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Planar Cubic Pythagorean-Hodograph Hyperbolic Curves

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Abstract The purpose of this paper is to develop a method to construct the Pythagorean-hodograph hyperbolic (PH-H) curves based on the good geometric properties of PH curves and algebraic hyperbolic curves. The definition of Pythagorean-hodograph hyperbolic curves is given and their properties are examined. By using hyperbolic basis functions and algebraic Bézier basis functions respectively, two different necessary and sufficient conditions for a planar cubic algebraic hyperbolic Bézier curve to be a PH curve are obtained. Moreover, cubic PH-H curves are applied in the problem of G^1 Hermite interpolation with determined closed form solutions. Several examples serve to illustrate our method.

Keywords Pythagorean-hodograph curve; algebraic hyperbolic Bézier curve; G^1 Hermite interpolation

MR(2020) Subject Classification 65D17; 65D18; 68U07

1. Introduction

Pythagorean-hodograph (PH) curves, a kind of polynomial curves $\boldsymbol{r}(t) = (x(t), y(t))$, $t \in [0, 1]$ with derivative components $x'(t)$, $y'(t)$ satisfying the condition

$$x'^2(t) + y'^2(t) = \sigma^2(t)$$

for some polynomial $\sigma(t)$, play significant role in the theory as well as in practical applications of polynomial curves [1]. Because of their unique properties, such as exact representation of polynomial arc length and rational offset, there has been a respectful amount of work devoted to the PH curves since their first appearance [2–4]. Moreover, they have been widely used in geometric modeling and CNC machining [5].

It is well known that Bézier basis is presented in the space spanned by $\{1, t, \ldots, t^{n-2}, t^{n-1}, t^n\}$, and it is often used to represent polynomial PH curves in the previous research work on PH curves. However, as shown by Mainar et al. [6], there still exist several limitations of Bézier model and B-Spline model. For instance, it cannot accurately represent transcendental curves such as hyperbola, catenary and exponential curves. In order to avoid the inconveniences of the rational Bézier model, many scholars have proposed many bases in new spaces other than

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the polynomial space. Sánchez-Reyes [7] gave a basis for the space of trigonometric polynomials $\{1, \sin t, \cos t, \ldots, \sin mt, \cos mt\}$. In [8], Chen and Wang constructed C-Bézier basis of the space spanned by $\{1, t, \ldots, t^{n-2}, \sin t, \cos t\}$ which can represent exactly transcendental curves such as helix and cycloid. Furthermore, the Pythagorean-hodograph cycloidal curves (PHC curves) were proposed and a necessary and sufficient condition for a cubic plane C-Bézier curve to be a PHC curve was obtained in [9]. Li and Wang [10] constructed a basis for the algebraic hyperbolic space spanned by $\{1, t, \ldots, t^{n-2}, \sinh t, \cosh t\}$ and studied the algebraic hyperbolic (AH) Bézier curves. Tan et al. [11, 12] discussed the problem of interpolation, splicing and segmentation by AH Bézier curve.

In recent years, a lot of work have been given in the literature to deal with the problem of G^1 Hermite interpolation. The G^1 interpolation problem by cubic optimized geometric Hermite curves was presented and a general algorithm to construct such interpolation was described in [13]. Wu and Yang [14] proposed techniques of interpolation of intrinsically defined curves to geometric Hermite data. Jaklič and Žagar [15] constructed cubic G^1 interpolatory splines by taking tangent directions as unknowns, thereby relaxing conditions on admissible regions for tangent directions. Various types of spirals that have monotone curvatures were also used for interpolation of G^1 Hermite data [16, 17]. Besides, Kozak et al. [18] gave a geometric characterization of PHC curves, and the Hermite interpolation problem by PHC curves was also studied.

In this paper, polynomial PH curves are extended to algebraic hyperbolic space. Firstly, we define Pythagorean-hodograph hyperbolic curves based on AH Bézier curves and PH curves. Secondly, two necessary and sufficient conditions for planar cubic AH Béziers to be PH-H curves are presented. Furthermore, the problem of G^1 Hermite interpolation with PH-H curves is outlined. The analysis shows that there may be one, two or no solutions depending on the data supplied, similar to the case of cubic PHC curves. Finally, some examples are used to demonstrate the practicability of our method.

The remainder of this paper is organized as follows. In Section 2, we review the properties of AH Bézier curves and introduce the definition of PH-H curves. Section 3 gives two necessary and sufficient conditions of planar cubic AH Bézier curves to be PH-H curves by two different bases. In Section 4, the problem of G^1 Hermite interpolation with cubic PH-H curves is outlined. Finally, we conclude the paper in Section 5.

2. Algebraic hyperbolic Bézier curves

Let us consider the PH condition applied to n-hyperbolic curves based upon the space $\Gamma_n = \mathrm{span}\{1, t, \ldots, t^{n-2}, \sinh t, \cosh t\}$. The definition of algebraic hyperbolic Bézier curve and its related theorems and corollaries are as follows.

The AH Bézier basis functions start with the two initial functions

$$Z_{0,1}(t) = \frac{\sinh(\alpha - t)}{\sinh\alpha},\ Z_{1,1}(t) = \frac{\mathrm{sinht}}{\sinh\alpha},\quad t \in [0, \alpha],\ \alpha \in (0, +\infty).$$

For $n>1$, the AH Bézier basis functions $\{Z_{i,n}\}$ of the space Γ_n are defined recursively as

$$\begin{aligned}
Z_{0,n}(t) =&1-\int_0^t \delta_{0,n-1}Z_{0,n-1}(s)\mathrm{d}s,\\
Z_{i,n}(t) =&\int_0^t \delta_{i-1,n-1}Z_{i-1,n-1}(s)-\delta_{i,n-1}Z_{i,n-1}(s)\mathrm{d}s,\quad i=1,2,\ldots,n-1,\\
Z_{n,n}(t) =&\int_0^t \delta_{n-1,n-1}Z_{n-1,n-1}(s)\mathrm{d}s,
\end{aligned}$$

where

$$\delta_{i,n}=\frac{1}{\int_0^\alpha Z_{i,n}(t)\mathrm{d}t}.$$

One can refer to [10] for the details of AH Bézier curves and their properties.

The cubic AH Bézier curve is a kind of special curves generated in algebraic hyperbolic mixed space $\Gamma_3=\mathrm{span}\{1,t,\mathrm{sinh}t,\mathrm{cosh}t\}$. It retains the advantages of Bézier curve and can represent transcendental curve such as catenary. It can be expressed as

$$\boldsymbol{P}(t)=\sum_{i=0}^{3}\boldsymbol{b}_iZ_{i,3},\quad t\in[0,\alpha],\ \alpha\in(0,+\infty), \tag{2.1}$$

where $\{\boldsymbol{b}_i\}\in\mathbb{R}^2$ are control points and $\{Z_{i,3}\}$ are the AH Bézier basis of Γ_3 defined as

$$\begin{aligned}
Z_{0,3}(t) =&\frac{(\alpha-t)-\mathrm{sinh}(\alpha-t)}{\alpha-\mathrm{sinh}\alpha},\quad Z_{3,3}(t)=\frac{t-\mathrm{sinht}}{\alpha-\mathrm{sinh}\alpha},\\
Z_{1,3}(t) =&M[\frac{\alpha\mathrm{cosh}\alpha-t\mathrm{cosh}\alpha-\mathrm{sinh}\alpha+\mathrm{sinh}t}{\alpha\mathrm{cosh}\alpha-\mathrm{sinh}\alpha}-Z_{0,3}(t)],\\
Z_{2,3}(t) =&M[\frac{\mathrm{sinh}(\alpha-t)+t\mathrm{cosh}\alpha-\mathrm{sinh}\alpha}{\alpha\mathrm{cosh}\alpha-\mathrm{sinh}\alpha}-Z_{3,3}(t)],
\end{aligned}$$

with

$$M=\frac{\alpha\mathrm{cosh}\alpha-\mathrm{sinh}\alpha}{\alpha\mathrm{cosh}\alpha-2\mathrm{sinh}\alpha+\alpha}.$$

Note that the functions $\{Z_{i,3}\}$ form the partition of unity and have symmetry, i.e.,

$$\sum_{i=0}^{3}Z_{i,3}(t)=1,\ Z_{0,3}(t)=Z_{3,3}(\alpha-t),\ Z_{1,3}(t)=Z_{2,3}(\alpha-t).$$

Definition 2.1 *A planar algebraic hyperbolic Bézier curve $\boldsymbol{P}(t)=(x(t),y(t))\in\Gamma_n$, that has derivative components $x'(t)$, $y'(t)$ satisfying $x'^2(t)+y'^2(t)=\sigma^2(t)$ for some $\sigma(t)\in\Gamma_{n-1}$, is called a Pythagorean-hodograph hyperbolic (PH-H) curve.*

Lemma 2.2 ([9]) *Three real polynomials $a(t)$, $b(t)$ and $c(t)$ satisfy the condition*

$$a^2(t)+b^2(t)=c^2(t)$$

iff

$$a(t)=w(t)(u^2(t)-v^2(t)),\ b(t)=2w(t)u(t)v(t),\ c(t)=w(t)(u^2(t)+v^2(t)),$$

where $u(t)$, $v(t)$ and $w(t)$ are real polynomials.

3. Planar cubic Pythagorean-hodograph hyperbolic curves

Two different necessary and sufficient conditions for planar cubic AH Bézier curves to be PH-H curves by two different bases are proposed in this section.

3.1. The necessary and sufficient condition of cubic PH-H curve based on hyperbolic basis functions $\{1, t, \sinh t, \cosh t\}$

In this section, we will discuss the necessary and sufficient condition that a cubic AH Bézier curve is PH-H curve using the hyperbolic basis functions $\{1, t, \sinh t, \cosh t\}$. A cubic AH Bézier curve $\boldsymbol{P}(t) \in \Gamma_3$ has the general form

$$\boldsymbol{P}(t) = \boldsymbol{v_0}\sinh t + \boldsymbol{v_1}\cosh t + \boldsymbol{v_2}t + \boldsymbol{v_3}, \tag{3.1}$$

where $\boldsymbol{v_i} = (v_{ix},\, v_{iy})$, $i = 0, 1, 2, 3$. Then by Definition 2.1 and Lemma 2.2, we have the following result.

Theorem 3.1 *Suppose that a cubic AH Bézier curve $\boldsymbol{P}(t)$, distinct from a line segment, is given in the form (3.1). The necessary and sufficient conditions for $\boldsymbol{P}(t)$ to be a PH-H curve is that there exist real numbers x_1, x_2, y_1, y_2 such that*

$$\begin{cases} (v_{0x},\, v_{1x},\, v_{2x}) = (\frac{1}{2}(y_1^2 - y_2^2 + x_1^2 - x_2^2),\, x_1y_1 - x_2y_2,\, \frac{1}{2}(y_1^2 - y_2^2 - x_1^2 + x_2^2)), \\ (v_{0y},\, v_{1y},\, v_{2y}) = (y_1y_2 + x_1x_2,\, x_1y_2 + x_2y_1,\, y_1y_2 - x_1x_2). \end{cases} \tag{3.2}$$

Proof If $\boldsymbol{P}(t) = (x(t), y(t))$ is a PH-H curve, by Definition 2.1, we have

$$\chi_1^2 + \chi_2^2 = \chi_3^2, \tag{3.3}$$

where $\chi_1(t) = x'(t)$, $\chi_2(t) = y'(t)$, $\chi_3(t) = \sigma(t) \in \Gamma_2 = \mathrm{span}\{1, \sinh t, \cosh t\}$. For simplicity, let us define

$$\boldsymbol{v_0} = (a_1, a_2),\ \boldsymbol{v_1} = (b_1, b_2),\ \boldsymbol{v_2} = (c_1, c_2),$$

$$\chi_3(t) = a_3\cosh(t) + b_3\sinh(t) + c_3 \in \Gamma_2.$$

Then by (3.1), we have

$$\chi_i(t) = a_i\cosh(t) + b_i\sinh(t) + c_i \in \Gamma_2, \quad i = 1, 2.$$

Let $s = \tanh(\frac{t}{2})$, according to hyperbolic function identities,

$$\chi_i = \frac{(a_i - c_i)s^2 + 2b_is + (a_i + c_i)}{1 - s^2} = \frac{g_i(s)}{1 - s^2}, \quad i = 1, 2, 3. \tag{3.4}$$

Substituting (3.4) into expression (3.3), we have

$$g_1^2(s) + g_2^2(s) = g_3^2(s). \tag{3.5}$$

Since $g_i(s)$, $i = 1, 2, 3$ are polynomials of degree $n \le 2$, from Lemma 2.2, the equation (3.5) holds iff there are real polynomials

$$u = x_1s + y_1, \quad v = x_2s + y_2$$

such that

$$\begin{aligned}
g_1(s) &= u^2 - v^2 = (x_1^2 - x_2^2)s^2 + 2(x_1y_1 - x_2y_2)s + y_1^2 - y_2^2,\\
g_2(s) &= 2uv = 2x_1x_2s^2 + 2(x_1y_2 - x_2y_1)s + 2y_1y_2,\\
g_3(s) &= u^2 + v^2 = (x_1^2 + x_2^2)s^2 + 2(x_1y_1 + x_2y_2)s + y_1^2 + y_2^2,
\end{aligned}$$

where x_1, x_2, y_1, y_2 are real numbers. Substituting $g_i(s)$ into (3.4), we have

$$\begin{aligned}
(a_1, b_1, c_1) &= (\frac{1}{2}(y_1^2 - y_2^2 + x_1^2 - x_2^2),\ x_1y_1 - x_2y_2,\ \frac{1}{2}(y_1^2 - y_2^2 - x_1^2 + x_2^2)),\\
(a_2, b_2, c_2) &= (y_1y_2 + x_1x_2,\ x_1y_2 + x_2y_1,\ y_1y_2 - x_1x_2),\\
(a_3, b_3, c_3) &= (\frac{1}{2}(y_1^2 + y_2^2 + x_1^2 + x_2^2),\ x_1y_1 + x_2y_2,\ \frac{1}{2}(y_1^2 + y_2^2 - x_1^2 - x_2^2)).
\end{aligned}$$

Then the necessary and sufficient condition for AH Bézier curve $\boldsymbol{P}(t)$ to be PH curve is obtained. $\square$

By the basis transformation formula, the AH Bézier control points $\{\boldsymbol{b}_i\}$ of the curve $\boldsymbol{P}(t)$ in (2.1) satisfy the following relations

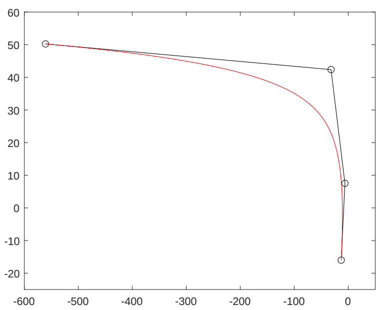
$$(\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)^T = A(\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)^T, \tag{3.6}$$

where

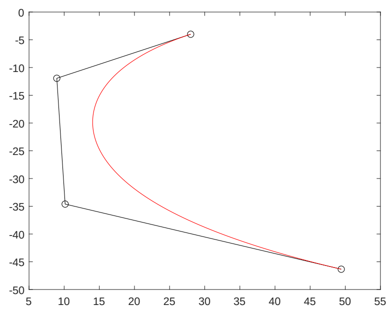
$$A = \frac{1}{\alpha - S}\begin{pmatrix} C & K - CM & M - CK & -1\\ -S & SM & SK & 0\\ -1 & M - CK & CK - M & 1\\ \alpha & -MS & -SK & 0 \end{pmatrix},$$

$$S = \sinh\alpha,\ C = \cosh\alpha,\ K = \frac{\alpha - S}{\alpha C - 2S + \alpha}.$$

The following two examples are used to illustrate Theorem 3.1. All of the calculations are carried out on a PC with 8 GHz cpu.



(a) Λ_1



(b) Λ_2

Figure 1 The cubic PH-H curves obtained by taking Λ_1 and Λ_2, respectively

Example 3.2 Consider the data

$$\Lambda_1 = \{\alpha, x_1, y_1, x_2, y_2, c_1, c_2\} = \{2\pi, -4, 4, -1, 3, 0, 0\},$$

and

$$\Lambda_2 = \{\alpha, x_1, y_1, x_2, y_2, c_1, c_2\} = \{\pi, 2, 1, 5, -5, 1, 1\},$$

respectively, in Figure 1. The solution in Figure 1(a) corresponds to the values

$$v_0 = (11,\ 16),\ v_1 = (-13,\ -16),\ v_2 = (-4,\ 8),\ v_3 = (0,\ 0),$$

and to four decimal places, the control points of the cubic PH-H curve are

$$\begin{aligned} \boldsymbol{b}_0 &= (-13.0000,\ -16.0000), \quad & \boldsymbol{b}_1 &= (-6.1387,\ 7.5245), \\ \boldsymbol{b}_2 &= (-31.8661,\ 42.3648), \quad & \boldsymbol{b}_3 &= (-560.6468,\ 50.2356), \end{aligned}$$

while the interpolant in Figure 1 (b) corresponds to the real solution

$$v_0 = (-22.5,\ 5),\ v_1 = (27,\ -5),\ v_2 = (-1.5,\ -15),\ v_3 = (1,\ 1),$$

with

$$\begin{aligned} \boldsymbol{b}_0 &= (28.0000,\ -4.0000), \quad & \boldsymbol{b}_1 &= (8.9505,\ -11.9373), \\ \boldsymbol{b}_2 &= (10.1365,\ -34.6055), \quad & \boldsymbol{b}_3 &= (49.4237,\ -46.3400). \end{aligned}$$

It can be easily checked that the resulting cubic curves accords with the definition of PH-H curves.

Note that the necessary and sufficient conditions on the control points in Theorem 3.1 are expressed as functions in x_1, x_2, y_1, y_2. But such expressions have less geometric meanings. Next we investigate the conditions to express the control points geometrically.

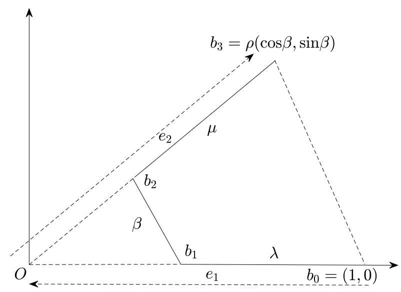


Figure 2 $\triangle \boldsymbol{b}_0\boldsymbol{O}\boldsymbol{b}_3$ and the control polygon $\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$

Given a triangle $\triangle \boldsymbol{b}_0\boldsymbol{O}\boldsymbol{b}_3$, let the length ratio of its two sides be $\frac{\|\boldsymbol{O}\boldsymbol{b}_3\|}{\|\boldsymbol{b}_0\boldsymbol{O}\|} = \rho > 0$, $\angle \boldsymbol{b}_0\boldsymbol{O}\boldsymbol{b}_3 = \beta > 0$ (see in Figure 2). If two points $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are obtained on $\boldsymbol{b}_0\boldsymbol{O}$ and $\boldsymbol{O}\boldsymbol{b}_3$ respectively, then $\boldsymbol{b}_0$ and $\boldsymbol{b}_3$ are the end control points, and $\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$ is the control polygon of AH Bézier curve. Note the vector $\boldsymbol{e}_1 = \boldsymbol{O} - \boldsymbol{b}_0$, $\boldsymbol{e}_2 = \boldsymbol{b}_3 - \boldsymbol{O}$, then the coordinates of $\boldsymbol{b}_0$ and $\boldsymbol{b}_3$ can be expressed as

$$\begin{cases} \boldsymbol{b}_1 = \lambda \boldsymbol{e}_1 + \boldsymbol{b}_0, \\ \boldsymbol{b}_2 = \boldsymbol{b}_3 - \mu \boldsymbol{e}_2, \end{cases}$$

where $0 \le \lambda, \mu \le 1$ are real numbers.

Next, we will discuss what conditions (λ, μ) meets when the AH Bézier curve obtained by the above method is a PH curve. Since the AH Bézier curve has rotation, translation, and expansion invariance, the vertex $\boldsymbol{O}$ of $\triangle \boldsymbol{b}_0\boldsymbol{O}\boldsymbol{b}_3$ is taken as the origin, $\boldsymbol{O}\boldsymbol{b}_0$ is the straight line where the x-axis is located and $\boldsymbol{b}_0 = (1,0)$. Then $\boldsymbol{e}_1 = (-1,0)$, $\boldsymbol{e}_2 = \rho(\cos\beta, \sin\beta)$ and the coordinates of the control vertices are

$$\boldsymbol{P}(\beta,\rho,\lambda,\mu) = (\boldsymbol{b}_0,\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3) = ((1,0),\ (1-\lambda,0),\ (1-\mu)\rho(\cos\beta,\sin\beta),\ \rho(\cos\beta,\sin\beta)). \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned} v_0 =& \frac{1}{\alpha - S}[C-(\lambda-1)(K-CM)-\rho\cos\beta-(\mu-1)(M-CK)\rho\cos\beta, \\ & -\rho\sin\beta-(\mu-1)(M-CK)\rho\sin\beta], \\ v_1 =& \frac{1}{\alpha - S}[-S-MS(\lambda-1)-KS(\mu-1)\rho\cos\beta,\ -KS(\mu-1)\rho\sin\beta], \\ v_2 =& \frac{1}{\alpha - S}[-(\lambda-1)(M-CK)-1+\rho\cos\beta+(\mu-1)(M-CK)\rho\cos\beta, \\ & \rho\sin\beta+(\mu-1)(M-CK)\rho\sin\beta]. \end{aligned} \tag{3.8}$$

If the curve is a PH curve, it can be known from Theorem 3.1 that

$$v_0 + v_2 = \frac{1}{\alpha - S}(C-1-(\lambda-1)(K-CM+M-CK),\ 0) = (y_1^2-y_2^2,\ 2y_1y_2),$$

so we have $y_1 = 0$, it is not difficult to verify that for a given (β,ρ), the system

$$F(\beta,\rho):\begin{cases} \boldsymbol{h}_1 = (v_{0x}+v_{2x})v_{0y} - v_{1x}v_{1y} = 0, \\ \boldsymbol{h}_2 = (v_{0x}^2 - v_{2x}^2) - v_{1x}^2 + v_{1y}^2 = 0, \end{cases} \tag{3.9}$$

has a solution $(\lambda,\mu) \in [0,1;0,1]$, where v_{ix}, v_{iy} is shown in (3.8), see Appendix I for details. That is, the AH Bézier curve obtained by using (3.7) as the control points and the curve obtained by the rotation, translation, and expansion of the curve are all PH curves. Therefore, the following theorem can be obtained.

Theorem 3.3 *Given a $\triangle(\beta,\rho) = \boldsymbol{b}_0\boldsymbol{O}\boldsymbol{b}_3$, $0<\beta<\pi$, $\rho>0$. Then as long as the points $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are selected on the two edges $\boldsymbol{b}_0\boldsymbol{O}$, $\boldsymbol{O}\boldsymbol{b}_3$, respectively, and satisfy*

$$\frac{\|\boldsymbol{b}_1-\boldsymbol{b}_0\|}{\|\boldsymbol{O}-\boldsymbol{b}_0\|} = \lambda,\quad \frac{\|\boldsymbol{b}_3-\boldsymbol{b}_2\|}{\|\boldsymbol{b}_3-\boldsymbol{O}\|} = \mu,$$

where (λ,μ) is the solution of the equation (3.9) in [0,1;0,1], the cubic AH Bézier curve generated by $\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$ is a PH curve.

The AH Bézier curve has a symmetrical property, that is, the AH Bézier curve generated by the control vertex $\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$ and the AH Bézier curve generated by the control vertex $\boldsymbol{b}_3\boldsymbol{b}_2\boldsymbol{b}_1\boldsymbol{b}_0$ are the same curve, and they only differ by one parameter transformation. Therefore, we have the following result.

Corollary 3.4 *Given $\triangle(\beta,\rho) = \boldsymbol{b}_0\boldsymbol{O}\boldsymbol{b}_3$, if the ratio (λ^*,μ^*) is selected according to Theorem 3.3, the AH Bézier curve generated by the control points is a PH curve, that is, (λ^*,μ^*) is the solution of the system $F(\beta,\rho)$ in [0,1;0,1]. Then for $\triangle(\beta,\frac{1}{\rho}) = \boldsymbol{b}_3\boldsymbol{O}\boldsymbol{b}_0$, the AH Bézier curve generated with the control points selected by the ratio (μ^*,λ^*) is also a PH curve, and (μ^*,λ^*) is the solution of the equation system $F(\beta,\frac{1}{\rho})$ in $[0,1;0,1]$.*

Example 3.5 Figure 3 is presented to illustrate Theorem 3.3. Taking $\alpha = \frac{\pi}{4}$, we can construct a cubic PH-H curve starting from a $\triangle(\beta, \rho) = \boldsymbol{b}_0\boldsymbol{O}\boldsymbol{b}_3$ with $\boldsymbol{b}_0 = (1, 0), \boldsymbol{b}_3 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, i.e., $(\beta, \rho) = (\frac{\pi}{4}, 1)$. By calculating the equation system $F(\beta, \rho)$, we can get

$$(\lambda, \mu) = (0.4373,\ 0.4373).$$

Then by (3.7), we have

$$\boldsymbol{b}_1 = (0.5627,\ 0.0000), \quad \boldsymbol{b}_2 = (0.7071,\ 0.7071).$$

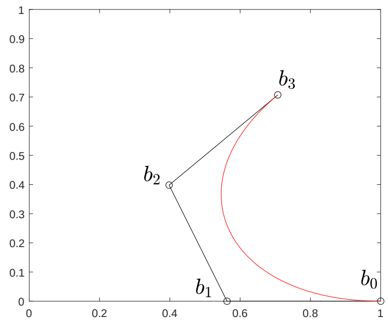


Figure 3 Example 3.5

3.2. The necessary and sufficient condition of cubic PH-H curve based on AH Bézier basis functions $\{Z_{i,3}\}_{i=0}^3$

Now we will discuss the necessary and sufficient condition that a cubic AH Bézier curve is PH-H curve using the AH Bézier basis functions $\{Z_{i,3}\}_{i=0}^3$. The curve $\boldsymbol{P}(t)$ in the form of (2.1) is a PH-H curve if its parametric speed σ satisfies

$$\sigma \in \Gamma_{n-1}, \quad \sigma^2 = \dot{\boldsymbol{P}}(t)^T \dot{\boldsymbol{P}}(t).$$

Here, $\boldsymbol{x}^T\boldsymbol{y}$ denotes the scalar product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$ the Euclidean norm of $\boldsymbol{x}$, and $\dot{\boldsymbol{P}}(t)$ denotes the derivation of $\boldsymbol{P}$ with respect to t.

We assume $\triangle\boldsymbol{b}_i = \boldsymbol{b}_{i+1} - \boldsymbol{b}_i$ a forward difference of control points. From (2.1) and the partition of unity, we obtain

$$\boldsymbol{P}(t) = \sum_{i=0}^{3} \boldsymbol{b}_i Z_i = \boldsymbol{b}_0 + \sum_{i=0}^{2} \triangle\boldsymbol{b}_i \sum_{j=i+1}^{3} Z_j.$$

Obviously, the hodograph of $\boldsymbol{P}(t)$ can be expressed as

$$\dot{\boldsymbol{P}}(t) = \triangle\boldsymbol{b}_0 w_0 + \triangle\boldsymbol{b}_1 w_1 + \triangle\boldsymbol{b}_2 w_2, \tag{3.10}$$

where

$$w_i = \sum_{j=i+1}^{3} \dot{Z}_j(t), \quad i = 0, 1, 2,$$

and further in the closed form

$$w_0 = \frac{1 - \cosh(\alpha - t)}{\alpha - \sinh\alpha}, \quad w_2 = \frac{1 - \cosh t}{\alpha - \sinh\alpha},$$

$$w_1=\frac{1-\cosh(\alpha-t)+\cosh\alpha-\cosh t}{\alpha-2\sinh\alpha+\alpha\cosh\alpha}.$$

Denote

$$\boldsymbol{w}=\begin{pmatrix}w_0\\w_1\\w_2\end{pmatrix},\quad G=\begin{pmatrix}\|\triangle\boldsymbol{b}_0\|^2 & \triangle\boldsymbol{b}_0^T\triangle\boldsymbol{b}_1 & \triangle\boldsymbol{b}_0^T\triangle\boldsymbol{b}_2\\ \triangle\boldsymbol{b}_0^T\triangle\boldsymbol{b}_1 & \|\triangle\boldsymbol{b}_1\|^2 & \triangle\boldsymbol{b}_1^T\triangle\boldsymbol{b}_2\\ \triangle\boldsymbol{b}_0^T\triangle\boldsymbol{b}_2 & \triangle\boldsymbol{b}_1^T\triangle\boldsymbol{b}_2 & \|\triangle\boldsymbol{b}_2\|^2\end{pmatrix},\tag{3.11}$$

and

$$\varphi_{ij}=\angle(\triangle\boldsymbol{b}_i,\triangle\boldsymbol{b}_j),\quad \varphi_{ij}\in[0,\pi].\tag{3.12}$$

Note that $\triangle\boldsymbol{b}_i^T\triangle\boldsymbol{b}_j=\|\triangle\boldsymbol{b}_i\|\|\triangle\boldsymbol{b}_j\|\cos\varphi_{ij}$, and the matrix introduced in the (3.11) is Gram matrix of the differences of control points, thus symmetric and positive semidefinite. If $(\triangle\boldsymbol{b}_i)_{i=0}^2$ are linearly independent, it is actually positive definite, and in this case $\boldsymbol{P}(t)$ is necessarily regular [18]. In this notation, the parametric speed σ of an AH Bézier curve satisfies

$$\sigma^2(t)=\dot{\boldsymbol{P}}(t)^T\dot{\boldsymbol{P}}(t)=(w_0,w_1,w_2)G\begin{pmatrix}w_0\\w_1\\w_2\end{pmatrix}=\boldsymbol{w}(t)^TG\boldsymbol{w}(t),\quad t\in[0,\alpha].\tag{3.13}$$

For cubic polynomial curves, there exist simple necessary and sufficient condition on Bézier control polygon to be a PH curve [1,19]. Kozak et al. [18] presented the necessary and sufficient condition of a cycloidal curve to be a PHC curve. Similarly, we can obtain the following result.

Theorem 3.6 *Suppose that a cubic AH Bézier curve $\boldsymbol{P}(t)$, distinct from a line segment, is given in the AH Bézier form (2.1). If all control points are pairwise distinct, i.e., $\boldsymbol{b}_{i+1}\neq\boldsymbol{b}_i$, $i=0,1,2$, the necessary and sufficient conditions for $\boldsymbol{P}(t)$ to be a PH-H curve are*

$$\varphi_{01}=\varphi_{12}\ \text{ and }\ \frac{\|\triangle\boldsymbol{b}_0\|\|\triangle\boldsymbol{b}_2\|}{\|\triangle\boldsymbol{b}_1\|^2}=\rho(\alpha)(\frac{1-\cos^2\varphi_{01}}{1-\cos\varphi_{02}}),\tag{3.14}$$

where

$$\rho(\alpha)=\frac{(2\alpha e^{\alpha}-e^{2\alpha}+1)^2e^{-\alpha}}{2(\alpha e^{\alpha}+\alpha+2-2e^{\alpha})^2}>0,$$

and the angles φ_{ij} are introduced in (3.12). For a cubic PH-H curve, the parametric speed σ is given as

$$\sigma=\sum_{i=0}^{2}\sigma_iw_i=\|\triangle\boldsymbol{b}_0\|w_0+\cos\varphi_{01}\|\triangle\boldsymbol{b}_1\|w_1+\|\triangle\boldsymbol{b}_2\|w_2.$$

If at least one of the control point difference $\triangle\boldsymbol{b}_i$, $i=0,1,2$ vanishes, a cubic PH-H curve $\boldsymbol{P}(t)$ reduces to a line segment.

Example 3.7 Figure 4 presents an example of cubic AH Bézier curve with the parameter $\alpha=\pi$. Its control points are

$$\boldsymbol{b}_0=(2.0000,\ -3.0000),\quad \boldsymbol{b}_1=(-0.3812,\ 0.1749),$$
$$\boldsymbol{b}_2=(-3.6182,\ 2.1153),\quad \boldsymbol{b}_3=(-9.4435,\ 3.0120).$$

By Theorems 3.1 or 3.3, it can be easily checked that it is a cubic PH-H curve. Moreover, we can easily get

$$\varphi_{01} = \varphi_{12} = 22.1895°, \quad \varphi_{02} = 44.3788°, \quad \rho(\alpha) = 3.2844,$$

$$\frac{\|\triangle \boldsymbol{b}_0\|\|\triangle \boldsymbol{b}_2\|}{\|\triangle \boldsymbol{b}_1\|^2} = 1.6422 = \rho(\alpha)(\frac{1-\cos^2\varphi_{01}}{1-\cos\varphi_{02}}).$$

Therefore, it satisfies the conditions in Theorem 3.6.

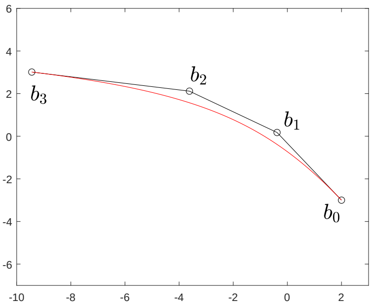


Figure 4 Example 3.7

Remark 3.8 For the two cubic PH-H curves in Example 3.2, it can be easily checked that

$$\varphi_{01} = \varphi_{12} = 52.7037°, \quad \varphi_{02} = 105.4074°,$$

$$\frac{\|\triangle \boldsymbol{b}_0\|\|\triangle \boldsymbol{b}_2\|}{\|\triangle \boldsymbol{b}_1\|^2} = \rho(\alpha)(\frac{1-\cos^2\varphi_{01}}{1-\cos\varphi_{02}}) = 6.9087,$$

and

$$\varphi_{01} = \varphi_{12} = 70.3751°, \quad \varphi_{02} = 140.7501°,$$

$$\frac{\|\triangle \boldsymbol{b}_0\|\|\triangle \boldsymbol{b}_2\|}{\|\triangle \boldsymbol{b}_1\|^2} = \rho(\alpha)(\frac{1-\cos^2\varphi_{01}}{1-\cos\varphi_{02}}) = 1.6422,$$

respectively. Obviously, they both satisfy the conditions in Theorem 3.6.

4. The G^1 Hermite interpolation with cubic PH-H curves

In this section, by recalling the representation of a cubic PH-H curve in Section 3, we consider the problem of interpolating planar G^1 Hermite data by a cubic PH-H curve.

4.1. Hermite interpolation

Consider the construction of a planar PH-H cubic curve $\boldsymbol{P}(t)$ with given end points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$ ($\boldsymbol{P}_0 \neq \boldsymbol{P}_1$), and two normalized tangent direction $\boldsymbol{d}_0$, $\boldsymbol{d}_1$. Find a cubic PH-H curve $\boldsymbol{P}(t)$ which interpolates these data in a geometric G^1 sense, i.e., the curve should satisfy the interpolation conditions

$$\boldsymbol{P}(0) = \boldsymbol{P}_0,\ \boldsymbol{P}(\alpha) = \boldsymbol{P}_1,\ \dot{\boldsymbol{P}}(0) = \lambda_0\boldsymbol{d}_0,\ \dot{\boldsymbol{P}}(\alpha) = \lambda_1\boldsymbol{d}_1,$$

where λ_0 and λ_1 are unknown tangent lengths that should be positive. Since

$$\boldsymbol{w}(0)=\begin{pmatrix}\frac{1-\cosh\alpha}{\alpha-\sinh\alpha}\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}\frac{1}{\nu(\alpha)}\\ 0\\ 0\end{pmatrix},\quad \boldsymbol{w}(\alpha)=\begin{pmatrix}0\\ 0\\ \frac{1}{\nu(\alpha)}\end{pmatrix},$$

from (3.10), the $\dot{\boldsymbol{P}}(0)$ and $\dot{\boldsymbol{P}}(\alpha)$ can be computed as

$$\dot{\boldsymbol{P}}(0)=w_0(0)\triangle\boldsymbol{b}_0=\frac{1}{\nu(\alpha)}\triangle\boldsymbol{b}_0,\quad \dot{\boldsymbol{P}}(\alpha)=w_2(\alpha)\triangle\boldsymbol{b}_2=\frac{1}{\nu(\alpha)}\triangle\boldsymbol{b}_2,$$

then the control points $\{\boldsymbol{b}_i\}$ can be expressed as

$$\boldsymbol{b}_0=\boldsymbol{P}_0,\quad \boldsymbol{b}_1=\boldsymbol{P}_0+\lambda_0\nu(\alpha)\boldsymbol{d}_0,\ \boldsymbol{b}_2=\boldsymbol{P}_1-\lambda_1\nu(\alpha)\boldsymbol{d}_1,\ \boldsymbol{b}_3=\boldsymbol{P}_1 \tag{4.1}$$

where two unknowns λ_0 and λ_1 can be determined by the PH-H condition (3.14). Let us define constants that depend entirely on the data

$$\delta=\|\triangle\boldsymbol{P}_0\|=\|\boldsymbol{P}_1-\boldsymbol{P}_0\|,\quad c_{ij}=\cos\theta_{ij},$$

where

$$\theta_{01}=\angle(\boldsymbol{d}_0,\triangle\boldsymbol{P}_0),\ \theta_{12}=\angle(\triangle\boldsymbol{P}_0,\boldsymbol{d}_1),\ \theta_{02}=\angle(\boldsymbol{d}_0,\boldsymbol{d}_1).$$

Since

$$\triangle\boldsymbol{b}_0=\lambda_0\nu(\alpha)\boldsymbol{d}_0,\ \triangle\boldsymbol{b}_1=\triangle\boldsymbol{P}_0-\nu(\alpha)(\lambda_0\boldsymbol{d}_0+\lambda_1\boldsymbol{d}_1),\ \triangle\boldsymbol{b}_2=\lambda_1\nu(\alpha)\boldsymbol{d}_1,$$

from the first equation in (3.14), we have the equations

$$\cos\varphi_{01}=\cos\varphi_{12}\Longrightarrow\frac{\triangle\boldsymbol{b}_0\triangle\boldsymbol{b}_1}{\|\triangle\boldsymbol{b}_0\|\|\triangle\boldsymbol{b}_1\|}=\frac{\triangle\boldsymbol{b}_1\triangle\boldsymbol{b}_2}{\|\triangle\boldsymbol{b}_1\|\|\triangle\boldsymbol{b}_2\|},$$

then we get the function for the unknowns λ_0 and λ_1

$$e_1(\lambda_0,\lambda_1)=(c_{01}-c_{12})\delta+(c_{02}-1)\nu(\alpha)(\lambda_0-\lambda_1).$$

Similarly, from the second equation in (3.14), we can get

$$\begin{aligned}e_2(\lambda_0,\lambda_1)=&\lambda_0\lambda_1\nu(\alpha)^2(1-c_{02})-\rho(\alpha)(\|\triangle\boldsymbol{b}_0\|^2-(\boldsymbol{d}_0^T\triangle\boldsymbol{b}_1)^2)\\ =&\rho(\alpha)\delta^2(c_{01}^2-1)+2\rho(\alpha)\nu(\alpha)\lambda_1\delta(c_{12}-c_{01}c_{02})+\\ &\lambda_0\lambda_1\nu(\alpha)^2(1-c_{02}).\end{aligned}$$

Since the first function is linear and the second one quadratic, the non-linear system $(e_1(\lambda_0,\lambda_1), e_2(\lambda_0,\lambda_1))=(0,0)$ has two solution pairs. With the help of the functions

$$\begin{aligned}h(z,\alpha)=&\rho(\alpha)(z^2-1)-z+1,\\ g_1(x,y,z,\alpha)=&2\rho(\alpha)(xz-y)+y-x,\\ g_2(x,y,z,\alpha)=&-4\rho(\alpha)(x^2-1)h(z,\alpha)+g_1(x,y,z,\alpha)^2,\\ \zeta^{\pm}(x,y,z,\alpha)=&\frac{2\rho(\alpha)\delta(x^2-1)}{\nu(\alpha)(g_1(x,y,z,\alpha)\mp\sqrt{g_2(x,y,z,\alpha)})}\end{aligned} \tag{4.2}$$

the two solutions $(\lambda_{0,i},\ \lambda_{1,i})$, $i=1,2$ are simplified to

$$\lambda_{0,1}=\zeta^{+}(c_{12},c_{01},c_{02},\alpha),\quad \lambda_{1,1}=\zeta^{+}(c_{01},c_{12},c_{02},\alpha),$$

$$\lambda_{0,2} = \zeta^-(c_{12}, c_{01}, c_{02}, \alpha), \quad \lambda_{1,2} = \zeta^-(c_{01}, c_{12}, c_{02}, \alpha). \tag{4.3}$$

Obviously, g_2 must be nonnegative to obtain real solutions. Moreover, to ensure λ_0, λ_1 to be positive numbers, the specific conditions must be imposed on c_{01}, c_{12}, c_{02}. The following theorem gives the necessary and sufficient condition for G^1 Hermite interpolation with cubic PH-H curves, together with the exact number of solutions.

Theorem 4.1 *Suppose that the data $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{P}_0$ and $\boldsymbol{P}_1$ are prescribed and let $|c_{01}| < 1, |c_{12}| < 1$, the interpolation problem has three cases as follows:*

- *There is precisely one interpolation determined by $(\lambda_{0,1},\ \lambda_{1,1})$ in (4.3), iff*

$$-1 \le c_{02} < \vartheta(\alpha) = -1 + \frac{1}{\rho(\alpha)} \quad \text{or} \quad c_{02} = \vartheta(\alpha),\ c_{01} + c_{12} > 0.$$

- *The interpolation problem has two solutions given by the pairs $(\lambda_{0,1},\ \lambda_{1,1})$ and $(\lambda_{0,2},\ \lambda_{1,2})$, iff*

$$\vartheta(\alpha) < c_{02} < 1,\ c_{01} + c_{12} > 0,\ g_2(c_{01}, c_{12}, c_{02}, \alpha) \ge 0.$$

The two solution pairs coincide iff $g_2(c_{01}, c_{12}, c_{02}, \alpha) = 0$.

- *Otherwise, there are no solutions.*

It is noted that the proof process is similar to the PHC curve in reference [18].

4.2. Examples

To illustrate our method in operation, some numerical examples are provided. To facilitate the analysis, it is convenient to use canonical form data with $\boldsymbol{P}_0 = (0, 0)$, $\boldsymbol{P}_1 = (1, 0)$.

Example 4.2 Let us consider a planar Hermite interpolation data

$$\boldsymbol{d}_0 = (\cos\frac{4\pi}{3},\ \sin\frac{4\pi}{3}),\ \boldsymbol{d}_1 = (\cos\frac{-7\pi}{4},\ \sin\frac{-7\pi}{4}),\ \alpha = \frac{\pi}{2}.$$

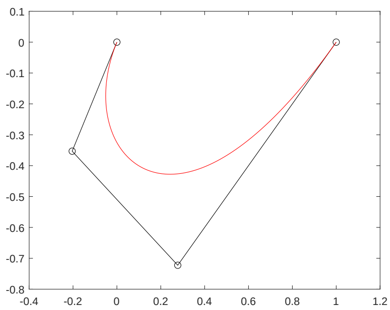


Figure 5 Example 4.2

For this example, since $c_{02} = -0.9659$, $\vartheta(\alpha) = -0.5582$, so $-1 \le c_{02} < \vartheta(\alpha)$. According to the first case of Theorem 4.1, there is only one solution

$$\lambda_{0,1} = 0.8424, \quad \lambda_{1,1} = 2.1109.$$

By (4.1), the corresponding control points are

$$\boldsymbol{b}_0 = (0.0000,\ 0.0000), \quad \boldsymbol{b}_1 = (-0.2039,\ -0.3531),$$
$$\boldsymbol{b}_2 = (0.2775,\ -0.7225), \quad \boldsymbol{b}_3 = (1.0000,\ 0.0000)$$

as shown in Figure 5.

Example 4.3 Suppose that the given data are

$$\boldsymbol{d}_0 = (\cos\frac{\pi}{3},\ \sin\frac{\pi}{3}),\ \boldsymbol{d}_1 = (\cos\frac{-0.16\pi}{4},\ \sin\frac{-0.16\pi}{4}),\ \alpha = \frac{\pi}{4},$$

as shown in Figure 6. It can be easily checked $c_{02} = 0.3875$, $\vartheta(\alpha) = -0.5152$, $c_{01}+c_{12} = 1.4921 > 0$, $g_2(c_{01}, c_{12}, c_{02}, \alpha) = 0.7907 > 0$. Therefore, by Theorem 4.1, there are two different solutions in this case. The red curve in Figure 6 corresponds to the parameters

$$\lambda_{0,1} = 0.1360, \quad \lambda_{1,1} = 3.2680,$$

and the resulting control points are

$$\boldsymbol{b}_0 = (0.0000,\ 0.0000), \quad \boldsymbol{b}_1 = (0.0174,\ 0.0302),$$
$$\boldsymbol{b}_2 = (0.1683,\ 0.1051), \quad \boldsymbol{b}_3 = (1.0000,\ 0.0000),$$

while the blue interpolant corresponds to the real solution

$$\lambda_{0,2} = 3.1753, \quad \lambda_{1,2} = 6.3074,$$

with the control points

$$\boldsymbol{b}_0 = (0.0000,\ 0.0000), \quad \boldsymbol{b}_1 = (0.4073,\ 0.7054),$$
$$\boldsymbol{b}_2 = (-0.6053,\ 0.2028), \quad \boldsymbol{b}_3 = (1.0000,\ 0.0000).$$

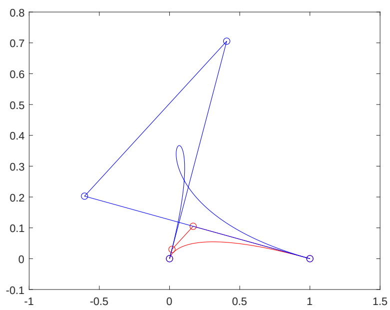


Figure 6 Example 4.3

Example 4.4 For the data

$$\boldsymbol{d}_0 = (\cos\frac{\pi}{3},\ \sin\frac{\pi}{3}),\ \boldsymbol{d}_1 = (\cos\pi,\ \sin\pi),\ \alpha = \frac{\pi}{4}.$$

Note that $c_{01} + c_{12} = -0.5000 < 0$, by Theorem 4.1, there exists no admissible solution. Namely, the tangent lengths $(\lambda_{0,i},\ \lambda_{1,i})$, $i = 1, 2$ are all non-positive numbers.

The above examples demonstrate that the PH-H cubic curves by our method achieve the desired results that preserve tangent directions at the end points with prescribed end points. This method can be applied to shape design since it can conveniently display the intended interpolation effects.

5. Conclusion

AH Bézier curve is a new kind of curve modeling tools proposed in recent years. As a generalization of Bézier curve, it has been widely used in CAD/CAM. By appealing to the advantageous features of PH curves, the present work proposes PH-H curve based on the properties of AH Bézier curve. Significantly, we give the necessary and sufficient conditions of cubic PH-H curve by two different methods. The first method is simple and extraordinary speedy to construct PH-H curves, while the second one is more convenient to extend the curve from the plane to the n-dimension space. Moreover, the problem of constructing a planar cubic G^1 Hermite interpolation with given end points and end tangents is also addressed in this paper using the second definition.

By appealing to the excellent characteristics of PH-H curves, this study is only a basic investigation into the planar cubic PH-H curve. There are several interesting directions in which the present results may possibly be extended, including interpolation of higher-order data, planar quintic PH-H curves, spatial PH-H curves and so on.

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Appendix I: Proof

For the equation (3.9), $\boldsymbol{h}_1$ is a first order equation for λ, which can be obtained as:

$$\begin{aligned}\lambda(\mu) =&\frac{(\mu-1)(\alpha-\mathrm{sinh}\alpha)^2(\rho\mu\mathrm{cos}\beta-\rho\mathrm{cos}\beta+1)}{(3-4\mathrm{cosh}\alpha+\mathrm{cosh}^2\alpha+2\alpha\mathrm{sinh}\alpha-\alpha^2\mathrm{cosh}\alpha)\mu+(2\alpha\mathrm{sinh}\alpha-\mathrm{cosh}^2\alpha-\alpha^2+1)}\\ =&\frac{m(\mu)}{n(\mu)}.\end{aligned} \tag{5.1}$$

Substituting it into $\boldsymbol{h}_2$, a fourth-order polynomial $\boldsymbol{h}_2(\mu)$ about μ can be obtained and

$$\boldsymbol{h}_2(0)=\boldsymbol{h}_{20}(\alpha)=\frac{\rho^2\mathrm{sinh}^2\alpha\mathrm{sin}^2\beta}{(\alpha-2\mathrm{sinh}\alpha+\alpha\mathrm{cosh}\alpha)^2},$$

$$\boldsymbol{h}_2(1)=\boldsymbol{h}_{21}(\alpha)=\frac{-\mathrm{sinh}^2\alpha}{(\alpha-2\mathrm{sinh}\alpha+\alpha\mathrm{cosh}\alpha)^2}.$$

So $\boldsymbol{h}_{20}(\alpha)>0$, $\boldsymbol{h}_{21}(\alpha)<0$ for $\alpha\in(0,+\infty)$. From the continuity of the function, we know that $\boldsymbol{h}_2(\mu)$ has at least one solution in [0,1]. Suppose one of the solutions is μ^*, then we can get $\lambda^*=\lambda(\mu^*)$ from (5.1).

Note that $n(\mu)$ in (5.1) is a linear function with respect to μ, and it can be easily checked that the coefficient of μ: $f(\alpha)=3-4\mathrm{cosh}\alpha+\mathrm{cosh}^2\alpha+2\alpha\mathrm{sinh}\alpha-\alpha^2\mathrm{cosh}\alpha$ is increasing for $\alpha\in[0,+\infty)$, so $f(\alpha)\geq f(0)=0$. Then $n(\mu)$ is increasing and $n(\mu^*)\leq n(1)=4\alpha\mathrm{sinh}\alpha-4\mathrm{cosh}\alpha-\alpha^2-$

$\alpha^2\cosh\alpha + 4 \leq 0$. On the other hand, when $\rho\cos\beta \leq 1$, $m(\mu^*) \leq 0$; when $\rho\cos\beta \geq 0$,

$$\begin{aligned} m(\mu^*) - n(\mu^*) =&(1-\mu^*)(\alpha - \sinh\alpha)^2((1-\mu^*)\rho\cos\beta - 1) - n(\mu^*) \\ \geq&(\mu^* - 1)(\alpha - \sinh\alpha)^2 - n(\mu^*) = -\mu^* \cdot n(1) \geq 0, \end{aligned}$$

so when $0 \leq \rho\cos\beta \leq 1$, there is $\lambda^* = \lambda(\mu^*) \in [0,1]$. When $\rho\cos\beta \geq 1$, consider constructing a curve from $\triangle(\beta, \frac{1}{\rho}) = \boldsymbol{b}_3\boldsymbol{O}\boldsymbol{b}_0$. The corresponding equation system is $F(\beta, \frac{1}{\rho})$. Since $\frac{1}{\rho}\cos\beta \leq 1$, there is a solution. According to Corollary 3.4, there is also a solution for the equation system $F(\beta, \rho)$.

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