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A Result on K-(2,1)-Total Choosability of Planar Graphs

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Abstract A list assignment of a graph G is a function $L: V(G) \cup E(G) \to 2^N$. A graph G is L-(2,1)-Total labeling if there exists a function c such that $c(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, $|c(u) - c(v)| \geq 1$ if $uv \in E(G)$, $|c(e_1) - c(e_2)| \geq 1$ if the edges e_1 and e_2 are adjacent, and $|c(u) - c(e)| \geq 2$ if the vertex u is incident to the edge e. A graph G is k-(2,1)-Total choosable if G is L-(2,1)-Total labeling for every list assignment L provided that $|L(x)| = k, x \in V(G) \cup E(G)$. The $(2,1)$-Total choice number of G, denoted by $C^T_{2,1}(G)$, is the minimum k such that G is k-(2,1)-Total choosable. In this paper, we prove that if G is a planar graph with $\Delta(G) \geq 11$, then $C^T_{2,1}(G) \leq \Delta + 4$.

Keywords L-(2,1)-total labeling; k-(2,1)-total choosable; planar graphs

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1. Introduction

In this paper, G is a finite simple graph. By $V(G)$, $E(G)$, $F(G)$, $\triangle(G)$, $\delta(G)$, we denote, respectively, the vertex set, the edge set, the face set, the maximum degree, and the minimum degree of G. Let $d(u)$ be the degree of a vertex u. If a vertex u is adjacent to a vertex v, then we call u a neighbor of v. A face $f \in F(G)$ is called a $(d(v_1), d(v_2), \ldots, d(v_k))$-face if $v_1, v_2, \ldots, v_k$ are all the boundary vertices arranged clockwise on f. Furthermore, we denote $(d(v_1), d(v_2), \ldots, d(v_k))$-face and $(d(v_{1'}), d(v_2), \ldots, d(v_k))$-face by $(d(v_1)(d(v_{1'})), d(v_2), \ldots, d(v_k))$-face. For example, we denote $(2,3,8,6)$-face and $(4,3,8,6)$-face by $(2(4),3,8,6)$-face. u is called a k-vertex, a k^+-vertex, or a k^--vertex, if $d(u) = k$, $d(u) \geq k$, or $d(u) \leq k$, respectively. Similarly, a k-face, a k^+-face, and a k^--face are also defined. P_k is called a path with $V(P_k) = \{v_1, v_2, \ldots, v_k\}$ and $E(P_k) = \{v_i v_{i+1} | i = 1, 2, \ldots, k-1\}$. Undefined notations are referred to [1].

The $(p,1)$-Total labeling problem of a graph G, which originated from the channel assignment problem, was proposed by Havet and Yu [2]. A graph G is said to be k-$(p,1)$-Total labeling if and only if there is a function c from $V(G) \bigcup E(G)$ to $\{0, 1, 2, \ldots, k\}$ such that $|c(u) - c(v)| \geq 1$ if $uv \in E(G)$, $|c(e_1) - c(e_2)| \geq 1$ if the edges e_1 and e_2 are adjacent, and $|c(u) - c(e)| \geq p$ if the vertex u is incident to the edge e. The $(p,1)$-Total labeling number of G, denoted dy $\lambda^T_p(G)$, is the minimum k such that G is k-$(p,1)$-Total labeling. Readers can refer to [3–7] for further research.

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Here we consider the $(p,1)$-Total labeling problem of the list version. Suppose that a list assignment of a graph G is a function L from $V(G)\bigcup E(G)$ to a subset of 2^N. We say G is L-$(p,1)$-Total labeling if there exists a $(p,1)$-Total labeling c such that $c(x)\in L(x)$ for all $x\in V(G)\bigcup E(G)$. If L is any list assignment of G such that $|L(x)|=k$ for all $x\in V(G)\bigcup E(G)$, then the function c is called a k-$(p,1)$-Total choosable function of G with respect to L. The $(p,1)$-Total choice number of G, denoted by $C^T_{p,1}(G)$, is the minimum k such that G has a k-$(p,1)$-Total choosable function c.

Clearly, L-$(p,1)$-Total labeling problem of graph is the list total coloring problem of graph where $p=1$. It is known that there is a List Total Coloring Conjecture $\chi''_l(G)=\chi''(G)$, we may conjecture $C^T_{p,1}(G)=\lambda^T_p(G)+1$. Unfortunately, we found some graphs satisfying $C^T_{p,1}(G)>\lambda^T_p(G)+1$ in [8]. So, Yu [8] proposed the following "Week List $(p,1)$-Total Labeling Conjecture".

Conjecture 1.1 ([8]) *If G is a simple graph with maximum degree Δ, then $C^T_{p,1}(G)\le\Delta+2p$.*

It is true that the conjecture is true for trees and paths. Yu [8] proved the following results:

(1) If G is a star graph $K_{1,n}$, where $n\ge3$ and $p\ge2$, then $C^T_{p,1}(G)\le\Delta+2p-1$.

(2) If G is an outerplanar graph with $\Delta(G)\ge p+3$, then $C^T_{p,1}(G)\le\Delta+2p-1$. Yu and Zhang [9] showed that if G is a graph embedded in surface with Euler characteristic ε and $\Delta(G)$ large enough, then $C^T_{p,1}(G)\le\Delta+2p$.

Especially, for the $(1,1)$-Total choice number of a planar graph, Hou et al. [10] proved that $C^T_{1,1}(G)\le\Delta+2$ where $\Delta(G)\ge9$. Borodin et al. [11] proved that $C^T_{1,1}(G)\le\Delta+1$ where $\Delta(G)\ge12$. Dong et al. [12] showed that if G is a planar graph without non-induced 7-cycles $\Delta(G)\ge9$, then $C^T_{1,1}(G)\le\Delta+2$ where $\Delta(G)\ge7$. In [13–15], Borodin et al. gave some sufficient conditions (in terms of Δ and g) to show that the $(1,1)$-Total choice number of a planar graph is equal to $\Delta+1$. These results mean that when $p=1$, Conjecture 1.1 holds for planar graphs with $\Delta(G)\ge9$.

For the $(1,1)$-Total choice number of a 1-planar graph, Zhang et al. [16] showed that each 1-planar graph with maximum degree Δ is $(\Delta+2)$-total-choosable if $\Delta\ge16$, and is $(\Delta+1)$-total-choosable if $\Delta\ge21$. The result means that when $p=1$, Conjecture 1.1 holds for 1-planar graphs with $\Delta(G)\ge16$.

For the $(2,1)$-Total choice number of a planar graph, Song and Sun [17] proved that

(1) If G is a planar graph with $\Delta(G)\ge7$ and 3-cycle is not adjacent to k-cycle, $k\in\{3,4\}$, then $C^T_{2,1}(G)\le\Delta+4$.

(2) If G is a planar graph with $\Delta(G)\ge8$ and i-cycle is not adjacent to j-cycle, where $i,j\in\{3,4,5\}$, then $C^T_{2,1}(G)\le\Delta+3$. We mainly studies the upper bound of $C^T_{p,1}(G)$ when $p=2$.

In this paper, we prove the following theorem.

Theorem 1.2 *If G is a planar graph with $\Delta(G)\ge11$, then $C^T_{2,1}(G)\le\Delta+4$.*

Theorem 1.2 implies that Conjecture 1.1 holds for planar graphs with $\Delta(G)\ge11$ when $p=2$.

2. Preliminaries

Suppose that L is a list assignment of G and $a = \min_{\cup_{x\in V(G)\cup E(G)}} L(x) > 1$. Let $L_1 = \{L_1(x)|x \in V(G)\cup E(G)\}$, where $L_1(x) = \{b-(a-1)|b \in L(x)\}$ for all $x \in V(G)\cup E(G)$. Then the function f is a k-$(2,1)$-Total choosable of G with respect to L if and only if the function f is also a k-$(2,1)$-Total choosable of G with respect to L_1. We know $\min_{\cup_{x\in V(G)\cup E(G)}} L_1(x) = 1$ and $|L_1(x)| = |L(x)|$ for all $x \in V(G)\cup E(G)$. So, suppose that the list assignment $L : V(G)\cup E(G) \to 2^N$ satisfies that $\min_{\cup_{x\in V(G)\cup E(G)}} L(x) = 1$ in this section.

Zhu et al. [18] gave two usable lemmas for the list L(2, 1)-labeling of planar graphs. According to their method, we got two similar lemmas which are very useful in the proofs of our theorems. For convenience, we do not distinguish between the concepts of "labeling" and "coloring" in the following sections.

Lemma 2.1 *Let L be a list assignment of $P_2 = v_1v_2$. If L satisfies $|L(v_1)| = 3$, $|L(v_1v_2)| = 3$, and $|L(v_2)| = 4$, then P_2 is $(2,1)$-Total choosable with respect to L.*

Proof Case 1. If $1 \in L(v_1v_2)$, then we define a function f such that $f(v_1v_2) = 1, f(v_1) \in L(v_1)\setminus\{1,2\}$ and $f(v_2) \in L(v_2)\setminus\{1,2,f(v_1)\}$. So the function f is a (2,1)-Total choosable function of P_2 with respect to L.

Case 2. If $1 \notin L(v_1v_2)$ and $1 \in L(v_1)$, then we define a function f such that $f(v_1) = 1$. Let $L_1(v_1v_2) = L(v_1v_2)\setminus\{2\}$, $L_1(v_2) = L(v_2)\setminus\{1\}$ and $m = \min\{L_1(v_1v_2)\bigcup L_1(v_2)\}$. Then $|L_1(v_1v_2)| \geq 2$ and $|L_1(v_2)| \geq 3$. When $m = 2$, let $f(v_2) = 2$ and $f(v_1v_2) \in L_1(v_1v_2)\setminus\{3\}$. When $m \neq 2$ and $m \in L_1(v_1v_2)$, let $f(v_1v_2) = m$ and $f(v_2) \in L_1(v_2)\setminus\{m, m+1\}$. When $m \neq 2$, $m \notin L_1(v_1v_2)$ and $m \in L_1(v_2)$, let $f(v_2) = m$ and $f(v_1v_2) \in L_1(v_1v_2)\setminus\{m+1\}$. So we can get a $(2,1)$-Total choosable function f of P_2 with respect to L.

Case 3. If $1 \notin L(v_1v_2)$, $1 \notin L(v_1)$ and $1 \in L(v_2)$, then we define a function f such that $f(v_2) = 1$. Let $L_1(v_1v_2) = L(v_1v_2)\setminus\{2\}$, $L_1(v_1) = L(v_1)$ and $m = \min\{L_1(v_1v_2)\bigcup L_1(v_2)\}$. Then $|L_1(v_1v_2)| \geq 2$ and $|L_1(v_1)| = 3$. When $m = 2$, let $f(v_1) = 2$ and $f(v_1v_2) \in L_1(v_1v_2)\setminus\{3\}$. When $m \neq 2$ and $m \in L_1(v_1v_2)$, let $f(v_1v_2) = m$ and $f(v_1) \in L_1(v_1)\setminus\{m, m+1\}$. When $m \neq 2$, $m \notin L_1(v_1v_2)$ and $m \in L_1(v_1)$, let $f(v_1) = m$ and $f(v_1v_2) \in L_1(v_1v_2)\setminus\{m+1\}$. So we always get a function P_2 is a (2,1)-Total choosable function of G with respect to L. □

Lemma 2.2 *Let L be a list assignment of star graph $K_{1,3}$, where $E(K_{1,3}) = \{vv_1, vv_2, vv_3\}$. If $|L(v)| = 4, |L(vv_1)| = 5, |L(vv_2)| = 4, |L(vv_3)| = 3$ and the color in set $L(x)$, where $x \in \{v, vv_1, vv_2, vv_3\}$, is the available color when v_i has been colored for $1 \leq i \leq 3$, then $K_{1,3}$ is $(2,1)$-Total choosable with respect to L.*

Proof Case 1. If $1 \in L(v)$, then we define a function f such that $f(v) = 1$. Let $L_1(vv_i) = L(vv_i)\setminus\{1,2\}$ for $i = 1,2,3$. Then $|L_1(vv_1)| \geq 3, |L_1(vv_2)| \geq 2, |L_1(vv_3)| \geq 1$. We can greedily color vv_3, vv_2 and vv_1. So we get a $(2,1)$-Total choosable function f of $K_{1,3}$ with respect to L.

Case 2. If $1 \notin L(v_1)$ and $1 \in L(vv_3)$. Let $L_1(vv_1) = L(vv_1)\setminus\{1\}$, $L_1(v) = L(v)\setminus\{2\}$, $L_1(vv_2) = L(vv_2)\setminus\{1\}$. Then $|L_1(vv_1)| \geq 4$, $|L_1(v)| \geq 3, |L_1(vv_2)| \geq 3$. By Lemma 2.1, there is a $(2,1)$-Total choosable function g of $K_{1,3}-\{vv_3\}$ with respect to L_1. Now we can extend g to $K_{1,3}$ by letting $f(vv_3)=1$.

Case 3. If $1\notin L(v_1)\cup L(vv_3)$ and $1\in L(vv_2)$. Let $L_1(vv_1)=L(vv_1)\setminus\{1\}, L_1(v)=L(v)\setminus\{2\}, L_1(vv_3)=L(vv_3)$. Then $|L_1(vv_1)|\ge 4, |L_1(v)|\ge 3, |L_1(vv_3)|\ge 3$. By Lemma 2.1, there is a $(2,1)$-Total choosable function g of $K_{1,3}-\{vv_2\}$ with respect to L_1. Now we can extend g to $K_{1,3}$ by letting $f(vv_2)=1$.

Case 4. If $1\notin L(v_1)\cup L(vv_3)\cup L(vv_2)$ and $1\in L(vv_1)$. Let $L_1(vv_2)=L(vv_2), L_1(v)=L(v)\setminus\{2\}, L_1(vv_3)=L(vv_3)$. Then $|L_1(vv_2)|\ge 4, |L_1(v)|\ge 3, |L_1(vv_3)|\ge 3$. By Lemma 2.1, there is a $(2,1)$-Total choosable function g of $K_{1,3}-\{vv_1\}$ with respect to L_1. Now we can extend g to $K_{1,3}$ by letting $f(vv_1)=1$. □

3. Proof of Theorem 1.2

We prove Theorem 1.2 by contradiction. Suppose that there is a graph G with minimal number of $|V(G)|+|E(G)|$ contradicting Theorem 1.2. That is, G is not Δ+4-$(2,1)$-Total choosable, but each proper subgraph of G is. By the minimality of G, G is connected.

3.1. Structural properties

In this part, we will give some properties of G as follows. For convenience, let $\Theta(x)\in L(x)$, where $x\in V(G)\cup E(G)$, be a partially $(2,1)$-Total choosable function of graph G, and the function satisfies the definition of L-$(2,1)$-Total labeling in the following sections. We denote the set of available colors of x for $x\in V(G)\cup E(G)$ under the partially $(2,1)$-Total choosable function $\Theta(x)$ by $A_\Theta(x)$.

Property 3.1.1 *The minimum degree of G is at least 3.*

Proof Suppose that $\delta(G)\le 2$. Obviously, $\delta(G)\ge 2$. And if there is a 2-vertex v, then $d(v_1)=d(v_2)=\Delta$, where v_1 and v_2 are the neighbors of v. Then $\Delta(G-vv_1)=\Delta(G)$. Thus, the graph $G-vv_1$ satisfies the condition of the Theorem 1.2. By the minimality of G, the graph $G-vv_1$ has a Δ+4-$(2,1)$-Total choosable function Θ. We first erase the color of the vertex v. Since $|A_\Theta(vv_1)|\ge \Delta+4-(d(v)-1+d(v_1)-1+3)\ge\Delta+4-(\Delta+3)\ge 1$ and $|A_\Theta(v)|\ge\Delta+4-(d(v)+3(d(v)-1))=\Delta+4-(2+3)\ge 10$, we can recolor the edge vv_1 and the vertex v in sequence. Therefore, G is Δ+4-$(2,1)$-Total choosable, a contradiction. □

Property 3.1.2 *Every 3-vertex in G is adjacent to 11^+-vertex.*

Proof Suppose that a 3-vertex u is adjacent to a 10^--vertex v. By the minimality of G, the graph $G-uv$ has a Δ+4-$(2,1)$-Total choosable function Θ. We first erase the color of the vertex u. Since $|A_\Theta(uv)|\ge\Delta+4-(d(u)-1+d(v)-1+3)=\Delta+4-(2+9+3)\ge 1$ and $|A_\Theta(u)|\ge\Delta+4-(d(u)+3(d(u)-1))=\Delta+3-(3+3\times 2)\ge 6$, we can recolor the edge uv and the vertex v in sequence. Therefore, G is Δ+4-$(2,1)$-Total choosable, a contradiction. □

Property 3.1.3 *Every 4-vertex in G is adjacent to 8^+-vertex.*

Proof Suppose that a 4-vertex u is adjacent to a 7^{-}-vertex v. By the minimality of G, the graph $G-uv$ has a Δ+4-$(2,1)$-Total choosable function Θ. We first erase the color of the vertex u since $|A_{\Theta}(uv)| \geq \Delta+4-(d(u)-1+d(v)-1+3) = \Delta+4-(3+6+3) \geq 3$ and $|A_{\Theta}(u)| \geq \Delta+4-(d(u)+3(d(u)-1)) = \Delta+3-(4+3\times 3) \geq 2$. Let $\Theta(uv) = \alpha \in A_{\Theta}(uv)$. If $A_{\Theta}(u) \neq \{\alpha-1, \alpha, \alpha+1\}$, then let $\Theta(u) \in A_{\Theta}(u) \setminus \{\alpha-1, \alpha, \alpha+1\}$. If $A_{\Theta}(u) = \{\alpha-1, \alpha, \alpha+1\}$, then let $\Theta(u) = \beta \in A_{\Theta}(u) \setminus \{\alpha\}$ and $\Theta(uv) \in A_{\Theta}(uv) \setminus \{\beta-1, \beta, \beta+1\}$. We can recolor the vertex v and the edge vv_1, easily. Then, G is Δ+4-$(2,1)$-Total choosable, a contradiction. □

Property 3.1.4 *If a 5-vertex v in G is adjacent to a 5-vertex, then v is adjacent to four 7^{+}-vertices.*

Proof Suppose that $d(v) = 5, d(v_1) = 5$, and $d(v_2) \leq 6$ for $vv_1, vv_2 \in E(G)$. By the minimality of G, the graph $G - \{vv_1, vv_2\}$ has a Δ+4-$(2,1)$-Total chooosable function Θ. We first erase the color of the vertex v. Since $|A_{\Theta}(vv_1)| \geq \Delta+4-(3+4+3) = 5$, and $|A_{\Theta}(v)| \geq \Delta+4-(5+3\times3) = 1$, and $|A_{\Theta}(vv_2)| \geq \Delta+4-(3+5+3) = 4$, we can recolor the vertex v and the edge vv_2, vv_1 in sequence. Therefore, G is Δ+4-$(2,1)$-Total choosable, a contradiction. □

Property 3.1.5 *If a 5-vertex v in G is adjacent to a 5-vertex and a 7-vertex, then v is adjacent to three 9^{+}-vertices.*

Proof Suppose that $d(v) = 5, d(v_1) = 5, d(v_2) = 7$, and $d(v_3) \leq 8$ for $vv_1, vv_2, vv_3 \in E(G)$. By the minimality of G, the graph $G - \{vv_1, vv_2, vv_3\}$ has a Δ+4-$(2,1)$-Total chooosable function Θ. We first erase the color of the vertex v. Since $|A_{\Theta}(v)| \geq \Delta+4-(5+3\times 2) = 4$, $|A_{\Theta}(vv_1)| \geq \Delta+4-(4+2+3) = 6$, $|A_{\Theta}(vv_2)| \geq \Delta+4-(6+2+3) = 4$, and $|A_{\Theta}(vv_3)| \geq \Delta+4-(7+2+3) = 3$, we can recolor the vertex v and the edges vv_i, where $i = 1, 2, 3$, at the same time by Lemma 2.2. Therefore, G is Δ+4-$(2,1)$-Total choosable, a contradiction. □

Property 3.1.6 *If a 5-vertex v in G is adjacent to two 6-vertices, then v is adjacent to three 9^{+}-vertices.*

Proof It is similar to the proof of Property 3.1.5. □

Property 3.1.7 *If a 5-vertex v in G is adjacent to a 6-vertex and a 7-vertex, then v is adjacent to three 9^{+}-vertices.*

Proof It is similar to the proof of Property 3.1.5. □

3.2. Discharging

According to Euler's formula $|V(G)|-|E(G)|+|F(G)| = 2$ and $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E|$, we get

$$\sum_{v\in V(G)} (d(v)-4) + \sum_{f\in F(G)} (d(f)-4) = -8.$$

Then, we define an initial charge ω on $V(G) \cup E(G)$ by setting $\omega(x) = d(x) - 4$ for every $x \in V(G) \cup F(G)$. So, we have $\sum_{x\in V(G)\cup F(G)} \omega(x) = -8$. Our aim is to obtain a new nonnegative charge $\omega'(x)$ for all $x \in V(G) \cup E(G)$ by designing discharging rules and redistributing the charges, then we can get a contradiction:

$$0 \leq \sum_{x \in V(G) \bigcup F(G)} \omega'(x) = \sum_{x \in V(G) \bigcup F(G)} \omega(x) = -8 < 0.$$

This contradiction proves the non-existence of G and completes the proof. So, we design discharging rules as follows.

(R1) Every 11^+-vertex sends $\frac{2}{3}$ to each of its incident $(3, 11^+, 11^+)$ -faces, $\frac{1}{3}$ to each of its incident $(11^+, 11^+, 11^+)$ -faces, $\frac{1}{3}$ to each of its incident 4^+-faces, and $\frac{1}{2}$ to other 3 -faces.

(R2) If $6 \leq d(v) \leq 10$, then v sends $\frac{d(v)-4}{d(v)}$ to each of its incident faces.

(R3) Every 5-vertex sends $\frac{1}{3}$ to each of its incident $(5, 5(6), 6(7))$-faces, $\frac{1}{4}$ to each of its incident $(5, 5(6), 8^+)$-faces, and $\frac{1}{7}$ to each of its incident $(5, 7, 7^+)$-faces.

(R4) Every 3^+-face redistributes its remaining charge after applying the previous rules equitably to each of its incident 3-vertices.

(R5) Let v be a 3-vertex. If v is incident with a cutedge, then it receives $\frac{1}{2}$ from each of its neighbors on cutedges and $\frac{1}{12}$ from each of its neighbors on 3^+-face.

Claim 3.2.1 *Every 3^+-face sends at least $\frac{1}{3}$ to each of its incident 3-vertices.*

Proof Let f be any 3^+-face. Suppose that f is incident with t 3-vertices, where $t \geq 1$. Since 3-vertices are not adjacent by Property 3.1.2, then $t \leq \lfloor \frac{d(f)}{2} \rfloor$. Thus, the number of 11^+-vertices on the f is at least t. By (R1), (R4), f sends to each of its incident 3-vertices at least $\frac{d(f)-4+\frac{t}{3}}{t} \geq \frac{1}{3}$, where $d(f) \geq 4$. Suppose that $f = v_1v_2v_3$ is a 3-face with $d(v_1) \leq d(v_2) \leq d(v_3)$. Let $d(v_1) = 3$, then v_2, v_3 are 11^+-vertices by Property 3.1.2. So f sends to each of its incident 3-vertices at least $d(f) - 4 + \frac{2}{3} \times 2 = \frac{1}{3}$ by (R1) and (R4). □

Checking $\omega'(x) \geq 0$ for $x \in V(G) \cup F(G)$.

We first check all the vertices in $V(G)$.

(1) $d(v) = 3$.

If v is only incident with a cutedge, then by $R5$ and Claim 3.2.1, we have $\omega'(v) \geq -1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{12} \times 2 = 0$. If v is incident with two cutedges, then the other edge is also a cutedge. By (R5), we have $\omega'(v) \geq -1 + \frac{1}{2} \times 3 = \frac{1}{2} > 0$. Suppose that v is a 3-vertex with three incident faces f_i, $i \in \{1, 2, 3\}$, in a cyclic order. By Claim 3.2.1, f_i, where $i \in \{1, 2, 3\}$, sends to v at least $\frac{1}{3}$. So, $\omega'(v) = -1 + \frac{1}{3} \times 3 = 0$.

(2) $d(v) = 4$.

Since the discharging rules does not involve 4-vertex, we have $\omega'(v) = \omega(v) = d(v) - 4 = 0$.

(3) $d(v) = 5$.

If v is adjacent with 5-vertex, then v is at most adjacent with a 5-vertex by Property 3.1.4. If v is adjacent with a 7-vertex, then the other neighbors of v are 9^+-vertices by Property 3.1.5. By (R3), we have $\omega'(v) \geq \min\{1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{7}, 1 - \frac{1}{7} \times 2 - \frac{1}{4} \times 2\} > 0$. If v is not adjacent with a 7-vertex, then the other neighbors of v are 8^+-vertices by Properties 3.1.4 and 3.1.5. By (R3), we have $\omega'(v) \geq 1 - \frac{1}{4} \times 2 = \frac{1}{2} > 0$.

If v is adjacent with 6-vertex, then v is at most adjacent with two 6-vertices by Property 3.1.6. If v is adjacent with two 6-vertices, then the other neighbors of v are 9^+-vertices by Property 3.1.6. By (R3), we have $\omega'(v) \geq \min\{1 - \frac{1}{3} - \frac{1}{4} \times 2, 1 - \frac{1}{4} \times 4\} = 0$. If v is adjacent with a 6-vertex, then v is at most adjacent with a 7-vertex by Property 3.1.7. Suppose v is adjacent with a 7-vertex. By (R3), we have $\omega'(v) \geq \min\{1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{7}, 1 - \frac{1}{7} \times 2 - \frac{1}{4} \times 2\} > 0$. Otherwise, the other neighbors of v are 8^+-vertices. By (R3), we have $\omega'(v) \geq 1 - \frac{1}{4} \times 2 > 0$.

If v is only adjacent with 7^+-vertices, then by (R3), we have $\omega'(v) \geq 1 - \frac{1}{7} \times 5 = \frac{2}{7} > 0$.

(4) $6 \leq d(v) \leq 10$.

By (R3), we have $\omega'(v) \geq d(v) - 4 - \frac{d(v)-4}{d(v)} \times d(v) = 0$.

(5) $d(v) = 11$.

If the 3-vertex in $(3, 11^+, 11^+)$-face is incident with a cutedge, then we say the face is a special 3-face. And v is at most incident with five special 3-faces.

Case 1. If v is incident with five special 3-faces, then v is at most incident with six 3-vertices. By (R1) and (R5), we have $\omega'(v) \geq 7 - \frac{2}{3} \times 5 - \frac{1}{12} \times 5 - \frac{1}{2} > 0$.

Case 2. If v is incident with four special 3-faces, then v is at most incident with nine faces. By (R1) and (R5), we have $\omega'(v) \geq 7 - \frac{2}{3} \times 4 - \frac{1}{12} \times 4 - \frac{2}{3} \times 5 = \frac{2}{3} > 0$.

Case 3. If v is incident with three special 3-faces, then v is at most incident with nine faces. By (R1) and (R5), we have $\omega'(v) \geq 7 - \frac{2}{3} \times 3 - \frac{1}{12} \times 3 - \frac{2}{3} \times 6 > 0$.

Case 4. If v is incident with two special 3-faces, then v is at most incident with ten faces. By (R1) and (R5), we have $\omega'(v) \geq 7 - \frac{2}{3} \times 2 - \frac{1}{12} \times 2 - \frac{2}{3} \times 8 = \frac{1}{6} > 0$.

Case 5. If v is only incident with one special 3-face, then v is at most incident with ten faces. By (R1) and (R5), we have $\omega'(v) \geq 7 - \frac{2}{3} \times 1 - \frac{1}{12} \times 1 - \frac{2}{3} \times 9 = \frac{3}{12} > 0$.

Case 6. If v is not incident with any special 3-face, then let k be the number of $(3, 11^+, 11^+)$-face. By Property 3.1.2, we have $k \leq 10$. Suppose $k = 10$, then v is not incident with any cutedge. By (R1), we have $\omega'(v) \geq 7 - \frac{2}{3} \times 10 - \frac{1}{3} = 0$. Suppose $1 \leq k \leq 9$, then by (R1) and (R5), we have $\omega'(v) \geq 7 - \frac{2}{3} \times 9 - \frac{1}{2} \times 2 = 0$. Suppose $k = 0$, then by (R1) and (R5), we have $\omega'(v) \geq 7 - \frac{1}{2} \times 11 > 0$.

(6) $d(v) \geq 12$.

Let s be the number of special 3-faces. Then $s \leq \lfloor \frac{d(v)}{2} \rfloor$.

Case 1. If $1 \leq s \leq \lfloor \frac{d(v)}{2} \rfloor$, then v is at most incident with $d(v)-1$ faces. By (R1) and (R5), we have $\omega'(v) \geq d(v) - 4 - \frac{2}{3} \times \lfloor \frac{d(v)}{2} \rfloor - \frac{1}{12} \times \lfloor \frac{d(v)}{2} \rfloor - \frac{2}{3} \times (d(v) - 1 - \lfloor \frac{d(v)}{2} \rfloor) \geq d(v) - 4 - \frac{2}{3} d(v) - \frac{d(v)}{24} + \frac{2}{3} = \frac{7d(v)}{24} - \frac{10}{3} \geq \frac{1}{6}$.

Case 2. If $s = 0$, then let k be the number of special $(3, 11^+, 11^+)$-face. Suppose $1 \leq k \leq d(v)$, then by (R1) and (R5), we have $\omega'(v) \geq d(v) - 4 - \frac{2}{3} \times d(v) \geq 0$. Otherwise $k = 0$, then by (R1) and (R5), we have $\omega'(v) \geq d(v) - 4 - \frac{1}{2} \times d(v) > 0$.

Next, we consider the discharge of the faces in G.

(1) $d(f) = 3$.

If f is incident with a 3-vertex, then it is incident with at least two 11^+-vertices by Property 3.1.2. By (R1), we have $\omega'(f) \geq d(f) - 4 + \frac{2}{3} \times 2 > 0$.

If f is incident with a 4-vertex, then it is incident with at least two 8^+-vertices by Property 3.1.3. By (R1) and (R3), we have $\omega'(f) \geq d(f) - 4 + \frac{1}{2} \times 2 = 0$.

If f is incident with a 5-vertex, then it is incident with at most two 5-vertices by Property 3.1.5. Suppose f is adjacent with two 5-vertices, so f is a $(5,5,7)$-face or $(5,5,8^+)$-face. By (R2) and (R3), we have $\omega'(f) \geq \min\{-1 + \frac{1}{3} \times 2 + \frac{3}{7}, -1 + \frac{1}{4} \times 2 + \frac{1}{2}\} = 0$. Otherwise, f is incident with a 5-vertex. Then, f is one of $(5,6,6)$-face, $(5,6,7)$-face, $(5,6,8^+)$-face, $(5,7,7^+)$-face, or $(5,8^+,8^+)$-face by Properties 3.1.6 and 3.1.7. By (R2) and (R3), we have $\omega'(f) \geq \min\{-1 + \frac{1}{3} + \frac{1}{3} \times 2, -1 + \frac{1}{3} + \frac{1}{3} + \frac{3}{7}, -1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{2}, -1 + \frac{1}{7} + \frac{3}{7} \times 2, -1 + \frac{1}{2} \times 2\} = 0$.

If f is only incident with 6^+-vertices, then by (R1)–(R3), we have $\omega'(f) \geq -1 + \frac{1}{3} \times 3 = 0$.

(2) $d(f) \geq 4$.

By (R1)–(R5), we have $\omega'(f) = \omega(f) = d(f) - 4 = 0$.

The proof of Theorem 1.2 is completed. □

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References

[1] J. A. BONDY, U. S. R. MURTY. *Graph Theory with Applications*. MacMilan, London, 1976.

[2] F. HAVET, Minli YU. *$(p,1)$-total labelling of graphs*. Discrete Math., 2008, **308**(4): 496–513.

[3] Xin ZHANG, Yong YU, Guizhen LIU. *On $(p,1)$-total labelling of plane graphs with independent crossings*. Filomat, 2012, **26**(6): 1901–1100.

[4] A. KCMNITZ, M. MARANGIO. *$[r,s,t]$-colorings of graphs*. Discrete Math., 2007, **307**(2): 199–207.

[5] Lin SUN, Jianliang WU. *On $(p,1)$-total labelling of planar graphs*. J. Comb. Optim., 2015, **33**(1): 317–325.

[6] Bei NIU, Xin ZHANG. *On $(p,1)$-total labelling of some 1-planar graphs*. Discuss. Math. Graph Theory, 2021, **41**(2): 431–558.

[7] Yong YU, Xin ZHANG, Guanghui WANG, et al. *$(2,1)$-total labelling of planar graphs with large maximum degree*. Discrete Math., 2017, **20**(8): 1625–1636.

[8] Yong YU. *$[r,s,t;f]$-colorings and $(p,1)$-total labelling of graphs*. D. Shandong University, 2012, **1**: 25–79.

[9] Yong YU, Xin ZHANG, Guizhen LIU. *List $(d,1)$-total labelling of graphs embedded in surfaces*. Oper. Res. Trans., 2011, **15**(3): 29–37.

[10] Jianfeng HOU, Guizhen LIU, Jianliang WU. *Some results on list total colorings of planar graphs*. Lecture Note in Computer Science, 2007, **4489**: 320–328.

[11] O. BORODIN, A. KOSTOCHKA, D. WOODALL. *List edge and list total colourings of multigraphs*. J. Combin. Theory Ser. B, 1997, **71**(2): 184–204.

[12] Aijun DONG, Guizhen LIU, Guojun LI. *List edge and list total colorings of planar graphs without non-induced 7-cycles*. Discrete Math. Theor. Comput. Sci., 2013, **15**(1): 101–106.

[13] Rui LI, Baogang XU. *Edge choosability and total choosability of planar graphs with no 3-cycles adjacent 4-cycles*. Discrete Math., 2011, **311**(20): 2158–2163.

[14] Qiuli LU, Zhengke MIAO, Yingqian WANG. *Sufficient conditions for a planar graph to be list edge Δ-colorable and list totally $(\Delta+1)$-colorable*. Discrete Math., 2013, **313**(5): 575–580.

[15] Huijun WANG, Lidong WU, Xin ZHANG, et al. *A note on the minimum number of choosability of planar graphs*. J. Comb. Optim., 2016, **31**(3): 1013–1022.

[16] Xin ZHANG, Jianliang WU, Guizhen LIU. *List edge and list total coloring of 1-planar graphs*. Front. Math. China, 2012, **7**(5): 1005–1018.

[17] Yan SONG, Lei SUN. *Two Results on K-(2,1)-Total Choosability of Planar Graphs*. Discrete Math. Algorithms Appl., 2020, **12**(6): 1–14.

[18] Haiyang ZHU, Lianying MIAO, Sheng CHEN, et al. *The list $L(2,1)$-labeling of planar graphs*. Discrete Math., 2018, **341**(8): 2211–2219.