterns that allow diagonalizability are identified below.

Theorem 4.6 *Let A be an upper triangular square sign pattern. Then A allows diagonalizability if and only if $c(A) = \mathrm{mr}(A)$.*

Proof Since A is an upper triangular square sign pattern, every matrix $B \in Q(A)$ has precisely $c(A)$ nonzero eigenvalues, so $\mathrm{mr}(A) \geq c(A)$.

Suppose that A allows diagonalizability. Then $c(A) \geq \mathrm{mr}(A)$. In view of the opposite inequality above, we get $c(A) = \mathrm{mr}(A)$.

Conversely, assume that $c(A) = \mathrm{mr}(A)$. Let $B \in Q(A)$ be such that $\mathrm{rank}(B) = \mathrm{mr}(A)$. Clearly, there is a diagonal matrix D with positive diagonal entries such that all the nonzero diagonal entries of $DB \in Q(A)$ are distinct. Thus every nonzero eigenvalue of DB has algebraic and geometric multiplicity 1. If 0 is an eigenvalue of DB, then its algebraic and geometric multiplicities are both equal to $n - c(A) = n - \mathrm{rank}(B) = n - \mathrm{rank}(DB)$. Hence, $DB \in Q(A)$ is diagonalizable, so that A allows diagonalizability. □

A square sign pattern is said to be idempotent if A^2 is unambiguously defined, and $A^2 = A$. More generally, we say a sign pattern is k-potent (where k is a positive integer) if A^{1+k} is unambiguously defined and $A^{1+k} = A$. Such sign patterns always allow diagonalizability.

Theorem 4.7 *Every sign k-potent sign pattern A allows diagonalizability with rank $\mathrm{mr}(A)$.*

Proof Let A be a k-potent sign pattern and let $B \in Q(A)$ be such that $\operatorname{rank}(B) = \operatorname{mr}(A)$. On the one hand, clearly $\operatorname{rank}(B^{1+k}) \le \operatorname{rank}(B)$. On the other hand, since $\operatorname{rank}(B) = \operatorname{mr}(A)$ and $B^{1+k} \in Q(A^{1+k}) = Q(A)$, we also have $\operatorname{rank}(B^{1+k}) \ge \operatorname{rank}(B)$. Thus, $\operatorname{rank}(B^{1+k}) = \operatorname{rank}(B)$. It follows that $\operatorname{rank}(B) = \operatorname{rank}(B^2) = \cdots = \operatorname{rank}(B^{1+k})$. By considering the Jordan canonical form of B, we see that either B is nonsingular or the eigenvalue 0 of B has index 1. Thus $\operatorname{rank}(B)$ is equal to the number of nonzero eigenvalues of B, which ensures that B is rank-principal. By Theorem 2.6, A allows diagonalizability with rank $\operatorname{mr}(A)$. □

5. Sign patterns in Frobenius normal form

The Frobenius normal form of a sign pattern $A \in Q_n$ is a sign pattern in block upper triangular form:

$$P^TAP = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ 0 & A_{22} & \dots & A_{2p} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{pp} \end{pmatrix},$$

where P is a permutation sign pattern and the diagonal blocks A_{ii} are irreducible (which are called the irreducible components of A).

By Lemma 2.7, permutational similarity preserves diagonalizability. So it suffices to consider which sign patterns in Frobenius normal form allow diagonalizability.

By considering the minimal polynomials, we get the following result.

Lemma 5.1 *If a sign pattern A in Frobenius normal form*

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{pp} \end{pmatrix}$$

allows diagonalizability, then each irreducible component $A_{ii}, 1 \le i \le p$, allows diagonalizability.

Theorem 5.2 *A square sign pattern A in Frobenius normal form*

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{pp} \end{pmatrix}$$

allows diagonalizability if and only if there exists a real matrix

$$B = \begin{pmatrix} B_{11} & \dots & B_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{pp} \end{pmatrix} \in Q(A) \text{ where } B_{ii} \in Q(A_{ii})$$

such that $\operatorname{rank}(B) = \operatorname{rank}(B_{11}) + \cdots + \operatorname{rank}(B_{pp})$, *and each B_{ii} is diagonalizable.*

Proof Sufficiency. Assume that

$$B = \begin{pmatrix} B_{11} & \dots & B_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{pp} \end{pmatrix} \in Q(A) \text{ where } B_{ii} \in Q(A_{ii}),$$

$\operatorname{rank}(B) = \operatorname{rank}(B_{11}) + \cdots + \operatorname{rank}(B_{pp})$, and each of $B_{11}, B_{22}, \dots, B_{pp}$ is diagonalizable. By Theorem 2.6, each B_{ii} has a rank-principal certificate. In view of $\operatorname{rank}(B) = \operatorname{rank}(B_{11}) + \cdots + \operatorname{rank}(B_{pp})$, the smallest principal submatrix containing all these certificates forms a rank-principal certificate of B. By Theorem 2.6, A allows diagonalizability.

Necessity. Assume that A allows diagonalizability. Let

$$B = \begin{pmatrix} B_{11} & \dots & B_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{pp} \end{pmatrix} \in Q(A),$$

be a diagonalizable matrix. Then the minimal polynomial of B has no repeated roots, and thus the same holds for each B_{ii}. Hence, each B_{ii} is diagonalizable. Further, $\operatorname{rank}(B)$ is equal to the number of nonzero eigenvalues of B, and hence, $\operatorname{rank}(B) = \operatorname{rank}(B_{11}) + \cdots + \operatorname{rank}(B_{pp})$. □

Corollary 5.3 *If a sign pattern in Frobenius normal form*

$$A = \begin{pmatrix} A_{11} & \dots & A_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{pp} \end{pmatrix}$$

allows diagonalizability, then the set of the ranks of diagonalizable matrices in $Q(A)$, is a subset of the set of ranks of diagonalizable matrices in the qualitative class of the block diagonal sign pattern $\begin{pmatrix} A_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{pp} \end{pmatrix}$.

The following fact is useful when studying sign patterns in block form that allow diagonalizability.

Lemma 5.4 *Let $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right) \in Q_n$. If $\operatorname{mr}(A) = \operatorname{mr}(A_{11})$, then there exist a real matrix $B = \left(\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{smallmatrix}\right) \in Q(A)$ and a nonsingular submatrix C of B_{11} such that*

$$\operatorname{rank}(C) = \operatorname{rank}(B_{11}) = \operatorname{rank}(B) = \operatorname{mr}(A).$$

Proof Pick a matrix $B = \left(\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{smallmatrix}\right) \in Q(A)$ with $\operatorname{rank}(B) = \operatorname{mr}(A)$, where each $B_{ij} \in Q(A_{ij})$. Then $\operatorname{rank}(B_{11}) \le \operatorname{rank}(B) = \operatorname{mr}(A) = \operatorname{mr}(A_{11})$. But of course we also have the opposite inequality $\operatorname{rank}(B_{11}) \ge \operatorname{mr}(A_{11})$. It follows that $\operatorname{rank}(B_{11}) = \operatorname{mr}(A_{11}) = \operatorname{mr}(A)$. Thus B_{11} has a nonsingular submatrix C of rank $\operatorname{mr}(A)$. □

We now phrase an interesting open combinatorial sufficient condition for a symmetrically partitioned block upper triangular sign pattern to allow diagonalizability.

Problem 5.5 Let A be a sign pattern in symmetrically partitioned block upper triangular form

$$A = \begin{pmatrix} A_{11} & \dots & A_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{pp} \end{pmatrix}.$$

Suppose that for each $i = 1, \ldots, p$, $\mathrm{mr}(A_{ii}) = \mathrm{mr}([A_{ii} \cdots A_{ip}])$ and each A_{ii} allows diagonalizability. Does it then necessarily follow that A allows diagonalizability?

A related open problem is the following.

Problem 5.6 Let A_1 be a square sign pattern that allows rank-principality. Is it true that for every sign pattern $A = (A_1 \;\; A_2)$ such that $\mathrm{mr}(A_1) = \mathrm{mr}(A)$, A allows rank-principality?

We note that an affirmative answer to Problem 5.6 implies an affirmative answer to Problem 5.5.

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References

[1] R. A. BRUALDI, H. J. RYSER. *Combinatorial Matrix Theory*. Cambridge University Press, Cambridge, 1991.

[2] C. A. ESCHENBACH, C. R. JOHNSON. *Sign patterns that require repeated eigenvalues*. Linear Algebra Appl., 1993, **190**: 169–179.

[3] F. J. HALL, Zhongshan LI. *Sign Pattern Matrices in Handbook of Linear Algebra, Chap. 33, L. Hogben, ed.* Simon and Hall/CRC Press, Boca Raton, 2007.

[4] R. A. HORN, A. K. LOPATIN. *The moment and Gram matrices, distinct eigenvalues and zeroes, and rational criteria for diagonalizability*. Linear Algebra Appl., 1999, **299**: 153–163.

[5] Yanling SHAO, Yubin GAO. *Sign patterns that allow diagonalizability*. Linear Algebra Appl., 2003, **359**: 113–119.

[6] Yanling SHAO, Yubin GAO. *On Eschenbach-Johnson conjecture about diagonalizability*. JP J. Algebra Number Theory Appl., 2006, **6**(3): 573–583.

[7] Xinlei FENG, Tingzhu HUANG, Zhongshan LI, et al. *Sign patterns that allow diagonalizability revisited*. Linear Multilinear Algebra, 2013, **61**(9): 1223–1233.

[8] Xinlei FENG, Wei GAO, F. J. HALL, et al. *Minimum rank conditions for sign patterns that allow diagonalizability*. Discrete Math., 2020, **343**(5): 1–11.