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Fractional Boundary Value Problems with Integral Boundary Conditions via Topological Degree Method

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Abstract This paper examines the existence and uniqueness of solutions for the fractional boundary value problems with integral boundary conditions. Banach's contraction mapping principle and Schaefer's fixed point theorem have been used besides topological technique of approximate solutions. An example is propounded to uphold our results.

Keywords fractional derivatives and integrals; fixed point theorems; degree theory; nonlinear operators

MR(2020) Subject Classification 26A33; 47H10; 47H11

1. Introduction

Fractional calculus has attracted broad attention as it may be applied in various fields of science and applications such like engineering, mechanics, electro chemistry, porous media, etc., [1–8]. Recently, using topological technique becomes very close to verify the existence of solutions for fractional differential equations [9, 10].

In 2013, Graef et al. [11], studied a type of nonlinear fractional boundary value problem with the integral boundary conditions by constructing an associated Green's function, spectral theory and applying fixed point theory on cones. The existence of mild solutions for fractional differential equations with integral boundary conditions and not instantaneous impulses was investigated by Li and Xu [12]. They also established the sufficient conditions for the existence and uniqueness of solutions by some fixed point theorems. Sudsutad and Tariboon [13] studied a boundary value problem of nonlinear fractional differential equations with three points fractional integral boundary conditions by applying standard fixed point theorems. The existence and uniqueness of solutions for fractional differential equations with nonlocal and fractional integral boundary conditions were studied by Derbazi and Hammouche [14]. Younis and Singh [15], found the sufficient conditions for the existence of solutions of some class of Hammerstein integral equations and fractional differential equations. Younis et al. [16], presented the notion of graphical extended b-metric spaces, blending the concepts of graph theory and metric fixed point theory. Existence and uniqueness results were established using the Banach contraction principle and some other

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existence results were obtained using O'Regan fixed point theorem and Burton and Kirk fixed point. A new technique, based on F-Reich contraction, was given for solving some models of real world problems, viz by Younis et al. [17].

Our aim during this paper is to verify some new results on the following boundary value problem (BVP) for fractional differential equations involving the Caputo fractional derivative by topological degree method and fixed point theorem.

$$\begin{cases} {}^{c}\mathcal{D}^{q}x(t) = \xi(t,x), \quad t \in \mathcal{J} := [0,T],\ 0<q<1, \\ x(0) = \mathcal{I}_0^T \eta(t,x), \quad x(T) = \mathcal{I}_0^T \zeta(t,x), \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}^{q}$ is the Caputo fractional derivative and $\xi, \eta, \zeta : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ are continuous functions.

2. Preliminaries

In this section, we introduce some necessary definitions, propositions and theorems which are needed throughout this paper.

We define a Banach space $\mathcal{C}(\mathcal{J},\mathcal{X})$ as the Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{X}$ with the norm $\|x\|_c := \sup\{\|x(t)\| : x \in \mathcal{C}(\mathcal{J},\mathcal{X})\}$ for $t \in \mathcal{J}$ and $\mathcal{J} = [0,T]$, $T > 0$.

Definition 2.1 ([18]) *For a given continuous function ξ on a closed interval $[a,b]$, the qth fractional order integral of ξ is defined by*

$$\mathcal{I}_{a+}^{q}\xi(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\xi(s)\mathrm{d}s, \tag{2.1}$$

where Γ is the gamma function.

Definition 2.2 ([18]) *For a given continuous function ξ on a closed interval $[a,b]$, the qth Riemann-Liouville fractional order derivative of ξ, is defined by*

$$(\mathcal{D}_{a+}^{q}\xi)(t) = \frac{1}{\Gamma(n-q)}(\frac{\mathrm{d}}{\mathrm{d}t})^n \int_a^t (t-s)^{n-q-1}\xi(s)\mathrm{d}s, \tag{2.2}$$

where $n = [q]+1$ and $[q]$ denotes the integer part of q.

Definition 2.3 ([18]) *For a given continuous function ξ on a closed interval $[a,b]$, the Caputo fractional order derivative of ξ, is defined by*

$$({}^{c}\mathcal{D}_{a+}^{q}\xi)(t) = \frac{1}{\Gamma(n-q)}\int_a^t (t-s)^{n-q-1}\xi^{(n)}(s)\mathrm{d}s, \tag{2.3}$$

where $n = [q]+1$.

Theorem 2.4 ([19]) *Let $\mathcal{X}$ be a Banach space, and $\psi, \varphi : \mathcal{X} \to \mathcal{X}$ be two operators such that ψ is a contraction operator and φ is a completely continuous operator, then the operator equation $\mathcal{F}x = \psi x + \varphi x = x$ has a solution $x \in \mathcal{X}$.*

Definition 2.5 ([19]) *Let $\Omega \subset X$ and $F : \Omega \to X$ be a continuous bounded map. One can say that F is α-Lipschitz if there exists $k \geq 0$ such that*

$$\alpha(F(B)) \leq k\alpha(B) \ \ (\forall)\ B \subset \Omega \text{ bounded.}$$

In case, $k<1$, then we call F is a strict α-contraction. One can say that F is α-condensing if

$$\alpha(F(B))<\alpha(B) \ \ (\forall)\ B\subset\Omega \text{ bounded with } \alpha(B)>0.$$

We recall that $F:\Omega\to X$ is Lipschitz if there exists $k>0$ such that

$$\|F_x-F_y\|\le k\|x-y\| \ \ (\forall)\ x,y\subset\Omega,$$

and if $k<1$ then F is a strict contraction.

Proposition 2.6 ([19]) *If $\psi,\varphi:\Omega\to X$ are α-Lipschitz maps with constants k, k' respectively, then $\psi+\varphi:\Omega\to X$ is α-Lipschitz with constant $k+k'$.*

Proposition 2.7 ([19]) *If $\psi:\Omega\to X$ is compact, then ψ is α-Lipschitz with zero constant.*

Proposition 2.8 ([19]) *If $\psi:\Omega\to X$ is Lipschitz with constant k, then ψ is α-Lipschitz with the same constant k.*

3. Main results

First, let us define the meaning of a solution of the BVP(1.1).

Definition 3.1 *A function $x\in\mathcal{C}(\mathcal{J},\mathcal{X})$ is said to be a solution of the fractional BVP(1.1) if x satisfies the equation ${}^c\mathcal{D}^q x(t)=\xi(t,x)$ almost everywhere on $\mathcal{J}$ and the conditions $x(0)=\mathcal{I}_0^T\eta(t,x)$ and $x(T)=\mathcal{I}_0^T\zeta(t,x)$.*

In order to treat the problem of existence for a solution of BVP(1.1), we need the following assumptions:

(H1) $\xi:\mathcal{J}\times\mathcal{X}\to\mathcal{X}$ is continuous.

(H2) For arbitrary $x,y\in\mathcal{X}$, there exists a constant $\delta_\xi>0$ such that

$$\|\xi(t,x)-\xi(t,y)\|\le\delta_\xi\|x-y\|.$$

For the existence of solutions for the BVP(1.1), we also need the following auxiliary lemma [1].

Lemma 3.2 *Let $0<q\le1$ and let $\xi,\eta,\zeta:\mathcal{J}\to\mathcal{X}$ be continuous. A function $x\in\mathcal{C}(\mathcal{J},\mathcal{X})$ is said to be a solution of the fractional integral equation*

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\xi(s,x(s))\mathrm{d}s-\frac{t}{T\Gamma(q)}\int_0^T(T-s)^{q-1}\xi(s,x(s))\mathrm{d}s-\\&(\frac{t}{T}-1)\int_0^T\eta(s,x(s))\mathrm{d}s+\frac{t}{T}\int_0^T\zeta(s,x(s))\mathrm{d}s,\end{aligned}\tag{3.1}$$

if and only if x is the solution of the fractional BVP(1.1).

Theorem 3.3 *Assume that (H2) holds and also the following hypotheses:*

(H3) For arbitrary $x,y\in\mathcal{C}(\mathcal{J},\mathcal{X})$ there exists a constant $\delta_\eta\in(0,1)$ such that

$$\|\eta(x)-\eta(y)\|\le\delta_\eta\|x-y\|.$$

(H4) For arbitrary $x, y \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ there exists a constant $\delta_\zeta \in (0,1)$ such that

$$\|\zeta(x) - \zeta(y)\| \leq \delta_\zeta \|x - y\|.$$

If

$$\frac{2\delta_\xi T^q}{\Gamma(q+1)} + T(\delta_\eta + \delta_\zeta) < 1,$$

then the fractional BVP(1.1) has a unique solution $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$.

Proof Consider the operator $\mathcal{F} : \mathcal{C}(\mathcal{J}, \mathcal{X}) \to \mathcal{C}(\mathcal{J}, \mathcal{X})$ defined by

$$\begin{aligned}\mathcal{F}(x)(t) =& \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \xi(s, x(s)) \mathrm{d}s - \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \xi(s, x(s)) \mathrm{d}s - \\ & (\frac{t}{T} - 1) \int_0^T \eta(s, x(s)) \mathrm{d}s + \frac{t}{T} \int_0^T \zeta(s, x(s)) \mathrm{d}s.\end{aligned}$$

It is clear that, the fixed points of the operator $\mathcal{F}$ are solutions of the problem BVP(1.1). Now, consider

$$\begin{aligned}\|\mathcal{F}(x)(t) - \mathcal{F}(y)(t)\| \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|\xi(s, x(s)) - \xi(s, y(s))\| \mathrm{d}s + \\ & \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \|\xi(s, x(s)) - \xi(s, y(s))\| \mathrm{d}s + \\ & (\frac{t}{T} - 1) \int_0^T \|\eta(s, x(s)) - \eta(s, y(s))\| \mathrm{d}s + \frac{t}{T} \int_0^T \|\zeta(s, x(s)) - \zeta(s, y(s))\| \mathrm{d}s \\ \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \delta_\xi \|x - y\| \mathrm{d}s + \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \delta_\xi \|x - y\| \mathrm{d}s + \\ & (\frac{t}{T} - 1) \int_0^T \delta_\eta \|x - y\| \mathrm{d}s + \frac{t}{T} \int_0^T \delta_\zeta \|x - y\| \mathrm{d}s \\ \leq & \frac{\delta_\xi}{\Gamma(q)} (\frac{t^q}{q}) \|x - y\| + \frac{t\delta_\xi}{T\Gamma(q)} (\frac{T^q}{q}) \|x - y\| + (\frac{t}{T} - 1) T\delta_\eta \|x - y\| + \frac{t}{T} T\delta_\zeta \|x - y\| \\ = & \frac{t^q \delta_\xi}{\Gamma(q+1)} \|x - y\| + \frac{t\delta_\xi T^{q-1}}{\Gamma(q+1)} \|x - y\| + (t - T)\delta_\eta \|x - y\| + t\delta_\zeta \|x - y\| \\ = & \frac{t(t^{q-1} + T^{q-1})\delta_\xi}{\Gamma(q+1)} \|x - y\| + (t - T)\delta_\eta \|x - y\| + t\delta_\zeta \|x - y\|\end{aligned}$$

as $0 \leq t \leq T$, then

$$\|\mathcal{F}(x)(t) - \mathcal{F}(y)(t)\| \leq [\frac{2\delta_\xi T^q}{\Gamma(q+1)} + T(\delta_\eta + \delta_\zeta)] \|x - y\|.$$

Thus, $\mathcal{F}$ is a contraction mapping on $\mathcal{C}(\mathcal{J}, \mathcal{X})$ with contraction constant $[\frac{2\delta_\xi T^q}{\Gamma(q+1)} + T(\delta_\eta + \delta_\zeta)]$. By applying Banach's contraction mapping principle, one can deduce that the operator $\mathcal{F}$ has a unique fixed point on $\mathcal{C}(\mathcal{J}, \mathcal{X})$ which implies the BVP(1.1) has a unique solution in $\mathcal{C}(\mathcal{J}, \mathcal{X})$. □

Theorem 3.4 *Assume that (H1), (H2) and the following hypotheses:*

(H5) For arbitrary $(t, x) \in \mathcal{J} \times \mathcal{X}$, there exist $\delta_1, \delta_2 > 0$, $q_1 \in [0, 1)$ such that

$$\|\xi(t, x)\| \leq \delta_1 \|x\|^{q_1} + \delta_2.$$

(H6) For arbitrary $(t, x) \in \mathcal{J} \times \mathcal{X}$, there exist $\delta_3, \delta_4 > 0$, $q_2 \in [0, 1)$ such that

$$\|\eta(t, x)\| \leq \delta_3 \|x\|^{q_2} + \delta_4.$$

(H7) For arbitrary $(t, x) \in \mathcal{J} \times \mathcal{X}$, there exist $\delta_5, \delta_6 > 0$, $q_3 \in [0, 1)$ such that

$$\|\zeta(t, x)\| \leq \delta_5 \|x\|^{q_3} + \delta_6,$$

hold then the fractional BVP(1.1) has at least one solution $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$.

Proof Step 1. Prove continuity of $\mathcal{F}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of a bounded set $\mathcal{B}_k \subseteq \mathcal{C}(\mathcal{J}, \mathcal{X})$ such that $\|x_n - x\| \to 0$ as $n \to \infty$ in $\mathcal{B}_k$ $(k > 0)$. For all $s \in [0, t], t \in \mathcal{J}$, we have to show that $\|\mathcal{F}x_n - \mathcal{F}x\| \to 0$ as $n \to \infty$ as follows:

$$\begin{aligned}
\|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)\| \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| \mathrm{d}s + \\
& \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| \mathrm{d}s + \\
& (\frac{t}{T} - 1) \int_0^T \|\eta(s, x_n(s)) - \eta(s, x(s))\| \mathrm{d}s + \frac{t}{T} \int_0^T \|\zeta(s, x_n(s)) - \zeta(s, x(s))\| \mathrm{d}s \\
\leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \delta_\xi \|(x_n - x)\| \mathrm{d}s + \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \delta_\xi \|(x_n - x)\|| \mathrm{d}s + \\
& (\frac{t}{T} - 1) \int_0^T \delta_\eta \|(x_n - x)\| \mathrm{d}s + \frac{t}{T} \int_0^T \delta_\zeta \|(x_n - x)\| \mathrm{d}s \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Step 2. Prove $\mathcal{F}$ map bounded sets into bounded sets in $\mathcal{C}(\mathcal{J}, \mathcal{X})$. For any $r > 0$, we have $x \in \mathcal{B}_r := \{x \in \mathcal{C}(\mathcal{J}, \mathcal{X}) : \|x\| \leq r\}$,

$$\begin{aligned}
\|(\mathcal{F}x)(t)\| \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|\xi(s, x(s))\| \mathrm{d}s + \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \|\xi(s, x(s))\| \mathrm{d}s + \\
& (\frac{t}{T} - 1) \int_0^T \|\eta(s, x(s))\| \mathrm{d}s + \frac{t}{T} \int_0^T \|\zeta(s, x(s))\| \mathrm{d}s \\
\leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\delta_1 \|x\|^{q_1} + \delta_2] \mathrm{d}s + \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} [\delta_1 \|x\|^{q_1} + \delta_2] \mathrm{d}s + \\
& (\frac{t}{T} - 1) \int_0^T [\delta_3 \|x\|^{q_2} + \delta_4] \mathrm{d}s + \frac{t}{T} \int_0^T [\delta_5 \|x\|^{q_3} + \delta_6] \mathrm{d}s \\
\leq & \frac{[\delta_1 r^{q_1} + \delta_2]}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathrm{d}s + \frac{t[\delta_1 r^{q_1} + \delta_2]}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \mathrm{d}s + \\
& (\frac{t}{T} - 1)[\delta_3 r^{q_2} + \delta_4] \int_0^T \mathrm{d}s + \frac{t}{T} [\delta_5 r^{q_3} + \delta_6] \int_0^T \mathrm{d}s \\
\leq & \frac{[\delta_1 r^{q_1} + \delta_2]}{\Gamma(q)} (\frac{t^q}{q}) + \frac{t[\delta_1 r^{q_1} + \delta_2]}{T\Gamma(q)} (\frac{T^q}{q}) + (\frac{t}{T} - 1)[\delta_3 r^{q_2} + \delta_4] T + \frac{t}{T} [\delta_5 r^{q_3} + \delta_6] T \\
\leq & \frac{t^q [\delta_1 r^{q_1} + \delta_2]}{\Gamma(q+1)} + \frac{t T^{q-1} [\delta_1 r^{q_1} + \delta_2]}{\Gamma(q+1)} + (t - T)[\delta_3 r^{q_2} + \delta_4] + t[\delta_5 r^{q_3} + \delta_6] \\
\leq & \frac{t(t^{q-1} + T^{q-1})[\delta_1 r^{q_1} + \delta_2]}{\Gamma(q+1)} + (t - T)[\delta_3 r^{q_2} + \delta_4] + t[\delta_5 r^{q_3} + \delta_6] := k.
\end{aligned}$$

Thus, $\mathcal{F}$ map bounded sets into bounded sets in $\mathcal{C}(\mathcal{J},\mathcal{X})$.

Step 3. Prove $\mathcal{F}(\mathcal{B}_r)$ is equicontinuous. For $t_1,t_2\in\mathcal{J}$ and $0\le t_1\le t_2\le 1$, let $x\in\mathcal{B}_r$, then,

$$\begin{aligned}
\|(\mathcal{F}x)(t_1)-(\mathcal{F}x)(t_2)\| \le{}& \|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}\xi(s,x(s))\mathrm{d}s-\frac{t_1}{T\Gamma(q)}\int_0^T(T-s)^{q-1}\xi(s,x(s))\mathrm{d}s- \\
&\frac{t_1}{T}\int_0^T\eta(s,x(s))\mathrm{d}s+\frac{t_1}{T}\int_0^T\zeta(s,x(s))\mathrm{d}s-\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}\xi(s,x(s))\mathrm{d}s+ \\
&\frac{t_2}{T\Gamma(q)}\int_0^T(T-s)^{q-1}\xi(s,x(s))\mathrm{d}s+\frac{t_2}{T}\int_0^T\eta(s,x(s))\mathrm{d}s-\frac{t_2}{T}\int_0^T\zeta(s,x(s))\mathrm{d}s\| \\
\le{}& \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]\|\xi(s,x(s))\|\mathrm{d}s+\frac{|t_1-t_2|}{T\Gamma(q)}\int_0^T(T-s)^{q-1}\|\xi(s,x(s))\|\mathrm{d}s+ \\
&\frac{|t_1-t_2|}{T}\int_0^T\|\eta(s,x(s))\|\mathrm{d}s+\frac{|t_1-t_2|}{T}\int_0^T\|\zeta(s,x(s))\|\mathrm{d}s+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|\xi(s,x(s))\|\mathrm{d}s.
\end{aligned}$$

As $t_1\to t_2$, one can deduce $(\mathcal{F}x)(t_1)\to(\mathcal{F}x)(t_2)$ that means $\mathcal{F}(\mathcal{B}_r)$ is equicontinuous.

As consequence of Steps 1 to 3 together with the Arzela Ascoli theorem, one can get $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is completely continuous.

Step 4. Consider the following set of solutions of the system (1.1)

$$\mathcal{S}=\{x\in\mathcal{C}(\mathcal{J},\mathcal{X}):\ \text{there exists }\lambda\in[0,1]\text{ such that }x=\lambda\mathcal{F}x\}.$$

We shall prove that $\mathcal{S}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. For $x\in S$ and $\lambda\in[0,1]$, we have

$$\begin{aligned}
\|x(t)\|=\lambda\|\mathcal{F}x(t)\|\le{}& \|\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\xi(s,x(s))\mathrm{d}s-\frac{t}{T\Gamma(q)}\int_0^T(T-s)^{q-1}\xi(s,x(s))\mathrm{d}s- \\
&(\frac{t}{T}-1)\int_0^T\eta(s,x(s))\mathrm{d}s+\frac{t}{T}\int_0^T\zeta(s,x(s))\mathrm{d}s\| \\
\le{}& \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|\xi(s,x(s))\|\mathrm{d}s+\frac{t}{T\Gamma(q)}\int_0^T(T-s)^{q-1}\|\xi(s,x(s))\|\mathrm{d}s+ \\
&(\frac{t}{T}-1)\int_0^T\|\eta(s,x(s))\|\mathrm{d}s+\frac{t}{T}\int_0^T\|\zeta(s,x(s))\|\mathrm{d}s \\
\le{}& \frac{t(t^{q-1}+T^{q-1})[\delta_1r^{q_1}+\delta_2]}{\Gamma(q+1)}+(t-T)[\delta_3r^{q_2}+\delta_4]+t[\delta_5r^{q_3}+\delta_6].
\end{aligned}$$

The above inequality together with $q_1,q_2,q_3\in[0,1)$ and Step 2 show that $\mathcal{S}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. As a consequence of Schaefer's fixed point theorem, one can conclude that $\mathcal{F}$ has a fixed point which is the solution of the BVP (1.1). □

Remark 3.5 If the growth conditions on ξ,η and ζ include linear growth case, then the set of solutions of the system (1.1) is conex.

Lemma 3.6 *The operator $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is compact. Consequently, $\mathcal{F}$ is α-Lipschitz with zero constant.*

Proof Consider a closed subset $\mathcal{M}\subseteq\mathcal{C}(\mathcal{J},\mathcal{X})$. As we prove in Theorem 3.4, $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is continuous and completely continuous, then by applying the Arzela Ascoli Theorem $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ implies $\mathcal{F}(\mathcal{M})$ is a relatively compact subset of $\mathcal{C}(\mathcal{J},\mathcal{X})$. Therefore,

$\mathcal{F} : \mathcal{C}(\mathcal{J},\mathcal{X}) \to \mathcal{C}(\mathcal{J},\mathcal{X})$ is compact. Consequently, by Proposition 2.7, $\mathcal{F}$ is α-Lipschitz with zero constant. $\square$

Example 3.7 Let us consider the following fractional BVP

$$\begin{cases} {}^c\mathcal{D}^{\frac{2}{3}}x(t) = \frac{e^{-t}|x(t)|}{(1+2e^t)(1+|x(t)|)} & t \in \mathcal{J} := [0,1],\ 0 < q < 1, \\ x(0) = \sum_{i=0}^{\infty} c_i x(t_i), & x(1) = \sum_{j=0}^{\infty} d_j x(t_j), \end{cases} \tag{3.2}$$

where $0 < t_0 < t_1 < \cdots < 1, c_i, d_j, i, j = 0, \ldots,$ are given positive constants with $\sum_{i=0}^{\infty} c_i x(t_i) < \infty$, $\sum_{j=0}^{\infty} d_j x(t_j) < \infty$ and $\sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j = \frac{3}{17}$.

Set $q = \frac{2}{3}$, for $(t,x) \in [0,1] \times [0,+\infty)$, we can define $\xi(t,x) = \frac{e^{-t}x}{(1+2e^t)(1+x)}$. Also, for $t \in [0,1]$ we have $x(t) = \frac{e^{-t}}{(1+2e^t)}$. For $x, y \in [0,+\infty)$, we have

$$\begin{aligned} |\xi(t,x) - \xi(t,y)| =& |\frac{e^{-t}x}{(1+2e^t)(1+x)} - \frac{e^{-t}y}{(1+2e^t)(1+y)}|, \quad t \in [0,1] \\ \le& \frac{1}{3}|\frac{x-y}{(1+x)(1+y)}| \le \frac{1}{3}|x-y| \Rightarrow \delta_\xi = \frac{1}{3}. \end{aligned}$$

Next, we shall check that (H3) and (H4) are satisfied with $T = 1$, then one can get $\delta_\eta = \sum_{i=0}^{\infty} c_i$ and $\delta_\zeta = \sum_{j=0}^{\infty} d_j$. Indeed,

$$\frac{2\delta_\xi T^q}{\Gamma(q+1)} + T(\delta_\eta + \delta_\zeta) = \frac{\frac{2}{3}}{\Gamma(\frac{5}{3})} + \frac{3}{17} < 1$$

which is satisfied for $q \in (0,1)$. Then by Theorem 3.3 the problem (3.2) has a unique solution on $[0,1]$.

4. Conclusion

In this article, we investigated some sufficient conditions for existence and uniqueness of solutions for the fractional boundary value problems with integral boundary conditions. Banach's contraction mapping principle and Schaefer's fixed point theorem have been utilized besides applying topological technique of approximate solutions. Finally, the given example confirmed our results.

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