Journal of Mathematical Research with Applications
Mar., 2022, Vol. 42, No. 2, pp. 189–198
DOI:10.3770/j.issn:2095-2651.2022.02.009
Http://jmre.dlut.edu.cn

Central Extensions and Deformations of Lie Triple Systems with a Derivation

Shuangjian GUO
School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou 550025*, P. R. China*

Abstract In this paper, we consider Lie triple systems with derivations. A pair consisting of a Lie triple system and a distinguished derivation is called a LietsDer pair. We define a cohomology theory for LietsDer pair with coefficients in a representation. We study central extensions of a LietsDer pair. In the next, we generalize the formal deformation theory to LietsDer pairs in which we deform both the Lie triple system bracket and the distinguished derivation. It is governed by the cohomology of LietsDer pair with coefficients in itself.

Keywords Lie triple system; derivation; representation; cohomology; central extension; deformation

MR(2020) Subject Classification 17A40; 17A36; 17B10; 17B56

1. Introduction

Lie triple systems arose initially in Cartan's study of Riemannian geometry. Jacobson [1] first introduced Lie triple systems and Jordan triple systems in connection with problems from Jordan theory and quantum mechanics, viewing Lie triple systems as subspaces of Lie algebras that are closed relative to the ternary product. Lister [2] investigated notions of the radical, semi-simplicity and solvability as defined for Lie triple systems, and determined all simple Lie triple systems over an algebraically closed field, the reader is referred to [3–11] and references cited therein.

Algebraic structures are useful via their derivations. Derivations are also useful in constructing homotopy Lie algebras [12], deformation formulas [13] and differential Galois theory [14]. They also play an essential role in control theory and gauge theories in quantum field theory [15]. In [16] the authors studied algebras with derivations from operadic point of view. Recently, Lie algebras with derivations (called LieDer pairs) are studied from cohomological point of view [17] and extensions, deformations of LieDer pairs are considered. The results of [17] have been extended to associative algebras and Leibniz algebras with derivations in [18] and [19].

The deformation is a tool to study a mathematical object by deforming it into a family of the same kind of objects depending on a certain parameter. The deformation theory was

Received December 29, 2020; Accepted April 28, 2021
Supported by the National Natural Science Foundation of China (Grant No. 12161013) and the General Project of Guizhou University of Finance and Economics (Grant No. 2021KYYB16).
E-mail address: shuangjianguo@126.com

introduced by Gerstenhaber for rings and algebras [20, 21], and by Kubo and Taniguchi for Lie triple systems [22]. They studied 1-parameter formal deformations and established the connection between the cohomology groups and infnitesimal deformations: the suitable cohomology groups for the deformation theory of associative algebras and Lie triple systems are the Hochschild cohomology [23] and the Yamaguti cohomology [24], respectively.

The paper is organized as follows. In Section 2, we define a cohomology theory for LietsDer pair with coefficients in a representation. In Section 3, we study central extensions of a LietsDer pair and show that isomorphic classes of central extensions are classified by the second cohomology of the LietsDer pair with coefficients in the trivial representation. In Section 4, we study formal one-parameter deformations of LietsDer pairs in which we deform both the Lie triple system bracket and the distinguished derivations.

Throughout this paper, we work over the field $\mathbb{F}$.

2. Cohomology of LietsDer pairs

In this section, we define a cohomology theory for LietsDer pair with coefficients in a representation.

Definition 2.1 ([2]) *A vector space T together with a trilinear map $(x, y, z) \mapsto [x, y, z]_T$ is called a Lie triple system if*

$$(T1) \qquad [x, x, y]_T = 0,$$

$$(T2) \qquad [x, y, z]_T + [y, z, x]_T + [z, x, y]_T = 0,$$

$$(T3) \qquad [u, v, [x, y, z]_T]_T = [[u, v, x]_T, y, z]_T + [x, [u, v, y]_T, z]_T + [x, y, [u, v, z]_T]_T,$$

for all $x, y, z, u, v \in T$.

Definition 2.2 ([24]) *Let $(T, [\cdot, \cdot, \cdot]_T)$ be a Lie triple system, M an $\mathbb{F}$-vector space. If $\theta : T \times T \to End(M)$ is a bilinear map such that for all $a, b, c, d \in T$,*

$$\theta(c, d)\theta(a, b) - \theta(b, d)\theta(a, c) - \theta(a, [b, c, d]_T) + D(b, c)\theta(a, d) = 0, \tag{2.1}$$

$$\theta(c, d)D(a, b) - D(a, b)\theta(c, d) + \theta([a, b, c]_T, d) + \theta(c, [a, b, d]_T) = 0, \tag{2.2}$$

where $D(a, b) = \theta(b, a) - \theta(a, b)$, then (M, θ) is called the representation of $(T, [\cdot, \cdot, \cdot]_T)$. $(M, 0)$ is called the trivial representation of $(T, [\cdot, \cdot, \cdot]_T)$.

Definition 2.3 ([25]) *Let $(T, [\cdot, \cdot, \cdot]_T)$ be a Lie triple system. A derivation on T is given by a linear map $\phi_T : T \to T$ satisfying*

$$\phi_T([x, y, z]_T) = [\phi_T(x), y, z]_T + [x, \phi_T(y), z]_T + [x, y, \phi_T(z)]_T, \quad \forall x, y, z \in T.$$

We call the pair (T, ϕ_T) composed of a Lie triple system and a derivation a LietsDer pair.

Example 2.4 Let $(T, [\cdot, \cdot])$ be a Lie algebra. We define $[\cdot, \cdot, \cdot]_T : T \times T \times T \to T$ and $\phi_T : T \to T$

by

$$[x,y,z]_T := [[x,y],z],$$

$$\phi_T([x,y,z]_T) = [\phi_T(x),y,z]_T + [x,\phi_T(y),z]_T + [x,y,\phi_T(z)]_T, \quad \forall x,y,z \in T.$$

Then (T,ϕ_T) becomes a LietsDer pair.

Definition 2.5 *Let (T,ϕ_T) be a LietsDer pair. A representation of it is given by (M,ϕ_M) in which M is a representation of T and $\phi_M : M \to M$ is a linear map satisfying*

$$\phi_M(\theta(x,y)(m)) = \theta(\phi_T(x),y)(m) + \theta(x,\phi_T(y))(m) + \theta(x,y)(\phi_M(m)),$$

$$\phi_M(D(x,y)(m)) = D(\phi_T(x),y)(m) + D(x,\phi_T(y))(m) + D(x,y)(\phi_M(m)),$$

for all $x,y \in T$ and $m \in M$.

Proposition 2.6 *Let (T,ϕ_T) be a LietsDer pair and (M,ϕ_M) be a representation of it. Then $(T\oplus M,\phi_T\oplus\phi_M)$ is a LietsDer pair where the Lie triple system bracket on $T\oplus M$ is given by the semi-direct product*

$$[(x,m),(y,n),(z,p)] = ([x,y,z]_T, \theta(y,z)(m) - \theta(x,z)(n) + D(x,y)(p)),$$

for any $x,y,z \in T$ and $m,n,p \in M$.

Proof It is known that $T\oplus M$ equipped with the above product is a Lie triple system. Moreover, we have

$$\begin{aligned}
&(\phi_T\oplus\phi_M)([(x,m),(y,n),(z,p)])\\
&\quad = (\phi_T([x,y,z]_T), \phi_M(\theta(y,z)(m)) - \phi_M(\theta(x,z)(n)) + \phi_M(D(x,y)(p)))\\
&\quad = ([\phi_T(x),y,z]_T, \theta(y,z)(\phi_M(m)) - \theta(\phi_T(x),z)(n) + D(\phi_T(x),y)(p)) +\\
&\qquad ([x,\phi_T(y),z]_T, \theta(\phi_T(y),z)(m) - \theta(x,z)(\phi_M(n)) + D(x,\phi_T(y))(p)) +\\
&\qquad ([x,y,\phi_T(z)]_T, \theta(y,\phi_T(z))(m) - \theta(x,\phi_T(z))(n) + D(x,y)(\phi_M(p)))\\
&\quad = [(\phi_T\oplus\phi_M)(x,m),(y,n),(z,p)] + [(x,m),(\phi_T\oplus\phi_M)(y,n),(z,p)] +\\
&\qquad [(x,m),(y,n),(\phi_T\oplus\phi_M)(z,p)].
\end{aligned}$$

Hence the proof is completed. □

Let θ be a representation of $(T,[\cdot,\cdot,\cdot]_T)$ on M. If an n-linear map $f : T\times T\times\cdots\times T \to M$ satisfies

(1) $f(x_1,\ldots,x,x,x_n) = 0$,

(2) $f(x_1,\ldots,x_{n-3},x,y,z) + f(x_1,\ldots,x_{n-3},y,z,x) + f(x_1,\ldots,x_{n-3},z,x,y) = 0$,

then f is called an n-cochain on T. Denote by $C^n(T,M)$ the set of all n-cochains.

Define the linear maps $d : C^n(T,M) \to C^{n+2}(T,M)$, $n \geq 1$ as follows:

for $f \in C^{2n-1}(T,M)$, $n = 1,2,3,\ldots$,

$$df(x_1,\ldots,x_{2n+1}) = \theta(x_{2n},x_{2n+1})f(x_1,\ldots,x_{2n-1}) - \theta(x_{2n-1},x_{2n+1})f(x_1,\ldots,x_{2n-2},x_{2n}) +$$

$$\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1},x_{2k})f(x_1,\ldots,\widehat{x_{2k-1}},\widehat{x_{2k}},\ldots,x_{2n+1})+$$
$$\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(x_1,\ldots,\widehat{x_{2k-1}},\widehat{x_{2k}},\ldots,[x_{2k-1},x_{2k},x_j],\ldots,x_{2n+1});$$

for $f \in C^{2n}(T,M)$, $n = 1,2,3,\ldots$,

$$df(y,x_1,\ldots,x_{2n+1}) = \theta(x_{2n},x_{2n+1})f(y,x_1,\ldots,x_{2n-1})-$$
$$\theta(x_{2n-1},x_{2n+1})f(y,x_1,\ldots,x_{2n-2},x_{2n})+$$
$$\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1},x_{2k})f(y,x_1,\ldots,\widehat{x_{2k-1}},\widehat{x_{2k}},\ldots,x_{2n+1})+$$
$$\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(y,x_1,\ldots,\widehat{x_{2k-1}},\widehat{x_{2k}},\ldots,[x_{2k-1},x_{2k},x_j],\ldots,x_{2n+1}),$$

where the sign$\widehat{\ }$indicates that the element below must be omitted, and $d\circ d = 0$.

In the next, we introduce a cohomology for a LietsDer pair with coefficients in a representation.

Let (T,ϕ_T) be a LietsDer pair and (M,ϕ_M) be a representation of it. For any $n \geq 1$, we define a new map δ :Hom$(T^{\otimes n},M)$ $\to$Hom$(T^{\otimes n+1},M)$ by

$$\delta : \mathrm{Hom}(T^{\otimes n},M) \to \mathrm{Hom}(T^{\otimes n+1},M)$$
$$\delta f(x_1,x_2,\ldots,x_{n+1})$$
$$=\sum_{k=1}^{n+1}(\sum_{i=1}^{n} f\circ(Id_T\otimes\cdots\otimes\phi_T\otimes\cdots\otimes Id_T)-\phi_M\circ f)(x_1,x_2,\ldots,\widehat{x}_k,\ldots,x_{n+1}).$$

The following lemma is useful to define the coboundary operator of the cohomology of LietsDer pair.

Lemma 2.7 *The map δ commutes with d, i.e., $d\circ\delta = \delta\circ d$.*

We are now in a position to define the cohomology of the LietsDer pair. Define the space $C^0_{\mathrm{LietsDer}}(T,M)$ of 0-cochains to be 0 and the space $C^1_{\mathrm{LietsDer}}(T,M)$ of 1-cochains to be Hom(T,M). The space of n-cochains $C^n_{\mathrm{LietsDer}}(T,M)$, for $n \geq 2$, is defined by

$$C^n_{\mathrm{LietsDer}}(T,M) := \mathrm{Hom}(T^{\otimes n},M)\otimes \mathrm{Hom}(T^{\otimes n-1},M).$$

We define a map $\partial : C^n_{\mathrm{LietsDer}}(T,M) \to C^{n+2}_{\mathrm{LietsDer}}(T,M)$ by

$$\partial f = (df,-\delta f), \quad \text{for any} \quad f\in C^1_{\mathrm{LietsDer}}(T,M),$$
$$\partial(f_n,\overline{f_n}) = (df_n, d\overline{f}_n - \delta f_n), \quad \text{for any} \quad (f_n,\overline{f}_n)\in C^{2n-1}_{\mathrm{LietsDer}}(T,M),$$
$$\partial(f_n,\overline{f_n}) = (df_n, d\overline{f_n} + \delta f_n), \quad \text{for any} \quad (f_n,\overline{f}_n)\in C^{2n}_{\mathrm{LietsDer}}(T,M).$$

Proposition 2.8 *The map ∂ satisfies $\partial\circ\partial = 0$.*

Proof For any $f \in C^1_{\mathrm{LietsDer}}(T,M)$, we have

$$(\partial\circ\partial)f = \partial(df,-\delta f) = ((d\circ d)f, -(d\circ\delta)f + (\delta\circ d)f) = 0.$$

Similarly, for any $(f_n, \overline{f}_n) \in C^n_{\mathrm{LietsDer}}(T, M)$, we have

$$\begin{aligned}(\partial \circ \partial)(f_n, \overline{f_n}) &= \partial(df_n, df_n + (-1)^n f_n)\\ &= (d^2 f_n, d^2\overline{f_n} + (-1)^n d\delta f_n + (-1)^{n+1}\delta df_n) = 0.\end{aligned}$$

Hence the proof is completed. □

Therefore, $(C^*_{\mathrm{LietsDer}}(T, M), \partial)$ forms a cochain complex. We denote the corresponding cohomology groups by $H^*_{\mathrm{LietsDer}}(T, M)$.

3. Central extensions of LietsDer pairs

In this section, we study central extensions of a LietsDer pair, we show that isomorphic classes of central extensions are classified by the cohomology of the LietsDer pair with coefficients in the trivial representation.

Let (T, ϕ_T) be a LietsDer pair and (M, ϕ_M) be an abelian LietsDer pair i.e., the Lie triple system bracket of M is trivial.

Definition 3.1 *A central extension of (T, ϕ_T) by (M, ϕ_M) is an exact sequence of LietsDer pairs*

$$0 \longrightarrow (M, \phi_M) \xrightarrow{\ i\ } (\hat{T}, \phi_{\hat{T}}) \xrightarrow{\ p\ } (T, \phi_T) \longrightarrow 0 \tag{3.1}$$

such that $[i(m), \hat{x}, \hat{y}] = 0$, for all $m \in M, \hat{x}, \hat{y} \in \hat{T}$.

In a central extension, we can identify M with the corresponding subalgebra of $\hat{T}$ and with this $\phi_M = \phi_{\hat{T}}|_M$.

Definition 3.2 *Two central extensions $(\hat{T}, \phi_{\hat{T}})$ and $(\hat{T}', \phi_{\hat{T}'})$ are said to be isomorphic if there is an isomorphism $\eta : (\hat{T}, \phi_{\hat{T}}) \to (\hat{T}', \phi_{\hat{T}'})$ of LietsDer pairs that makes the following diagram commutative*

$$\begin{tikzcd}
0 \arrow[r] & (M, \phi_M) \arrow[r, "i"] \arrow[d, "Id_M"] & (\hat{T}, \phi_{\hat{T}}) \arrow[r, "p"] \arrow[d, "\eta"] & (T, \phi_T) \arrow[r] \arrow[d, "Id_T"] & 0 \\
0 \arrow[r] & (M, \phi_M) \arrow[r, "i'"] & (\hat{T}', \phi_{\hat{T}'}) \arrow[r, "q"] & (T, \phi_T) \arrow[r] & 0
\end{tikzcd}$$

Diagram 1 Two central extensions isomorphic

Let Eq. (3.1) be a central extension. A section of the map p is given by a linear map $s : T \to \hat{T}$ such that $p \circ s = Id_T$.

For any section s, we define maps $\psi : T \otimes T \otimes T \to M$ and $\chi : T \to M$ by

$$\psi(x, y, z) := [s(x), s(y), s(z)]_{\hat{T}} - s([x, y, z]_T),\ \chi(x) = \phi_{\hat{T}}(s(x)) - s(\phi_T(x)),\quad \forall x, y, z \in T.$$

Note that the vector space $\hat{T}$ is isomorphic to the direct sum $T \oplus M$ via the section s. Therefore, we may transfer the structures of $\hat{T}$ to $T \oplus M$. The product and linear maps on $T \oplus M$ are given by

$$[(x, m), (y, n), (z, p)]_\psi = ([x, y, z]_T, \psi(x, y, z)),$$

$$\phi_{T\oplus M}(x,m)=(\phi_T(x),\phi_M(m)+\chi(x)).$$

Proposition 3.3 *The vector space $T\oplus M$ equipped with the above product and linear maps $\phi_{T\oplus M}$ forms a LietsDer pair if and only if (ψ,χ) is a 3-cocycle in the cohomology of the LietsDer pair T with coefficients in the trivial representation M. Moreover, the cohomology class of (ψ,χ) does not depend on the choice of the section s.*

Proof The tuple $(T\oplus M,\phi_{T\oplus M})$ is a LietsDer pair if and only if

$$[(x,m),(x,m),(y,n)]_\psi=0, \tag{3.2}$$

$$[(x,m),(y,n),(z,p)]_\psi+[(y,n),(z,p),(x,m)]_\psi+[(z,p),(x,m),(y,n)]_\psi=0, \tag{3.3}$$

$$\begin{aligned}&[(x,m),(y,n),[(z,p),(v,k),(w,l)]_\psi]_\psi\\ &\quad=[[(x,m),(y,n),(z,p)]_\psi,(v,k),(w,l)]_\psi+[(z,p),[(x,m),(y,n),(v,k)]_\psi,(w,l)]_\psi+\\ &\qquad[(z,p),(v,k),[(x,m),(y,n),(w,l)]_\psi]_\psi,\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\phi_{T\oplus M}[(x,m),(y,n),(z,p)]_\psi\\ &\quad=[\phi_{T\oplus M}(x,m),(y,n),(z,p)]_\psi+[(x,m),\phi_{T\oplus M}(y,n),(z,p)]_\psi+\\ &\qquad[(x,m),(y,n),\phi_{T\oplus M}(z,p)]_\psi,\end{aligned} \tag{3.5}$$

for all $x\oplus m,\ y\oplus n,\ z\oplus p,\ v\oplus k,\ w\oplus l\in T\oplus M$. The condition Eq. (3.2) is equivalent to

$$\psi(x,x,y)=0.$$

The condition Eq. (3.3) is equivalent to

$$\psi(x,y,z)+\psi(y,z,x)+\psi(z,x,y)=0.$$

The condition Eq. (3.4) is equivalent to

$$\psi(x,y,[z,v,w]_T)=\psi([x,y,z]_T,v,w)+\psi(z,[x,y,v]_T,w)+\psi(z,v,[x,y,w]_T).$$

Hence, $d(\psi)=0$. The condition Eq. (3.5) is equivalent to

$$\phi_M(\psi(x,y,z))+\chi([x,y,z]_T)=\psi(\phi_T(x),y,z)+\psi(x,\phi_T(y),z)+\psi(x,y,\phi_T(z)).$$

This is the same as $d(\chi)+\delta\psi=0$.

Let s_1,s_2 be two sections of p. Define a map $u:T\to M$ by $u(x):=s_1(x)-s_2(x)$. Then we have

$$\begin{aligned}\psi(x,y,z)=&[s_1(x),s_1(y),s_1(z)]_{\hat T}-s_1([x,y,z]_T)\\ =&[s_2(x)+u(x),s_2(y)+u(y),s_2(z)+u(z)]_{\hat T}-s_2([x,y,z]_T)-u([x,y,z]_T)\\ =&\psi'(x,y,z)-u([x,y,z]_T),\end{aligned}$$

and

$$\begin{aligned}\chi(x)&=\phi_{\hat T}(s_1(x))-s_1(\phi_T(x))=\phi_{\hat T}(s_2(x)+u(x))-s_2(\phi_T(x))-u(\phi_T(x))\\ &=\chi'(x)+\phi_M(u(x))-u(\phi_T(x)).\end{aligned}$$

This shows that $(\psi,\chi)-(\psi',\chi')=\partial u$. Hence they correspond to the same cohomology class. □

Theorem 3.4 *Let (T, ϕ_T) be a LietsDer pair and (M, ϕ_M) be an abelian LietsDer pair. Then the isomorphism classes of central extensions of T by M are classified by the cohomology group $H^3_{\text{LietsDer}}(T, M)$.*

Proof Let $(\hat{T}, \phi_{\hat{T}})$ and $(\hat{T}', \phi_{\hat{T}'})$ be two isomorphic central extensions and the isomorphism is given by $\eta : \hat{T} \to \hat{T}'$. Let $s : T \to \hat{T}$ be a section of p. Then

$$p' \circ (\eta \circ s) = (p' \circ \eta) \circ s = p \circ s = Id_T.$$

This shows that $s' := \eta \circ s$ is a section of p'. Since η is a morphism of LietsDer pairs, we have $\eta|_M = Id_M$. Thus,

$$\begin{aligned}\psi'(x, y) &= [s'(x), s'(y), s'(z)]_{\hat{T}} - s'([x, y, z]_T)\\ &= \eta([s(x), s(y), s(z)]_{\hat{T}} - [x, y, z]_T) = \psi(x, y, z),\end{aligned}$$

and

$$\begin{aligned}\chi'(x) &= \phi^{\hat{T}'}(s'(x)) - s'(\phi_T(x)) = \phi_{\hat{T}'}(\eta \circ s(x)) - \eta \circ s(\phi_T(x))\\ &= \phi_{\hat{T}}(s(x)) - s(\phi_T(x)) = \chi(x).\end{aligned}$$

Therefore, isomorphic central extensions give rise to same 3-cocycle, hence, correspond to same element in $H^3_{\text{LietsDer}}(T, M)$.

Conversely, let (ψ, χ) and (ψ', χ') be two cohomologous 3-cocycles. Therefore, there exists a map $v : T \to M$ such that

$$(\psi, \chi) - (\psi', \chi') = \partial v.$$

The LietsDer pair structures on $T \oplus M$ corresponding to the above 3-cocycles are isomorphic via the map $\eta : T \oplus M \to T \oplus M$ given by $\eta(x, m) = (x, m + v(x))$. Hence the proof is completed. □

4. Deformations of LietsDer pairs

In this section, we study formal one-parameter deformations of LietsDer pairs in which we deform both the Lie triple system bracket and the distinguished derivations.

Let (T, ϕ_T) be a LietsDer pair. We denote the Lie triple system bracket on T by μ, i.e., $\mu(x, y, z) = [x, y, z]_T$, for all $x, y, z \in T$. Consider the space $T[[t]]$ of formal power series in t with coefficients from T. Then $T[[t]]$ is an $\mathbb{F}[[t]]$-module.

A formal one-parameter deformation of the LietsDer pair (T, ϕ_T) consists of formal power series

$$\mu_t = \sum_{i=0}^{\infty} t^i \mu_i \in \text{Hom}(T^{\otimes 3}, T)[[t]] \text{ with } \mu_0 = \mu,$$

$$\phi_t = \sum_{i=0}^{\infty} t^i \phi_i \in \text{Hom}(T, T)[[t]] \text{ with } \phi_0 = \phi_T,$$

such that $T[[t]]$ together with the bracket μ_t forms a Lie triple system over $\mathbb{F}[[t]]$ and ϕ_t is a derivation on this Lie triple system.

Therefore, in a formal one-parameter deformation of LietsDer pair, the following relations hold:

$$\mu_t(x,x,y)=0, \tag{4.1}$$

$$\mu_t(x,y,z)+\mu_t(y,z,x)+\mu_t(z,x,y)=0, \tag{4.2}$$

$$\mu_t(x,y,\mu_t(z,v,w))=\mu_t(\mu_t(x,y,z),v,w)+\mu_t(z,\mu_t(x,y,v),w)+\mu_t(z,v,\mu_t(x,y,w)), \tag{4.3}$$

$$\phi_t(\mu_t(x,y,z))=\mu_t(\phi_t(x),y,z)+\mu_t(x,\phi_t(y),z)+\mu_t(x,y,\phi_t(z)). \tag{4.4}$$

Conditions Eqs. (4.1) and (4.2) are equivalent to the following equations:

$$\begin{aligned}&\mu_n(x,x,y)=0,\\&\mu_n(x,y,z)+\mu_n(y,z,x)+\mu_n(z,x,y)=0\end{aligned}$$

respectively, for $n=0,1,2,\ldots$. Conditions Eqs. (4.3) and (4.4) are equivalent to the following equations:

$$\begin{aligned}&\sum_{i+j=n}\mu_i(x,y,\mu_j(z,v,w))\\&=\sum_{i+j=n}\mu_i(\mu_j(x,y,z),v,w)+\mu_i(z,\mu_j(x,y,v),w)+\mu_i(z,v,\mu_j(x,y,w)),\\&\sum_{i+j=n}\phi_i(\mu_j(x,y,z))=\sum_{i+j=n}\mu_i(\phi_j(x),y,z)+\mu_i(x,\phi_j(y),z)+\mu_i(x,y,\phi_j(z)).\end{aligned}$$

All the identities hold for $n=0$ as (T,ϕ_T) is a LietsDer pair. For $n=1$, we have

$$\begin{aligned}&\mu_1(x,y,[z,v,w]_T)+[x,y,\mu_1(z,v,w)]_T\\&\quad=\mu_1([x,y,z]_T,v,w)+[\mu_1(x,y,z),v,w]_T+[z,\mu_1(x,y,v),w]_T+\\&\qquad\mu_1(z,[x,y,v]_T,w)+[z,v,\mu_1(x,y,w)]_T+\mu_1(z,v,[x,y,w]_T),\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\phi_1([x,y,z]_T)+\phi_T(\mu_1(x,y,z))\\&\quad=\mu_1(\phi_T(x),y,z)+[\phi_1(x),y,z]_T+\mu_1(x,\phi_T(y),z)+[x,\phi_1(y),z]_T+\\&\qquad\mu_1(x,y,\phi_T(z))+[x,y,\phi_1(z)]_T.\end{aligned} \tag{4.6}$$

The condition Eq. (4.5) is equivalent to $d(\mu_1)=0$ whereas the condition Eq. (4.6) is equivalent to $d(\phi_1)+\delta(\mu_1)=0$. Therefore, we have

$$\partial(\mu_1,\phi_1)=0.$$

Hence, we have the following.

Proposition 4.1 *Let (μ_t,ϕ_t) be a formal one-parameter deformation of a LietsDer pair (T,ϕ_T). Then the linear term (μ_1,ϕ_1) is a 3-cocycle in the cohomology of the LietsDer pair T with coefficients in itself.*

The 3-cocycle (μ_1,ϕ_1) is called the infinitesimal of the deformation. In particular, if $(\mu_1,\phi_1)=\cdots=(\mu_{n-1},\phi_{n-1})=0$ and (μ_n,ϕ_n) is non-zero, then (μ_n,ϕ_n) is a 3-cocycle.

Next we define a notion of equivalence between formal deformations of LietsDer pairs.

Definition 4.2 *Two deformations (μ_t, ϕ_t) and (μ'_t, ϕ'_t) of a LietsDer pair (T, ϕ_T) are said to be equivalent if there exists a formal isomorphism $\Phi_t = \sum_{i=0}^{\infty} t^i \Phi_i : T[[t]] \to T[[t]]$ with $\Phi_0 = Id_T$ such that*

$$\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t \otimes \Phi_t), \quad \Phi_t \circ \phi_t = \phi'_t \circ \Phi_t.$$

By equating coefficients of t^n, we get

$$\sum_{i+j=n} \Phi_i \circ \mu_j = \sum_{p+q+r+l=n} \mu'_p \circ (\Phi_q \otimes \Phi_r \otimes \Phi_l),$$

$$\sum_{i+j=n} \phi'_i \circ \Phi_j = \sum_{p+q=n} \Phi_p \circ \phi_q.$$

The above identities hold for $n = 0$, for $n = 1$, we obtain

$$\mu_1 + \Phi_1 \circ \mu = \mu'_1 + \mu \circ (\Phi_1 \otimes Id \otimes Id) + \mu \circ (Id \otimes Id \otimes \Phi_1) + \mu \circ (Id \otimes \Phi_1 \otimes Id), \tag{4.7}$$

$$\phi_T \circ \Phi_1 + \phi'_1 = \phi_1 + \Phi_1 \circ \phi_T. \tag{4.8}$$

These two identities together imply that

$$(\mu_1, \phi_1) - (\mu'_1, \phi'_1) = \partial \Phi_1.$$

Thus, we have the following.

Proposition 4.3 *The infinitesimals corresponding to equivalent deformations are cohomologous. Hence, they correspond to the same cohomology class.*

Definition 4.4 *A deformation (μ_t, ϕ_t) of a LietsDer pair is said to be trivial if it is equivalent to the undeformed deformation $(\mu'_t = \mu, \phi'_t = \phi_T)$.*

Definition 4.5 *A LietsDer pair (T, ϕ_T) is called analytically rigid, if every 1-parameter formal deformation μ_t is trivial.*

Theorem 4.6 *If $H^3_{\mathrm{LietsDer}}(T, T) = 0$, then every formal deformation of the LietsDer pair (T, ϕ_T) is trivial.*

Proof Let (μ_t, ϕ_t) be a deformation of the LietsDer pair (T, ϕ_T). Then by Proposition 4.1, the linear term (μ_1, ϕ_1) is a 3-cocycle. Therefore, $(\mu_1, \phi_1) = \partial \Phi_1$ for some $\Phi_1 \in C^1_{\mathrm{LietsDer}}(T, T) = \mathrm{Hom}(T, T)$.

We set $\Phi_t = Id_T + t\Phi_1 : T[[t]] \to T[[t]]$ and define

$$\mu'_t = \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t \otimes \Phi_t), \qquad \phi'_t = \Phi_t^{-1} \circ \phi_t \circ \Phi_t. \tag{4.9}$$

By definition, (μ'_t, ϕ'_t) is equivalent to (μ_t, ϕ_t). Moreover, it follows from Eq. (4.9) that

$$\mu'_t = \mu + t^2 \mu'_2 + \cdots \quad \text{and} \quad \phi'_t = \phi_T + t^2 \phi'_2 + \cdots .$$

In other words, the linear terms vanish. By repeating this argument, we conclude the result. □

Acknowledgements The author is grateful to the referees for carefully reading the manuscript and for many valuable comments which largely improved the article.

References

[1] N. JACOBSON. *Lie and Jordan triple systems.* Amer. J. Math. Soc., 1949, **71**(1): 149–170.

[2] W. LISTER. *A structure theory of Lie triple systems.* Trans. Amer. Math. Soc., 1952, **72**(2): 217–242.

[3] M. BREMNER, R. FELIPE, J. SÁNCHEZ-ORTEGA. *Jordan triple disystems.* Comput. Math. Appl., 2012, **63**(6): 1039–1055.

[4] A. J. CALDERÓN, C, MARTIN. *The Banach-Lie group of Lie triple automorphisms of an H^*-algebra.* Acta Math. Sci. Ser. B (Engl. Ed.), 2010, **30**(4): 1219–1226.

[5] A. J. CALDERÓN, C. MARTIN. *A linear approach to Lie triple automorphisms of H^*-algebras.* J. Korean Math. Soc., 2011, **48**(1): 117–132.

[6] B. HARRIS. *Cohomology of Lie triple systems and Lie algebras with involution.* Trans. Amer. Math. Soc., 1961, **98**(1): 148–162.

[7] T. HODGE, B. PARSHALL. *On the representation theory of Lie triple systems.* Trans. Amer. Math. Soc., 2002, **354**(11): 4359–4391.

[8] E. STITZINGER. *On derivation algebras of Malcev algebras and Lie triple systems.* Proc. Amer. Math. Soc., 1976, **55**(1): 9–13.

[9] Tao ZHANG. *Notes on cohomologies of Lie triple systems.* J. Lie Theory, 2014, **24**(4): 909–929.

[10] Zhixue ZHANG, Liangyun CHEN, Wenli LIU, et al. *The Frattini subsystem of a Lie triple system.* Comm. Algebra, 2009, **37**(10): 3750–3759.

[11] Zhixue ZHANG, Yiqian SHI, Lina ZHAO. *Invariant symmetric bilinear forms on Lie triple systems.* Comm. Algebra, 2002, **30**(11): 5563–5573.

[12] T. VORONOV. *Higher derived brackets and homotopy algebras.* J. Pure Appl. Algebra, 2005, **202**: 133–153.

[13] V. COLL, M. GERSTENHABER, A. GIAQUINTO. *An Explicit Deformation Formula with Noncommuting Derivations.* Weizmann, Jerusalem, 1989.

[14] A. MAGID. *Lectures on Differential Galois Theory.* American Mathematical Society, Providence, RI, 1994.

[15] V. AYALA, E. KIZIL, I. DE AZEVEDO TRIBUZY. *On an algorithm for finding derivations of Lie algebras.* Proyecciones, 2012, **31**(1): 81–90.

[16] M. DOUBEK, T. LADA. *Homotopy derivations.* J. Homotopy Relat. Struct., 2016, **11**(3): 599–630.

[17] Rong TANG, Y. FRÉGIER, Yunhe SHENG. *Cohomologies of a Lie algebra with a derivation and applications.* J. Algebra, 2019, **534**: 65–99.

[18] A. DAS, A. MANDAL. *Extensions, deformations and categorifications of AssDer pairs.* arxiv. 2002.11415.

[19] A. DAS. *Leibniz algebras with derivations.* J. Homotopy Relat. Struct., 2021, **16**(2): 245–274.

[20] M. GERSTENHABER. *The cohomology structure of an associative ring.* Ann. of Math. (2), 1963, **78**: 267–288.

[21] M. GERSTENHABER. *On the deformation of rings and algebras.* Ann. of Math. (2), 1964, **79**: 59–103.

[22] F. KUBO, Y. TANIGUCHI. *A controlling cohomology of the deformation theory of Lie triple systems.* J. Algebra, 2004, **278**(1): 242–250.

[23] G. HOCHSCHILD. *On the cohomology groups of an associative algebra.* Ann. of Math. (2), 1945, **46**: 58–67.

[24] K. YAMAGUTI. *On the cohomology space of Lie triple system.* Kumamoto J. Sci. Ser. A, 1960, **5**: 44–52.

[25] Jia ZHOU, Liangyun CHEN, Yao MA. *Generalized derivations of Lie triple systems.* Open Math., 2016, **14**(1): 260–271.