Journal of Mathematical Research with Applications
Mar., 2022, Vol. 42, No. 2, pp. 153–161
DOI:10.3770/j.issn:2095-2651.2022.02.006
Http://jmre.dlut.edu.cn

Finite Groups Whose Norm Quotient Groups Have Cyclic Sylow Subgroups

Songliang CHEN[1], **Yun FAN**[2,*]
1. *School of Mathematics and Big Data, Guizhou Education University, Guizhou* 550018*, P. R. China;*
2. *School of Mathematics and Statistics, Central China Normal University, Hubei* 430079*, P. R. China*

Abstract Let G be a finite group and $N(G)$ be its norm. Then $N(G)$ is a characteristic subgroup of G which normalizes every subgroup of G. In this paper, we will study the structure of G under one of the following conditions: 1) norm quotient group $G/N(G)$ is cyclic; 2) all Sylow subgroups of $G/N(G)$ are cyclic and in particular if the order of $G/N(G)$ is a square-free number.

Keywords norm; Dedekind group; Hamiltonian group; π-group; structure of finite group; Sylow subgroup

MR(2020) Subject Classification 20D20; 20D25; 20E34

1. Introduction

Let G be a group. Then the norm of G, denoted by $N(G)$, consists of all those elements of G which normalize every subgroup of the group. Clearly, $N(G)$ is a characteristic subgroup of G and G is a Dedekind group if and only if $G = N(G)$; and it is well-known that the center, denoted by $Z(G)$, of G is contained in the norm $N(G)$. This concept was introduced in 1935 by Baer [1,2], who gave the basic properties of the norm such as: $N(G) = N(A) \times N(B)$ whenever $G = A \times B$ and $(|A|, |B|) = 1$ (see [1, Satz 4]); G is a 2-group with Hamiltonian norm, then G is Hamiltonian [2]; G is a group with $Z(G) = 1$, then $N(G) = 1$ (see [3]). In [4], let G be a finite p-group and $N < G$, Baer described the automorphism of N which is induced by an element $g \in G - N$, provided $N \lhd \langle N, g\rangle$ and $N \le N(\langle N, g\rangle) < \langle N, g\rangle$; and so on. Afterwards, many authors have investigated intermittently the property of $N(G)$ and the influence of $N(G)$ on the structure of G. For instance, Schenkman first gave the result that the norm $N(G)$ is contained in its second center $Z_2(G)$, but his proof is incorrect [5], and soon Kappe [6] and Cooper [7] gave independently the correct proof, respectively. In 2004, Beidleman, Heineken and Newell showed that at least one of the following two statements is true: (A) $N(G)/Z(G)$ is cyclic; (B) $[G, N(G)]$ is cyclic, whenever G is a finite p-group for every prime p (see [8]). In 2007, Wang and Guo proved that G is p-nilpotent if $\Omega_1(P) \le N(N_G(P))$, and when $p = 2$, $\Omega_2(P) = \langle \Omega_1(P),\ x | x \in P$ is quasi-central in $N_G(P)$ and $|x| = 4\rangle$ where $p \mid |G|$, and they determined the structure of finite groups

Received February 5, 2021; Accepted October 16, 2021
Supported by the National Natural Science Foundation of China (Grant No. 11661023) and the Service Industry Development Guide of Guizhou Province (Grant No. QianFaGaiFuWu[2018]1181).
* Corresponding author
E-mail address: chsl_2013@aliyun.com (Songliang CHEN); yfan@mail.ccnu.edu.cn (Yun FAN)

whose norm quotient $G/N(G)$ is a cyclic group of prime order p (see [9]). In 2009, Wang and Guo determined the structure of a finite group with norm quotient group of order pq, where p, q are two distinct primes [10, 11]. In 2016, Wang determined the structure of finite groups with a cyclic norm quotient [12]. In this paper, we will determine again the structure of the finite groups with a cyclic norm quotient (Theorem 2.4 and Corollary 2.6) by a new proof which is different from the Wang's proof in [12]. Additionally, we will determine the structure of finite groups whose norm quotient $G/N(G)$ is a finite group with cyclic Sylow subgroups (Theorem 3.1) and in particular of square-free-order (Remark 3.3).

Throughout this paper, all terminology and notations employed agree with standard usage, as in Robinson [13]. For example, we let p be a prime, P a finite p-group and A_p an arbitrary abelian p-group. Recall that $N(P)$, $Z(P)$ is the norm and center of P, respectively; and $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of all positive integers and $|g|$ denotes the order of element g in an arbitrary group; for a positive integer n and a prime p, n_p is the p-part of n, $\pi(G)$ is the set of prime factors of $|G|$, etc.

2. Cyclic norm quotient group

In order to study the structure of finite groups G with cyclic norm quotient, we consider the case that G are p-groups first.

Lemma 2.1 *Let P be a finite p-group and suppose that $P/N(P)$ is nontrivial. Then $N(P)$ is abelian.*

Proof See [2] or [7], and [13, Theorem 5.3.7]. □

Lemma 2.2 *Let π be a nonempty set of primes. Then for an arbitrary group G, $G/N(G)$ is a π-group if and only if either G is a π-group or $G = H \times D$ where H is a π-group with $G/N(G) \cong H/N(H)$ and D is a Dedekind π'-group.*

Proof It suffices to show that G is not a π-group. If $G/N(G)$ is a π-group, then Hall π'-subgroup D of $N(G)$ is a Dedekind π'-group. Clearly, D char $N(G) \trianglelefteq G$, hence there exists a complement H of D in G such that $G = HD$ by Schur-Zassenhaus Theorem. Clearly, $H \trianglelefteq G$ since $D \leq N(G)$. Then $G = H \times D$, and thus $N(G) = N(H) \times D$ by [1, Satz 4]. It follows that $G/N(G) \cong H/N(H)$. The converse is trivial. □

By Lemma 2.2, in order to determine the structure of a finite group G whose norm quotient group $G/N(G)$ is a finite π-group, we need only to consider the group G with $\pi(G) = \pi(G/N(G))$.

Lemma 2.3 *Suppose that element g and subgroup N of the finite p-group G satisfy:*

(a) *N is a normal subgroup of $\langle N, g\rangle$;*

(b) *$N \leq N(\langle N, g\rangle) < \langle N, g\rangle$.*

Then

(i) *$n(x) \leq m(g) = n(g) - a(g)$ for every x in N and $1 < m(g)$ in case $p = 2$;*

(ii) *$N(\langle N, g\rangle) = \langle N, g^{p^{a(g)}}\rangle$,* *where $n(x) = \log_p |x|$, $n(g) = \log_p |g|$ and $p^{a(g)}$ is the order of the automorphism of N induced by g.*

Proof See Satz 7, §5 of [1] or (1.1) and (1.2) of [4]. □

Theorem 2.4 *Let P be a finite p-group and $P/N(P)$ be a cyclic group of order p^a, $a \in \mathbb{N}$. Then $P = N(P)\langle x\rangle$, and P can be written as a semidirect product of a cyclic group $\langle x\rangle$ of order p^m by an abelian p-group T, where $T = \langle A_p, g\rangle$, $[x, g] = x^{p^{m-a}}$, $\exp T \le p^{m-a}$, $A_p \le Z(P)$, $x^{p^a} \in Z(P)$, $p^a \le |g| \le p^{m-a}$, $m \ge 2a$ if $p > 2$ or $a > 1$ and $m \ge 2a + 1$ if $p = 2$ and $a = 1$.*

Proof We may assume that $P = \langle x,\ N(P)\rangle$, then $P = N(P)\langle x\rangle$ since $N(P) \trianglelefteq P$ and so $\langle x\rangle \trianglelefteq P$. Clearly, $x^{p^a} \in N(P)$. Since $N(P)$ is abelian by Lemma 2.1, we have that $x^{p^a} \in Z(P)$. Let $|x| = p^m$ and suppose that x has the largest order among the elements of $P - N(P)$. We observe that $x \notin Z(P)$ and $\langle x\rangle \cap Z(P) \ne 1$ by nilpotence of P. So $m > a$.

We have that $Z(P) < N(P)$ because $P/N(P)$ is cyclic and P is non-abelian. Now, for every element $g \in N(P) - Z(P)$, there exists a positive integer t to be coprime with p such that $g^{-1}xg = x^t$, then we have $x^{p^a} = (g^{-1}xg)^{p^a} = x^{tp^a}$ as $x^{p^a} \in Z(P)$, hence $tp^a \equiv p^a \pmod{p^m}$, i.e., $t \equiv 1 \pmod{p^{m-a}}$. It follows that there exists an integer $r \in \mathbb{Z}$ such that $t = 1 + rp^{m-a}$, then $g^{-p^a}xg^{p^a} = x^{t^{p^a}} = x^{(1+rp^{m-a})^{p^a}} = x$, so $g^{p^a} \in Z(P)$, i.e., $\exp\{N(P)/Z(P)\} \le p^a$. Clearly, the number of the automorphisms of $\langle x\rangle$ whose form is

$$x \longmapsto x^{1+rp^{m-a}}$$

is exactly p^a (where $r = 0, 1, 2, \ldots, p^a - 1$), and it is easy to see that $N(P)/Z(P)$ faithfully acts on $\langle x\rangle$ by conjugate action, therefore, $|N(P)/Z(P)| \le p^a$. If, for every $g \in N(P) - Z(P)$ we have that $p|r$, then $g^{-1}x^{p^{a-1}}g = (g^{-1}xg)^{p^{a-1}} = x^{(p^{a-1}+rp^{m-1})} = x^{p^{a-1}}$. Hence, $x^{p^{a-1}} \in Z(P) < N(P)$, in contradiction to $|P/(N(P)| = p^a$. Thus there is element $g \in N(P) - Z(P)$ such that $g^{-1}xg = x^{1+rp^{m-a}}$ and $(p, r) = 1$. Clearly, we have $[x, g] = x^{rp^{m-a}} \in Z(P)$ by $N(P) \le Z_2(P)$, so $x^{p^{m-a}} \in Z(P)$. Since $x^{p^{a-1}} \notin Z(P)$ while $1 \ne x^{p^a} \in Z(P)$, we have that $m - a \ge a$, i.e., $m \ge 2a$. Now, we can prove that $N(P)/Z(P)$ is a cyclic group of order p^a generated by $gZ(P)$. If $a = 1$, this is clear. Suppose that $a > 1$, noting that $m \ge 2a > 2$, we have that $g^{-p^{a-1}}xg^{p^{a-1}} = x^{1+rp^{m-1}}$ for every prime p, so the order of automorphism of $\langle x\rangle$ induced by g is p^a, and it follows that $|g| \ge p^a$, as required.

Now, we have $N(P) = \langle g, Z(P)\rangle$ and $g^{-1}xg = x^{1+rp^{m-a}}$, $(p, r) = 1$. Now, there exists $r' \in \mathbb{Z}$ such that $rr' \equiv 1 \pmod{p^a}$, thus we can assume that $g^{-1}xg = x^{1+p^{m-a}}$ by replacing g by $g^{r'}$. It is clear that x induces an automorphism of order of p^a in $N(P)$ by Lemma 2.3, we have that $\exp N(P) \le p^{m-a}$, where $m \ge 2a$ if $p > 2$ or $a > 1$ and $m \ge 2a + 1$ if $p = 2$ and $a = 1$.

It is easy to see that x^{p^a} is an element of maximal order in $Z(P)$, then there exists a subgroup A_p of $Z(P)$ by [13, Theorem 4.2.7] such that $Z(P) = \langle x^{p^a}\rangle \times A_p$, $\exp A_p \le p^{m-a}$. We can assume that $g^{p^a} = x^{kp^{2a}}b$ as $g^{p^a} \in Z(P)$ and $|g| \le p^{m-a}$, where $b \in A_p$, $|b| \le p^{m-2a}$, $k \in \mathbb{Z}$. Let $g_1 = gx^{-kp^a}$. Then $g_1 \in N(P)$, $g_1^{-1}xg_1 = g^{-1}xg = x^{1+p^{m-a}}$, $|g_1| = |g| \le p^{m-a}$,

$g_1^{p^a} = b \in A_p$. Moreover, we have

$$P = (\langle x\rangle \times A_p)\langle g\rangle = \langle x\rangle(A_p\langle g_1\rangle),$$

$$|P| = p^{2a}|\langle x^{p^a}\rangle| \cdot |A_p| = p^{m+a}|A_p|, \quad |A_p\langle g_1\rangle| = p^a|A_p|,$$

hence $\langle x\rangle \cap A_p\langle g_1\rangle = 1$, and thus $P = \langle x\rangle \rtimes A_p\langle g_1\rangle$. Set $T = A_p\langle g_1\rangle$, clearly, T is an abelian group. Finally, by replacing g by g_1, we have $P = \langle x\rangle \rtimes T$, as required. □

Example 2.5 In the Theorem 2.4, the signs of equality in the inequalities among the conditions can be attained. If $p > 2$, $a = 1$ and $m = 2$, we have example: $P = \langle x, g | x^{p^2} = 1 = g^p, x^g = x^{1+p}\rangle$; If $p = 2$ and $a = 2 > 1$, $m = 4$, we have an example: $P = \langle x, g | x^{16} = 1 = g^4, x^g = x^5\rangle$; If $p = 2$, $a = 1$ and $m = 3$, we have example: $P = \langle x, g | x^8 = 1 = g^2, x^g = x^5\rangle$.

Corollary 2.6 *Let $\pi = \{p_1, p_2, \ldots, p_k\}$, where $p_i, 1 \le i \le k, k \in \mathbb{N}$, are distinct primes, G be a finite group such that $G/N(G)$ is a cyclic π-group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then,*

$$G = P_1 \times P_2 \times \cdots \times P_k \times H$$

where P_i is the Sylow p_i-subgroup of G with cyclic norm quotient group of order $p_i^{\alpha_i}$, $1 \le i \le k$, and H is a Dedekind π'-group.

Proof In the first place G is a finite nilpotent group by $N(G) \le Z_2(G)$ (see [5–7]) and commutativity of $G/N(G)$, then $G = P_1 \times P_2 \times \cdots \times P_k \times H$, where $P_i \in \mathrm{Syl}_{p_i}(G)$, and H is a Dedekind π'-group by Lemma 2.2. Moreover, $G/N(G) = P_1/N(P_1) \times P_2/N(P_2) \times \cdots \times P_k/N(P_k)$ by [1, Satz 4], then $P_i/N(P_i)$ is cyclic with order $p_i^{\alpha_i}$, $1 \le i \le k$. Therefore, the structure of G is completely determined by Theorem 2.4. □

3. Norm quotient group with cyclic Sylow subgroups

By Lemma 2.2, if G is a finite group whose norm quotient group $G/N(G)$ is a finite π-group with cyclic Sylow subgroups, we have $G = H \times D$, where D is a Dedekind π'-group and H is a finite π-group with $G/N(G) \cong H/N(H)$. In order to determine the structure of a finite group G, we need only to determine the structure of H.

Theorem 3.1 *Let H be a finite group such that every Sylow subgroup of H cannot be direct product factor of H and $\pi(H) = \pi(\overline{H})$, where $\overline{H} = H/N(H)$. Write $\pi = \pi(\overline{H})$, $\pi_1 = \pi(\overline{H}')$ and $\pi_2 = \pi - \pi_1$. Then $\overline{H}$ is a group with cyclic Sylow subgroups if and only if $H = R \rtimes S$ is a semidirect product of the Hall π_1-subgroups R and the Hall π_2-subgroups S, where $R = A \times \langle a\rangle$ is abelian of odd order and $|a| = |\overline{H}'|$, S is nilpotent with cyclic norm quotient, $[A, S] = 1$, and $\langle a\rangle$ is normalized by S. Moreover, $2 \notin \pi_1$, for any $q \in \pi_2 = \pi(S)$ and $S_q \in \mathrm{Syl}_q(S)$, S_q is one of the following types:*

(i) $S_q = B \times \langle b_q\rangle$ is abelian, where $[a, B] = 1$, $[a, b_q] \ne 1$ and $|\langle b_q\rangle / C_{\langle b_q\rangle}(a)| = |\overline{H}|_q$;

(ii) $q = 2$ and $S_2 = E_{2^\beta} \times Q_8$, where $[a, E_{2^\beta}] = 1$ and $|Q_8/C_{Q_8}(a)| = 2 = |\overline{H}|_2$;

(iii) $q = 2$ and $S_2 = E_{2^\beta} \times Q_8 \times \langle b_2\rangle$, where $[a, E_{2^\beta} \times Q_8] = 1$, $[a, b_2] \ne 1$ and $|b_2| = 2 = |\overline{H}|_2$;

(iv) $S_q = \langle b_q\rangle \rtimes T_q$ *and* $q \le |S_q/N(S_q)| \le |\overline{H}|_q$, *where* $N(S_q) \ge S_q \cap N(H) = \langle b_q^{|\overline{H}|_q}, T_q\rangle \le C_{S_q}(a)$, $|S_q/N(S_q)| = q^{\alpha}$, $|b_q| \ge \max(q^{2\alpha}, |\overline{H}|_q)$ *if* $q > 2$ *or* $q = 2$ *and* $\alpha > 1$, *and* $|b_q| \ge \max(8, |\overline{H}|_2)$ *if* $q = 2$ *and* $\alpha = 1$, T_q *is abelian with* $\exp T_q \le |b_q|/q^{\alpha}$, *and* $S_q/C_{S_q}(a)$ *is a cyclic group with* $q \le |S_q/C_{S_q}(a)| \le |\overline{H}|_q$.

Proof Suppose that $\overline{H} = H/N(H)$ is a finite group with cyclic Sylow subgroups, by Hölder, Burnside, Zassenhaus Theorem in [13, Theorem 10.1.10], we have $a, b \in H$ such that

$$\overline{H} = \langle \bar{a}, \bar{b} | \bar{a}^m = 1 = \bar{b}^n, \bar{b}^{-1}\bar{a}\bar{b} = \bar{a}^r\rangle$$

where $r^n \equiv 1 \pmod m$, m is odd, $0 \le r < m$, and m and $n(r-1)$ are coprime. Since H has no direct factor which is a Sylow subgroup, we know $r \ne 0$ or 1, so $|\overline{H}'| = m$ and $|\overline{H}/\overline{H}'| = n$, hence π_1 and π_2 are the sets of prime factors of m, n, respectively. Clearly, $2 \notin \pi_1$ since m is odd.

Clearly, H is solvable, let R, S be the Hall π_1-subgroup and Hall π_2-subgroup of H, respectively, then

$$\overline{R} = RN(H)/N(H) \cong \langle \bar{a}\rangle, \quad \overline{S} = SN(H)/N(H) \cong \langle \bar{b}\rangle$$

are the Hall π_1-subgroup and Hall π_2-subgroup of $\overline{H}$, respectively. Noting that $\langle \bar{a}\rangle \lhd \overline{H}$, we have $RN(H) \trianglelefteq H$. Furthermore R char $RN(H)$ and so $R \trianglelefteq H$. Therefore,

$$R \cap N(H) \lhd H \quad \text{and} \quad R/R \cap N(H) \cong RN(H)/N(H) \cong \langle \bar{a}\rangle.$$

Clearly, $R \cap N(H) \le N(R)$, then $|R/N(R)| \le m = |\bar{a}|$.

Similarly, since $S \cap N(H)$ is the normal Hall π_2-subgroup of $N(H)$, we have

$$S \cap N(H) \text{ char } N(H) \trianglelefteq H,$$

hence

$$S \cap N(H) \trianglelefteq H \quad \text{and} \quad S/S \cap N(H) \cong SN(H)/N(H) \cong \langle \bar{b}\rangle.$$

Thus, by $\pi_1 \cap \pi_2 = \emptyset$, we have

$$[S \cap N(H), R] = 1 = [S, R \cap N(H)]. \tag{3.1}$$

It follows that the order of the automorphism group of R induced by S is a divisor of $|S/S \cap N(H)| = n$. Now, $R/R\cap N(H) \cong \langle \bar{a}\rangle$ and $R\cap N(H) \le N(R)$, then $R/N(R)$ is a cyclic group, thus R is nilpotent. It follows that every Sylow subgroup P of R is normal in H, and $P/P\cap N(H) \cong \overline{P}$ is a cyclic group of the order m_p, where p is an odd prime and $p \in \pi_1$, so $P/N(P)$ is also a cyclic group. We conclude that P is abelian, hence R is abelian of odd order. Otherwise, we can assume that $\overline{P} = \langle \bar{x}\rangle$, i.e., $P/P \cap N(H) = \langle \bar{x}\rangle$ for some $x \in P$, then $P/N(P) = \langle \bar{x}\rangle$, so $P = \langle x\rangle \rtimes T$ by Theorem 2.4, where T is an abelian p-group and $T \le N(P)$. If $\langle x\rangle \cap (P \cap N(H)) = 1$, we have $P = \langle x\rangle \times (P \cap N(H))$, then P is abelian. If $\langle x\rangle \cap (P \cap N(H)) \ne 1$, we can show that P is also abelian. In fact, we may set $|P/P \cap N(H)| = p^{\alpha}$ and $P_k = \langle x^{p^{\alpha-k}}, P \cap N(H)\rangle$, $k = 1, 2, \ldots, \alpha$, then $P_k/P \cap N(H) = \langle \bar{x}^{p^{\alpha-k}}\rangle$. Clearly, $\langle \bar{x}^{p^{\alpha-k}}\rangle$ char $\langle \bar{x}\rangle$ char $\langle \overline{H}\rangle$, so $P_k \lhd H$. Now, $P_1 = \langle x^{p^{\alpha-1}}, P \cap N(H)\rangle$, $|x^{p^{\alpha}}| \ge p$ and $x^{p^{\alpha}} \in P \cap N(H)$. If $P_1 \le N(P)$, P_1 is abelian, then

$|hx^{p^{\alpha-1}}| \geq p^2$ for every element h in $P \cap N(H)$, hence $P \cap N(H)$ contains every element of order p in P_1. If $P_1 \nleq N(P)$, we have $N(P) = P \cap N(H)$ and $P/N(P)$ is a cyclic group of order p^α. By Theorem 2.4, we can assume that $|x| = p^u \geq p^{2\alpha}$, $\exp N(P) \leq p^{u-\alpha}$, $P = T\langle x\rangle$, $T = \langle A_p, g\rangle$, A_p is an abelian p-group and $[x, A_p] = 1$, $[x, g] = x^{p^{u-\alpha}} \in Z(P)$. Now, $\forall h \in A_p$ and $\forall t \in \mathbb{N}$, $(hg^t x^{p^{\alpha-1}})^p = h^p g^{tp} x^{p^\alpha} [x^{p^{\alpha-1}}, g^t]^{\frac{p(p-1)}{2}} = h^p g^{tp} x^{p^\alpha} \neq 1$, so $P \cap N(H)$ contains every element of order p in P_1. We have showed that $\Omega(P_1) \leq P \cap N(H) \leq R \cap N(H)$, but $[\Omega(P_1), S] = 1$ by (3.1), hence P_1 is centralized by S in [14, IV.5.12]. Clearly, P_1 contains every element of the order p in P_2 as P_2/P_1 is a cyclic group of order p, i.e., $\Omega(P_2) \leq P_1$, but $[P_1, S] = 1$, hence P_2 is centralized by S in [14, IV.5.12] again. We repeat the process above, and finally, we gain that $P = P_\alpha$ is centralized by S. Consequently, $\overline{P} \leq Z(\overline{H})$. On the other hand, it is easy to show that $\overline{H}' = \langle \overline{a}\rangle$, then $\overline{P} \leq \langle \overline{a}\rangle \cap Z(\overline{H}) = 1$ by Taunt Theorem [13, Theorem 10.1.7], a contradiction. In a word, P is abelian. Noting that the action of S on R is coprime, by [15, Theorem 8.4.2], we have that

$$R = C_R(S) \times [R, S].$$

Clearly, $C_R(S) \geq R \cap N(H)$, and $[R, S] \cap N(H) \subseteq R \cap N(H) \subseteq C_R(S)$, then $[R, S] \cap N(H) \subseteq C_R(S) \cap [R, S] = 1$, thus $[R, S] \cong [R, S]N(H)/N(H) = [\overline{R}, \overline{S}] = \langle \overline{a}\rangle$. Recall that $R/R \cap N(H) \cong \langle \overline{a}\rangle$, so we have $C_R(S) = R \cap N(H)$ is an abelian π_1-group and $[R, S] \cong R/C_R(S)$ is a cyclic group of order $|\langle \overline{a}\rangle|$. Without loss of generality, we can assume that $[R, S] = \langle a\rangle$. We know that $H = R \rtimes S$ by $R \lhd H$, hence

$$H = A \times (\langle a\rangle \rtimes S) \tag{3.2}$$

where $A = C_R(S)$ is an abelian π_1-group, $|a| = m$, it is easy to check that S is nilpotent.

It remains only to investigate S. To achieve this, we consider the Sylow q-subgroup S_q of S for arbitrary prime q in π_2. Clearly, S_q is not normal in H.

Case 1. $S_q = N(S_q)$.

Subcase 1.1. S_q is abelian.

Let $S_1 = S \cap N(H)$. Then $|S_q/(S_1 \cap S_q)| = n_q$. Put $b_q = b^{|b|/|b|_q}$, the q-component of b, then $b_q \in S_q$ and $\overline{S_q} = \langle \overline{b_q}\rangle$. Now, $S_q/(S_1 \cap S_q) = \langle \overline{b_q}\rangle$ is a cyclic group of order n_q, then $S_q/N(S_q)$ is a cyclic group by $(S_1 \cap S_q) \subseteq N(S_q)$, hence we can assume that b_q is an element of maximal order in S_q by $S_q = \langle b_q, N(S_q)\rangle$ and Theorem 2.4. We show that there exists a subgroup B such that $S_q = \langle b_q\rangle \times B$ and $B \subset (S_1 \cap S_q)$. In fact, there is a subgroup B which is maximal subject to $B \cap \langle b_q\rangle = 1$ and $B \subset (S_1 \cap S_q)$. If $S_q = \langle b_q\rangle B$, then $S_q = \langle b_q\rangle \times B$ and the proof is completed. Assume, therefore, that $S_q \neq \langle b_q\rangle B$ and let y be an element of minimal order in $S_q \backslash (\langle b_q\rangle B)$. By choice of y we have $y^q \in \langle b_q\rangle B$ and thus $y^q = z b_q^l$ where $z \in B$. Since b_q has maximal order q^k, we have $1{=}y^{q^k} = z^{q^{k-1}} b_q^{lq^{k-1}}$ and $b_q^{lq^{k-1}} \in B \cap \langle b_q\rangle = 1$. Consequently, q^k divides $q^{k-1}l$ and q divides l. Now write $l = qj$, so that $(yb_q^{-j})^q = z \in B$, while $yb_q^{-j} \notin B$ since $y \notin B\langle b_q\rangle$. From the maximality of B we know that $\langle yb_q^{-j}, B\rangle \cap \langle b_q\rangle \neq 1$, which implies that there exist integers u and v and an element z' of B such that $1 \neq b_q^u = (yb_q^{-j})^v z'$. Hence $y^v \in B\langle b_q\rangle$. Suppose that $q|v$, $(yb_q^{-j})^q \in B$ implies that $(yb_q^{-j})^v \in B$ and thus $b_q^u = 1$. Hence $(q, v) = 1$. However $y^q \in B\langle b_q\rangle$, so $y \in B\langle b_q\rangle$, a contradiction. Thus we have that $S_q = \langle b_q\rangle \times B$ and $B \leq S_1 \cap S_q$, i.e., $[a, B] = 1$.

It follows that

$$RS_q = C_R(S_q) \times (\langle a \rangle \rtimes (\langle b_q \rangle \times B)) = A \times B \times (\langle a \rangle \rtimes \langle b_q \rangle) \tag{3.3}$$

where A, B are some abelian π_1-group and q-group, respectively, and it is not difficult to prove that b_q induces an automorphism with order n_q on $\langle a \rangle$, so $[a, b_q] \neq 1$ and $|\langle b_q \rangle / C_{\langle b_q \rangle}(a)| = |\overline{H}|_q$. In a word, S_q satisfies condition (i).

Subcase 1.2. S_q is non-abelian.

Now, S_q is a non-abelian Dedekind group, hence $q = 2$, and $S_2 = Q_8 \times E_{2^\beta}$ is the direct product of quaternion group Q_8 and an elementary abelian 2-group E_{2^β}, $\beta \in \mathbb{N}$ (see [13, Theorem 5.3.7]).

(a) $E_{2^\beta} \leq S_2 \cap N(H)$.

Since $S_2/S_2 \cap N(H)$ is a cyclic 2-group, we have $|S_2/S_2 \cap N(H)| = n_2$ is either 2 or 4. If $n_2 = 2$, then we have $S_2 \cap N(H) = (Q_8 \times E_{2^\beta}) \cap N(H) = E_{2^\beta} \times (Q_8 \cap N(H)) = E_{2^\beta} \times (Q_8 \cap S_1)$, hence $|Q_8/Q_8 \cap S_1| = 2$. Therefore

$$RS_2 = A \times E_{2^\beta} \times (\langle a \rangle \rtimes Q_8) \tag{3.4}$$

where A is some abelian π_1-group and $Q_8/C_{Q_8}(a) \cong \langle \overline{b_2} \rangle$ is a group of order 2 where $C_{Q_8}(a) = Q_8 \cap S_1$. In a word, S_2 satisfies condition (ii).

If $n_2 = 4$, then $S_2/S_2 \cap N(H) \cong Q_8/Q_8 \cap S_1$ is a cyclic group of order 4. But Q_8 has no cyclic quotient group of order 4, a contradiction.

(b) $E_{2^\beta} \nleq S_2 \cap N(H)$.

Now, as $S_2/S_2 \cap N(H)$ has to be a cyclic group of order 2, $b_2 \in E_{2^\beta}$ such that $S_2 = \langle b_2 \rangle (S_2 \cap N(H)), E_{2^\beta} = \langle b_2 \rangle \times (E_{2^\beta} \cap N(H))$, where b_2 acts on $\langle a \rangle$ non-trivially, $[a, b_2] \neq 1$ and $|b_2| = 2$. If $Q_8 \leq S_2 \cap N(H)$, then

$$RS_2 = A \times B_2 \times (\langle a \rangle \rtimes \langle b_2 \rangle) \tag{3.5}$$

where $B_2 = E_{2^\beta} \cap N(H) \times Q_8$ is a non-abelian Dedekind 2-group. In a word, S_2 satisfies the condition (iii).

If $Q_8 \nleq S_2 \cap N(H)$, then $|Q_8/Q_8 \cap N(H)| = 2$, and thus there exists an element x of order 4 in Q_8 such that $[a, x] \neq 1$ and $[a, x^2] = 1$. Let $y \in Q_8 \cap N(H)$ and $|y| = 4$. Then $a^y = a$. Put $x^* = xb_2$, then $a^{x^*} = a$. Write $Q_8^* = \langle x^*, y \rangle$. Clearly, Q_8^* is also a quaternion group, and $Q_8 \times \langle b_2 \rangle = Q_8^* \times \langle b_2 \rangle$, thus $RS_2 = A \times B_2 \times (\langle a \rangle \rtimes \langle b_2 \rangle)$, where $B_2 = E_{2^\beta} \cap N(H) \times Q_8^*$. This is the same as (3.5).

Case 2. $S_q > N(S_q)$.

Now $S_q/N(S_q)$ and $S_q/(S_q \cap N(H))$ are cyclic q-groups. Since $S_q/(S_q \cap N(H)) \cong \langle \overline{b_q} \rangle$, we can assume that $S_q = \langle b_q \rangle \rtimes T_q$ by Theorem 2.4, where T_q is an abelian q-group and $T_q \leq N(S_q)$, $|S_q/N(S_q)| = q^\alpha$, $|S_q/(S_q \cap N(H))| = q^\beta$ and $|b_q| = q^k$, where $\alpha \leq \beta \leq k$, $\exp T_q \leq q^{k-\alpha}$, and $N(S_q) = \langle b_q^{q^\alpha}, T_q \rangle = \langle b_q^{q^\alpha} \rangle \times T_q$ is abelian. Clearly, $|\langle b_q^{q^\alpha}, S_q \cap N(H) \rangle| = |N(S_q)|$, then $N(S_q) = \langle b_q^{q^\alpha}, S_q \cap N(H) \rangle$ by $b_q^{q^\alpha} \in N(S_q)$ and $S_q \cap N(H) \subset N(S_q)$. Noting that $b_q^{q^\alpha}$ is an element of maximal order in $N(S_q) = \langle b_q^{q^\alpha} \rangle (S_q \cap N(H))$ by Theorem 2.4, we have a subgroup B in $S_q \cap N(H)$ such that $N(S_q) = \langle b_q^{q^\alpha} \rangle \times B$ by using the same proof in Subcase 1.1. Therefore

$S_q\cap N(H)=\langle b_q^{q^\beta}\rangle\times B$. Without loss of generality, we can replace T_q by B, then we have that $\langle b_q^{q^\beta}\rangle\times T_q\le C_{S_q}(a)$ by (3.1). Therefore,

$$RS_q=A\times(\langle a\rangle\rtimes(\langle b_q\rangle\rtimes T_q)) \tag{3.6}$$

where $N(S_q)\ge S_q\cap N(H)=\langle b_q^{|\overline{H}|_q},T_q\rangle\le C_{S_q}(a)$, $|S_q/N(S_q)|=q^\alpha$, $|b_q|\ge\max(q^{2\alpha},|\overline{H}|_q)$ if $q>2$ or $q=2$ and $\alpha>1$ and $|b_q|\ge\max(8,|\overline{H}|_2)$ if $q=2$ and $\alpha=1$, T_q is abelian with $\exp T_q\le|b_q|/q^\alpha$ by Theorem 2.4, and $S_q/C_{S_q}(a)$ is a cyclic group with $q\le|\langle b_q\rangle/C_{\langle b_q\rangle}(a)|\le|\overline{H}|_q=n_q$. In a word, S_q satisfies condition (iv).

Conversely assume that H has the given structure, then all Sylow subgroups of $H/N(H)$ are cyclic.

In fact, it is easy to see that $A\le Z(H)\le N(H)$ and $a\notin N(H)$, then $\overline{H}=H/N(H)\cong\langle\overline{a}\rangle\rtimes\overline{S}$. Hence every Sylow p-subgroup of $\overline{H}$ is cyclic for any $p\in\pi_1$, and it remains only to prove that every Sylow q-subgroup of $\overline{S}$ is cyclic for any $q\in\pi_2$. Since S is nilpotent, we have only to prove $\overline{S_q}$ is cyclic for any $q\in\pi_2$ and $S_q\in\mathrm{Syl}_q(S)$.

If S_q satisfies condition (i), $B\le Z(H)\le N(H)$ and $b_q\notin N(H)$, then $\overline{S_q}$ is isomorphic to a factor group of $S_q/B\cong\langle b_q\rangle$, hence $\overline{S_q}$ is cyclic.

If S_q satisfies condition (ii), then $E_{2^\beta}\times C_{Q_8}(a)$ is centralized by every Sylow p-subgroup of H whenever $p\ne q=2$. Now S_q is a Dedekind group, so $E_{2^\beta}\times C_{Q_8}(a)\le N(H)$, and $\overline{S_q}\cong Q_8/C_{Q_8}(a)$ is a cyclic group of order 2.

If S_q satisfies condition (iii), it is easy to see that $E_{2^\beta}\times Q_8\le N(H)$ similar to paragraph above, and $b_2\notin N(H)$, so $\overline{S_q}\cong\langle\overline{b_2}\rangle$ is a cyclic group of order 2.

If S_q satisfies condition (iv), T_q is centralized by every Sylow p-subgroup of H whenever $p\ne q$ by the hypothesis, then T_q normalizes every subgroup of the prime power order in H, hence $T_q\le N(H)$. And $b_q\notin N(H)$ clearly, thus $\overline{S_q}\cong S_q/S_q\cap N(H)$ is a cyclic group. □

Remark 3.2 Let H be a group whose norm quotient group has cyclic Sylow subgroups and $\pi(H)=\pi(H/N(H))$. Then some Sylow subgroups of H may be its direct product factors. Now, set M to be the direct product of all Sylow subgroups of H which are direct product factors, then M is a normal nilpotent Hall subgroup of H. Clearly, there exists a Hall complement K of M in H such that $H=K\times M$, hence $N(H)=N(K)\times N(M)$, and M has a cyclic norm quotient with $\pi(M)=\pi(M/N(M))$, so the structure of M is determined by Corollary 2.6, where K is a group with $\pi(K)=\pi(K/N(K))$ such that Sylow subgroups cannot be a direct product factor of K and $K/N(K)$ is a group with cyclic Sylow subgroups, so K possesses the structure described by Theorem 3.1. In a word, the structure of H can be determined.

Remark 3.3 Prominent among the groups whose norm quotient groups have cyclic Sylow subgroups are the groups whose norm quotient groups are of square-free order: such groups are therefore classified by Theorem 3.1, Corollary 2.6 and Lemma 2.2. Certainly, this is a generalization of Wang and Guo's result [11, Theorem 2.9].

Acknowledgements We thank the referees for their time and comments.

References

[1] R. BAER. *Der kern, eine charakteristische untergruppe*. Compositio Math., 1935, **1**(1): 254–283.

[2] R. BAER. *Gruppen mit Hamiltonschen kern*. Compositio Math., 1935, **1**(2): 241–246.

[3] R. BAER. *Norm and hypernorm*. Publ. Math. Debrecen., 1956, **4**: 347–350.

[4] R. BAER. *Groups with abelian norm quotient group*. Amer. J. Math., 1939, **61**: 700–708.

[5] E. SCHENKMAN. *On the norm of a group*. Illinois J. Math., 1960, **4**: 150–152.

[6] W. KAPPE. *Die a-norm eine gruppe*. Illinois J. Math., 1961, **5**: 187–197.

[7] C. D. H. COOPER. *Power automorphisms of a group*. Math. Z., 1968, **107**: 335–356.

[8] J. C. BEIDLEMAN, H. HEINEKEN, M. NEWELL. *Centre and norm*. Bull. Austral. Math. Soc., 2004, **69**: 457–464.

[9] Junxin WANG, Xiuyun GUO. *On the norm of finite groups*. Algebra Colloquium, 2007, **14**(4): 605–612.

[10] Xiuyun GUO, Junxin WANG. *On generalized Dedekind groups*. Acta Math. Hungar., 2009, **122**(1-2):37–44.

[11] Junxin WANG, Xiuyun GUO. *Finite groups with its power automorphism groups having small indices*. Acta Math. Sin. (Engl. Ser.), 2009, **25**(7): 1097–1108.

[12] Junxin WANG. *Finite groups with a cyclic norm quotient*. Bull. Korean Math. Soc., 2016, **53**(2): 479–486.

[13] D. J. S. ROBINSON. *A Course in the Theory of Groups*. Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[14] B. HUPPERT. *Endliche Gruppen I*. Springer-Verlag, Berlin, 1967.

[15] H. KURZWELL, B. STELLMACHER. *The Theory of Finite Groups*. Springer-Verlag Inc., New York, 2004.