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Degree Sum Conditions for Traceable Quasi-Claw-Free Graphs

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Abstract A traceable graph is a graph containing a Hamilton path. Let $N[v] = N(v) \cup \{v\}$ and $J(u,v) = \{w \in N(u) \cap N(v) : N(w) \subseteq N[u] \cup N[v]\}$. A graph G is called quasi-claw-free if $J(u,v) \neq \emptyset$ for any $u,v \in V(G)$ with distance of two. Let $\sigma_k(G) = \min\{\sum_{v\in S} d(v) : S$ is an independent set of $V(G)$ with $|S| = k\}$, where $d(v)$ denotes the degree of v in G. In this paper, we prove that if G is a connected quasi-claw-free graph of order n and $\sigma_3(G) \geq n-2$, then G is traceable; moreover, we give an example to show the bound in our result is best possible. We obtain that if G is a connected quasi-claw-free graph of order n and $\sigma_2(G) \geq \frac{2(n-2)}{3}$, then G is traceable.

Keywords traceable graph; quasi-claw-free graphs; degree sum

MR(2020) Subject Classification 05C07; 05C38

1. Introduction

We consider only finite and simple graphs in this paper. For notation and terminology not defined here we refer to [1]. For a graph G and a subset S of $V(G)$, let $N_S(v)$ denote the set of neighbors of v in S and $d_S(v) = |N_S(v)|$, where $v \in V(G)$; moreover, if $S = V(G)$, then $N(v)$ and $d(v)$ denote $N_G(v)$ and $d_G(v)$ for simplicity, respectively. For a vertex subset H of G, let $G[H]$ denote the subgraph induced by H, and $G - H = G[V(G) \setminus V(H)]$. For a graph G, let $\sigma_k(G) = \min\{\sum_{v\in S} d(v) : S$ is an independent vertex set of G with $|S| = k\}$ if $k \leq \alpha(G)$, and set $\sigma_k(G) = +\infty$ if $k > \alpha(G)$, where $\alpha(G)$ denotes the independence number of G. For two vertices u and v in a graph G, the distance between u and v, denoted by $d(u,v)$, is the number of edges in a shortest path connecting u and v in G. If a graph G contains a Hamilton path, then G is traceable, and G is Hamiltonian if G contains a Hamilton cycle.

Let P be a path with a given direction. If $u,v \in V(P)$, then uPv denotes the consecutive vertices on P from u to v along the positive direction, and vP^-u denotes the same vertices in reverse order. We will consider uPv and vPu both as sub-paths and vertex subsets of P. For a vertex $v \in V(P)$, let v^+ and v^- denote the successor and predecessor of v on P, respectively;

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similarly, v^{++} and v^{--} denote the successor and predecessor of v^+ and v^- on P, respectively; moreover, we denote the set of successors and predecessors of all the neighbors of v on P by $N^+(v)$ and $N^-(v)$, respectively. We also use analogous notation for a cycle C.

A claw-free or $K_{1,3}$-free graph is a graph without induced subgraphs isomorphic to $K_{1,3}$. For two vertices x and y, define $J(x,y)=\{u: u\in N(x)\cap N(y), N(u)\subseteq N[x]\cup N[y]\}$. A graph G is quasi-claw-free if $J(u,v)\neq\emptyset$ for each pair of vertices $u,v\in V(G)$ with $d(u,v)=2$. Obviously, each claw-free graph is quasi-claw-free, but the converse is not true. Note that the graph in Figure 1 given in [2] is a quasi-claw-free graph but not claw-free.

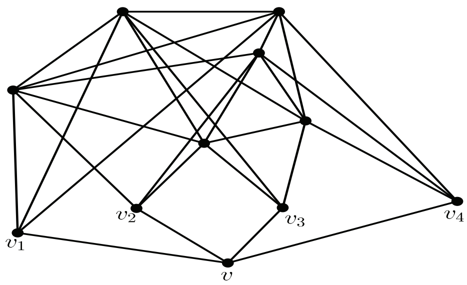

Figure 1 A quasi-claw-free graph G with a claw $G[v,v_1,v_2,v_3]$

Though, quasi-claw-free graphs are generalization of claw-free graphs, the two classes of graphs have a lot of analogous properties, especially on Hamiltonicity. As follows, we present some degree conditions on Hamiltonicity of claw-free graphs and quasi-claw-free graphs.

Theorem 1.1 ([3]) *If G is a 3-connected claw-free graph of order n and $\delta(G)\geq(n+5)/5$, then G is Hamiltonian.*

Theorem 1.2 ([4]) *Let G be a 3-connected quasi-claw-free graph of order n. If $\delta(G)\geq(n+5)/5$, then G is Hamiltonian.*

The family $\mathfrak{F}$ of graphs is defined as follows: if G is in $\mathfrak{F}$, then G can be decomposed into three vertex disjoint subgraphs G_1, G_2 and G_3 such that $V(G_i)\cap V(G_j)=\emptyset$ and $E_G(V(G_i),V(G_j))=\{u_iu_j, v_iv_j\}$, where $1\leq i\neq j\leq 3, u_i,v_i\in V(G_i)$ and $u_i\neq v_i, 1\leq i\leq 3$.

Theorem 1.3 ([5]) *If G is a 2-connected claw-free graph of order n and $\delta(G)\geq n/4$, then G is Hamiltonian or $G\in\mathfrak{F}$.*

Theorem 1.4 ([6]) *Let G be a 2-connected quasi-claw-free graph of order n. If $\delta(G)\geq n/4$, then G is Hamiltonian or $G\in\mathfrak{F}$.*

2. Main results

Clearly, if a graph is Hamiltonian, then it is 2-connected and traceable; but if a graph is traceable, then it may be neither Hamiltonian nor 2-connected. In this paper, we give the following degree sum conditions for traceable quasi-claw-free graphs.

Theorem 2.1 *If G is a connected quasi-claw-free graph of order n with $\sigma_3(G) \geq n-2$, then G is traceable.*

Proof of Theorem 2.1 To the contrary, suppose G is a non-traceable quasi-claw-free graph satisfying the conditions of Theorem 2.1. Assume $P := v_1v_2\cdots v_m$ is a longest path of G with a given positive orientation from v_1 to v_m. Clearly, $3 \leq m \leq n-1$, which implies $V(G-P) \neq \emptyset$. Since G is connected, there is a vertex $v \in V(G-P)$ such that $N_P(v) \neq \emptyset$. By the length of P, it is easy to obtain the following two results.

Claim 1. $N(v_1) \cup N(v_m) \subseteq V(P)$ and $v_1v_m, v_1v, v_mv \notin E(G)$.

Claim 2. There is no cycle C in G such that $V(C) = V(P)$.

Claim 3. For each $u \in N_P(v), \{u\} = J(v, u^-) = J(v, u^+)$ and $u^-u^+ \in E(G)$.

Proof of Claim 3 Since P is a longest path in G, $u^-v, u^+v \notin E(G)$ for each vertex $u \in N_P(v)$, and then $d(u^-, v) = d(u^+, v) = 2$. Since G is quasi-claw-free, $J(v, x) \neq \emptyset, x \in \{u^-, u^+\}$. To the contrary, suppose $w \in J(v, u^-)$ and $w \neq u$ for some vertex $u \in N_P(v)$. If $w \notin V(P)$, then we can obtain a longer path $v_1Pu^-wvuPv_m$ than P, a contradiction. Thus $w \in V(P)$. Clearly, $\{w^-, w^+\} \subseteq N[u^-] \cup N(v)$ by the definition of quasi-claw-free graphs. Moreover, $w^-v, w^+v \notin E(G)$ by $vw \in E(G)$ and the choice of P, which implies $w^-u^-, w^+u^- \in E(G)$ (otherwise, $w^- \notin N[u^-] \cup N[v]$, contradicting to $N(w) \in N[u^-] \cup N[v]$ as $w^- \in N(w)$). Suppose $w \in v_1Pu^-$. Then we can obtain a path $P' := v_1Pw^-u^-P^-wvuPv_m$, which is longer than P, a contradiction. Suppose $w \in u^+Pv_m$. Then we can obtain a longer path $P'' := v_1Pu^-w^-P^-uvwPv_m$ than P, a contradiction. It follows that $\{u\} = J(v, u^-)$, and hence $u^+ \in N(u^-) \cup N(v)$ by $u^+u \in E(G)$. Clearly, $u^+v \notin E(G)$, and hence $u^-u^+ \in E(G)$. By symmetry, we can obtain that $\{u\} = J(v, u^+)$. □

Claim 4. $N^-(v_1), N(v_m), N_P(v), N_P^-(v)$ are pairwise disjoint.

Proof of Claim 4 To the contrary, suppose $u_1 \in N^-(v_1) \cap N(v_m)$. By Claim 1, $u_1 \in V(P)$. Since u_1 is the predecessor of a neighbor of v_1, it implies $u_1^+v_1 \in E(G)$. Then we can obtain a cycle $C_1 := v_1Pu_1v_mP^-u_1^+v_1$ such that $V(C_1) = V(P)$, a contradiction to Claim 2. Thus $N^-(v_1) \cap N(v_m) = \emptyset$. Suppose $u_2 \in N^-(v_1) \cap N_P(v)$. Then there exists a longer path $P_1 := vu_2P^-v_1u_2^+Pv_m$ than P, a contradiction. Thus $N^-(v_1) \cap N_P(v) = \emptyset$. Suppose $u_3 \in N^-(v_1) \cap N_P^-(v)$. Then $u_3^+ \in N(v_1) \cap N_P(v)$. By Claim 3, $u_3u_3^{++} \in E(G)$. Then we can obtain a path $P_2 := vu_3^+v_1Pu_3u_3^{++}Pv_m$, which is longer than P, a contradiction. Thus $N^-(v_1) \cap N_P^-(v) = \emptyset$. Suppose $u_4 \in N(v_m) \cap N_P(v)$. By Claim 3, $u_4^-u_4^+ \in E(G)$. There exists a longer path $P_3 := vu_4v_mP^-u_4^+u_4^-P^-v_1$ than P, a contradiction. Thus $N(v_m) \cap N_P(v) = \emptyset$. Suppose $u_5 \in N(v_m) \cap N_P^-(v)$. Then $u_5^+v \in E(G)$, and we can obtain a longer path $P_4 := vu_5^+Pv_mu_5P^-v_1$ than P, a contradiction. Thus $N(v_m) \cap N_P^-(v) = \emptyset$. By the choice of P, it is easy to obtain that $N_P(v) \cap N_P^-(v) = \emptyset$. It follows that the claim is true. □

It is easy to obtain the following result.

Claim 5. $v_m \notin N^-(v_1) \cup N(v_m)$.

Next, we complete the proof of Theorem 2.1. By Claims 4 and 5, we have $N_P(v) \cup N_P^-(v) \subseteq$

$V(P) - N^-(v_1) \cup N[v_m]$, $N^-(v_1) \cap N[v_m] = \emptyset$ and $N_P(v) \cap N_P^-(v) = \emptyset$, which implies $2d_P(v) \leq |P| - (d_P(v_1) + d_P(v_m) + 1)$. Thus $d_P(v_1) + d_P(v_m) + d_P(v) \leq |P| - d_P(v) - 1$, and we denote the inequality by $(*)$. By Claim 1, $N_{G-P}(v_1) = N_{G-P}(v_m) = \emptyset$, and hence $d_{G-P}(v_1) = d_{G-P}(v_m) = 0$. Moreover, $d_{G-P}(v) \leq |G - P| - 1 = n - |P| - 1$. Thus $d_{G-P}(v_1) + d_{G-P}(v_m) + d_{G-P}(v) \leq n - |P| - 1$, which we denote by $(**)$. Since G is connected, $d_P(v) \geq 1$. By the inequalities $(*)$ and $(**)$, $\sigma_3(G) \leq d(v_1) + d(v_m) + d(v) \leq n - d_P(v) - 2 \leq n - 3$, a contradiction. Thus Theorem 2.1 is true. □

Remark 2.2 Suppose each G_i is a complete graph of order n_i with $u_i \in V(G_i)$, where n_i is a positive integer at least $3, 1 \leq i \leq 3$. Let G (See Figure 2) be a graph with $V(G) = V(G_1) \cup V(G_2) \cup V(G_3), E(G) = E(G_1) \cup E(G_2) \cup E(G_2) \cup \{u_1u_2, u_1u_3, u_2u_3\}$. Clearly, G is a quasi-claw-free graph with $\sigma_3(G) = |G| - 3$, and there is no Hamilton path in G. Thus the bound in Theorem 2.1 is best possible.

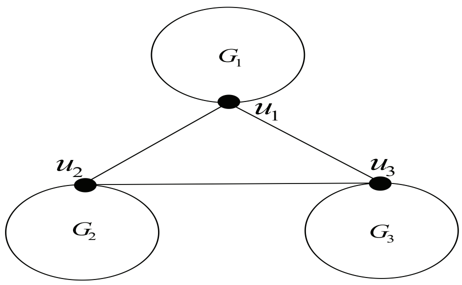

Figure 2 A connected and non-traceable quasi-claw-free graph G with $\sigma_3(G) = |G| - 3$

Since each claw-free graph is quasi-claw-free, we can get the following result.

Corollary 2.3 *Let G be a connected claw-free graph of order n with $\sigma_3(G) \geq n - 2$. Then G is traceable. The following result gives function relations between $\sigma_{k'}(G)$ and $\sigma_k(G)$ in a graph G, $1 \leq k \leq k' \leq |G|$.*

Lemma 2.4 ([7]) *Let G be a graph of order n and $1 \leq k \leq k' \leq n$. Then $\sigma_{k'}(G) \geq \frac{k'}{k}\sigma_k(G)$.*

By Theorem 2.1 and Lemma 2.4, we can obtain the following result.

Corollary 2.5 *If G is a connected quasi-claw-free graph of order n with $\sigma_2(G) \geq \frac{2(n-2)}{3}$, then G is traceable.*

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