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# Weighted EP of Block Operator Matrices

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**Abstract** In this paper, we introduce and study the weighted Moore-Penrose invertible and weighted-EP of block operator matrices in the context of Hilbert spaces, based on the weighted generalized Schur complement. Furthermore, an application of the weighted EP operator in operator equations is given.

**Keywords** EP operator; weighted-EP operator; operator matrix; weighted generalized inverse

**MR(2020) Subject Classification** 15A09; 47A05; 15A27

## 1. Introduction

The basic properties and many applications of Moore-Penrose inverse of a Hilbert space operator are well known. A bounded operator on a Hilbert space is EP, if it is a Moore-Penrose invertible operator (with closed range) and commutes with its Moore-Penrose inverse [1, 2]. In fact, a bounded operator $A$ is EP if and only if the range $\mathcal{R}(A)$ of $A$ is closed and $\mathcal{R}(A) = \mathcal{R}(A^*)$. It is noted that EP objects have been studied in many contexts such as matrices, rings, $C^*$-algebras, Banach spaces and Hilbert $C^*$-modules [3–11].

In recent years, the weighted-EP matrices (that commute with their weighted Moore-Penrose inverse) were introduced and investigated by Tian and Wang in [12]. The notion of weighted-EP matrices was extended to elements of $C^*$-algebras in [13–15] and Banach algebras in [16, 17]. The objective of this article is to introduce and study the weighted Moore-Penrose inverse and weighted-EP properties of block operator matrices on Hilbert spaces. For the study of block operator matrix, please refer to references [18, 19]. The rest of this paper is organized as follows. In Section 2, we recall the basic known definitions and some properties which are used throughout the paper. Furthermore, in Section 3, we discuss the weighted Moore-Penrose inverse of block operator matrices on Hilbert spaces, and some necessary and sufficient conditions for block operator matrices to be weighted-EP operators are investigated on Hilbert spaces. Here we consider the triangular and four block types of partitioned operators. Finally, in Section 4 an application of the weighted EP operator in operator equations is given.

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## 2. Preliminaries

In what follows, $\mathcal{H}$ and $\mathcal{K}$ denote (complex) Hilbert spaces. $\mathcal{B}(\mathcal{H},\mathcal{K})$ stands for the Hilbert space of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we use the notation $\mathcal{B}(\mathcal{H})$ instead of $\mathcal{B}(\mathcal{H},\mathcal{H})$. Let $\mathcal{B}^+(\mathcal{H})$ be the set of positive definite (invertible positive) operators of $\mathcal{B}(\mathcal{H})$. The identity operator on a subspace $\mathcal{M}$ of $\mathcal{H}$ is denoted by $I_{\mathcal{M}}$ (or $I$ when it is clear from the context of the space on which it acts). Given $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, we write $\mathcal{R}(T)$, $\mathcal{N}(T)$, $T^*$, $T^{-1}$, $T^{\frac{1}{2}}$ and $T^-$ for the range, the kernel, the adjoint, the inverse, the square root and the inner inverse of $T$, respectively. For the sake of completeness, we recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is positive if $\langle Th, h\rangle \geq 0$ for all $h \in \mathcal{H}$.

We now let $M \in \mathcal{B}^+(\mathcal{K})$ and $N \in \mathcal{B}^+(\mathcal{H})$. The inner-product on $\mathcal{K}$ induced by $M$ is given by

$$\langle x, y\rangle_M = \langle x, My\rangle \quad \text{for every} \quad x, y \in \mathcal{K}.$$

For each $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$,

$$\langle Tx, y\rangle_M = \langle x, T^\sharp y\rangle_N \quad \text{for every} \quad x \in \mathcal{H},\ y \in \mathcal{K},$$

where $T^\sharp = N^{-1}T^*M \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is called the weighted adjoint operator of $T$ (see [20, Remark 1.1]).

As an extension of the Moore-Penrose inverse, the weighted Moore-Penrose inverse is defined as follows:

**Definition 2.1** ([20,21]) *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be arbitrary and let $M \in \mathcal{B}^+(\mathcal{K})$ and $N \in \mathcal{B}^+(\mathcal{H})$. The weighted Moore-Penrose inverse $T^\dagger_{M,N}$ (if it exists) is the unique element $X$ of $\mathcal{B}(\mathcal{K},\mathcal{H})$, which satisfies*

$$TXT = T,\ XTX = X,\ (MTX)^* = MTX \text{ and } (NXT)^* = NXT.$$

If $M = I_K$ and $N = I_H$, then $T^\dagger_{M,N}$ is denoted simply by $T^\dagger$, which is called the Moore-Penrose inverse of $T$. Clearly, the weighted Moore-Penrose inverse $T^\dagger_{M,N}$ exists if and only if $T$ has a closed range. Then $TT^\dagger_{M,N}$ and $T^\dagger_{M,N}T$ are idempotents, so $\mathcal{R}(T) = \mathcal{R}(TT^\dagger_{M,N})$ and $\mathcal{R}(T^\dagger_{M,N}) = \mathcal{R}(T^\dagger_{M,N}T)$ are closed. Similar to [12, Lemma 3.1] and [20, Remark 1.2], it is not difficult to prove the following facts.

**Lemma 2.2** *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ with a closed range and let $M \in \mathcal{B}^+(\mathcal{K})$ and $N \in \mathcal{B}^+(\mathcal{H})$. Then*

*(1)* $T^\dagger_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}TN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}$;

*(2)* $(T^\dagger_{M,N})^\dagger_{N,M} = T$;

*(3)* $(T^\dagger_{M,N})^* = (T^*)^\dagger_{N^{-1},M^{-1}}$;

*(4)* $\mathcal{R}(T^\dagger_{M,N}) = \mathcal{R}(T^\sharp) = \mathcal{R}(N^{-1}T^*) = N^{-1}\mathcal{R}(T^*)$;

*(5)* $\mathcal{N}(T^\dagger_{M,N}) = \mathcal{N}(T^\sharp) = \mathcal{N}(T^*M) = M^{-1}\mathcal{N}(T^*)$;

*(6)* $\mathcal{N}(T^\dagger_{M,N}T) = \mathcal{N}(T)$, $\mathcal{N}(TT^\dagger_{M,N}) = \mathcal{N}(T^\dagger_{M,N})$.

The definition of generalized Schur complement is given below for later weighted generalized inverse representation of block operator matrices.

Suppose $H \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ is an operator matrix partitioned into the form

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.1}$$

where $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $D \in \mathcal{B}(\mathcal{K})$. The positive definite operators $M$, $N$ are given by

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2^* & M_3 \end{pmatrix} \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K}), \quad N = \begin{pmatrix} N_1 & N_2 \\ N_2^* & N_3 \end{pmatrix} \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K}), \tag{2.2}$$

where $M_1 \in \mathcal{B}^+(\mathcal{H})$, $M_2 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $M_3 \in \mathcal{B}^+(\mathcal{K})$, $N_1 \in \mathcal{B}^+(\mathcal{H})$, $N_2 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $N_3 \in \mathcal{B}^+(\mathcal{K})$.

**Definition 2.3** *Let $H$ be a block operator matrix of the form (2.1) and let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2). Then*

$$S := (H/A)_W = D - CA^\dagger_{M_1,N_1}B, \quad G := (H/D)_W = A - BD^\dagger_{M_3,N_3}C \tag{2.3}$$

*are called the generalized Schur complements of $A$ and $D$ in $H$ in the weighted Moore-Penrose inverse sense, respectively.*

The formula (2.3) has previously appeared in papers dealing with the weighted generalized inverses of partitioned matrices [22]. The generalized Schur complements of $A$ and $D$ in $H$ are

$$S_1 := H/A = D - CA^-B, \quad G_1 := H/D = A - BD^-C \tag{2.4}$$

in the inner inverse (non-weighted) sense, where $A^-$ and $D^-$ are the inner inverse of $A$ and $D$, respectively. The formula (2.4) has previously appeared in papers dealing with generalized inverses of partitioned matrices [23–25].

Similar to [26, Lemma 2.2.4], we have the following conclusion on Hilbert spaces.

**Lemma 2.4** *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Suppose that the inner inverse $A^-$ of $A$ exists. Then*

*(1) $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ if and only if $C = CA^-A$;*

*(2) $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ if and only if $B = AA^-B$.*

The following properties immediately follow from the definition of EP operators.

**Proposition 2.5** ([1, 2]) *Let $T \in \mathcal{B}(\mathcal{H})$ with a closed range. Then the following statements are equivalent:*

*(1) $T$ is an EP operator;*

*(2) $\mathcal{N}(T) = \mathcal{N}(T^*)$;*

*(3) $T$ is Moore-Penrose invertible and $T^\dagger T = TT^\dagger$.*

As an extension of the EP operator, the weighted EP operator is defined as follows:

**Definition 2.6** *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be weighted-EP with respect to $M, N \in \mathcal{B}^+(\mathcal{H})$, or shortly said to be weighted-EP w.r.t. $(M, N)$, if both $MT$ and $TN^{-1}$ are EP, that is,*

$$\mathcal{R}(MT) = \mathcal{R}((MT)^*), \ \mathcal{R}(TN^{-1}) = \mathcal{R}((TN^{-1})^*)$$

*hold and $\mathcal{R}(T)$ is closed.*

**Remark 2.7** It follows from the definitions of EP and weighted-EP operators that every weighted-EP operator needs to be an operator with a closed range.

In the following proposition, similar to [12, Theorem 3.5] and [13, Theorem 2.2], a number of necessary and sufficient conditions for an operator to be weighted-EP are presented on Hilbert spaces.

**Proposition 2.8** *Let $T \in \mathcal{B}(\mathcal{H})$ with a closed range and let $M, N \in \mathcal{B}^+(\mathcal{H})$. Then the following statements are equivalent:*

*(1) $T$ is weighted-EP w.r.t. $(M, N)$, i.e., $MT$ and $TN^{-1}$ are EP (or both $MT$ and $NT$ are EP);*

*(2) $T$ is weighted-EP w.r.t. $(N, M)$, i.e., $NT$ and $TM^{-1}$ are EP (or both $TN^{-1}$ and $TM^{-1}$ are EP);*

*(3) $T$ is weighted-EP both w.r.t. $(M, M)$ and w.r.t. $(N, N)$;*

*(4) $\mathcal{R}(MT) = \mathcal{R}(NT) = \mathcal{R}(T^*)$;*

*(5) $\mathcal{R}(M^{-1}T^*) = \mathcal{R}(N^{-1}T^*) = \mathcal{R}(T)$;*

*(6) $\mathcal{R}(T^\dagger_{M,N}) = \mathcal{R}(T)$ and $\mathcal{R}((T^\dagger_{M,N})^*) = \mathcal{R}(T^*)$;*

*(7) $\mathcal{N}(T^*M) = \mathcal{N}(T^*N) = \mathcal{N}(T)$;*

*(8) $\mathcal{N}(TM^{-1}) = \mathcal{N}(TN^{-1}) = \mathcal{N}(T^*)$;*

*(9) $T^*$ is weighted-EP w.r.t. $(M^{-1}, N^{-1})$;*

*(10) $T^\dagger$ is weighted-EP w.r.t. $(M^{-1}, N^{-1})$;*

*(11) $T^\dagger_{M,N}$ is weighted-EP w.r.t. $(M, N)$;*

*(12) $TT^\dagger_{M,N} = T^\dagger_{M,N}T$.*

## 3. Weighted-EP of block operator matrices

In this section, the weighted Moore-Penrose inverse and weighted-EP of block operator matrices will be investigated in the context of Hilbert spaces, based on the generalized Schur complement.

First, to get the main conclusions, the weighted Moore-Penrose inverse of block operator matrices will be considered in the context of Hilbert spaces, based on the generalized Schur complement.

Extending [22, Theorem 4 and Corollary 7] to infinite dimensional spaces, we have the following two conclusions on Hilbert spaces.

**Lemma 3.1** *Let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2). Assume that $H$ is an operator matrix of the form (2.1) with $\mathcal{R}(A)$ and $\mathcal{R}(S)$ being closed such that*

$$(M_2SS^\dagger_{M_3,N_3})^* = M_2^*AA^\dagger_{M_1,N_1} + M_2^*(I - AA^\dagger_{M_1,N_1})BS^\dagger_{M_3,N_3}CA^\dagger_{M_1,N_1}, \tag{3.1}$$

$$(N_2S^\dagger_{M_3,N_3}S)^* = N_2^*A^\dagger_{M_1,N_1}A + N_2^*A^\dagger_{M_1,N_1}BS^\dagger_{M_3,N_3}C(I - A^\dagger_{M_1,N_1}A), \tag{3.2}$$

$$S^\dagger_{M_3,N_3}M_3(I - SS^\dagger_{M_3,N_3}) = (I - S^\dagger_{M_3,N_3}S)M_3S^\dagger_{M_3,N_3}. \tag{3.3}$$

*Then*

$$H^{\dagger}_{M,N} = \begin{pmatrix} A^{\dagger}_{M_1,N_1} + A^{\dagger}_{M_1,N_1} B S^{\dagger}_{M_3,N_3} C A^{\dagger}_{M_1,N_1} & -A^{\dagger}_{M_1,N_1} B S^{\dagger}_{M_3,N_3} \\ -S^{\dagger}_{M_3,N_3} C A^{\dagger}_{M_1,N_1} & S^{\dagger}_{M_3,N_3} \end{pmatrix}$$

*if and only if*

$$CA^{\dagger}_{M_1,N_1} A = C,\ AA^{\dagger}_{M_1,N_1} B = B,\ BS^{\dagger}_{M_3,N_3} S = B,\ SS^{\dagger}_{M_3,N_3} C = C.$$

**Proof** Since $\mathcal{R}(A)$ and $\mathcal{R}(S)$ are closed, $A^{\dagger}_{M_1,N_1}$ and $S^{\dagger}_{M_3,N_3}$ of $A$ and $S$ exist, respectively. Therefore, the proof is analogous to the proof of [22, Theorem 4]. $\square$

**Lemma 3.2** *Let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2). Assume that $H$ is an operator matrix of the form (2.1) with $\mathcal{R}(A)$ and $\mathcal{R}(S)$ being closed such that*

$$CA^{\dagger}_{M_1,N_1} A = C,\ AA^{\dagger}_{M_1,N_1} B = B,\ BS^{\dagger}_{M_3,N_3} S = B,\ SS^{\dagger}_{M_3,N_3} C = C,$$

$$M_2 SS^{\dagger}_{M_3,N_3} = (M_2^* AA^{\dagger}_{M_1,N_1})^*,\ N_2 S^{\dagger}_{M_3,N_3} S = (N_2^* A^{\dagger}_{M_1,N_1} A)^*.$$

*Then*

$$H^{\dagger}_{M,N} = \begin{pmatrix} A^{\dagger}_{M_1,N_1} + A^{\dagger}_{M_1,N_1} B S^{\dagger}_{M_3,N_3} C A^{\dagger}_{M_1,N_1} & -A^{\dagger}_{M_1,N_1} B S^{\dagger}_{M_3,N_3} \\ -S^{\dagger}_{M_3,N_3} C A^{\dagger}_{M_1,N_1} & S^{\dagger}_{M_3,N_3} \end{pmatrix}.$$

Using the generalized Schur complement $G = A - BD^{\dagger}_{M_3,N_3} C$, similar to Lemma 3.2, one can get the following corollary.

**Corollary 3.3** *Let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2). Assume that $H$ is an operator matrix of the form (2.1) with $\mathcal{R}(D)$ and $\mathcal{R}(G)$ being closed such that*

$$BD^{\dagger}_{M_3,N_3} D = B,\ DD^{\dagger}_{M_3,N_3} C = C,\ CG^{\dagger}_{M_1,N_1} G = C,\ GG^{\dagger}_{M_1,N_1} B = B,$$

$$M_2^* GG^{\dagger}_{M_1,N} = (M_2 DD^{\dagger}_{M_3,N_3})^*,\ N_2^* G^{\dagger}_{M_1,N_1} G = (N_2 D^{\dagger}_{M_3,N_3} D)^*.$$

*Then*

$$H^{\dagger}_{M,N} = \begin{pmatrix} G^{\dagger}_{M_1,N_1} & -G^{\dagger}_{M_1,N_1} B D^{\dagger}_{M_3,N_3} \\ -D^{\dagger}_{M_3,N_3} C G^{\dagger}_{M_1,N_1} & D^{\dagger}_{M_3,N_3} + D^{\dagger}_{M_3,N_3} C G^{\dagger}_{M_1,N_1} B D^{\dagger}_{M_3,N_3} \end{pmatrix}.$$

Considering two special positive definite operator matrices $M$ and $N$, we have the following concise conclusion.

**Theorem 3.4** *Let $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_3 \end{pmatrix} \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$, $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_3 \end{pmatrix} \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$. Assume that $H$ is an operator matrix of the form (2.1) with $\mathcal{R}(A)$ and $\mathcal{R}(S)$ being closed. Then*

$$H^{\dagger}_{M,N} = \begin{pmatrix} A^{\dagger}_{M_1,N_1} + A^{\dagger}_{M_1,N_1} B S^{\dagger}_{M_3,N_3} C A^{\dagger}_{M_1,N_1} & -A^{\dagger}_{M_1,N_1} B S^{\dagger}_{M_3,N_3} \\ -S^{\dagger}_{M_3,N_3} C A^{\dagger}_{M_1,N_1} & S^{\dagger}_{M_3,N_3} \end{pmatrix}$$

*if and only if*

$$CA^{\dagger}_{M_1,N_1} A = C,\ AA^{\dagger}_{M_1,N_1} B = B,\ BS^{\dagger}_{M_3,N_3} S = B,\ SS^{\dagger}_{M_3,N_3} C = C.$$

**Proof** The proof is analogous to Lemma 3.1. It is worth noting that the conditions (3.1) and (3.2) of Lemma 3.1 are naturally satisfied, and the condition (3.3) of Lemma 3.1 can be dropped in combination with the present diagonal weights in Theorem 3.4. $\square$

Using the generalized Schur complement $G = A - BD^{\dagger}_{M_3,N_3}C$, similar to Theorem 3.4, one can get the following theorem.

**Corollary 3.5** *Let $M = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_3 \end{smallmatrix}\right) \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$, $N = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_3 \end{smallmatrix}\right) \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$. Assume that $H$ is an operator matrix of the form (2.1) with $\mathcal{R}(D)$ and $\mathcal{R}(G)$ being closed. Then*

$$H^{\dagger}_{M,N} = \begin{pmatrix} G^{\dagger}_{M_1,N_1} & -G^{\dagger}_{M_1,N_1}BD^{\dagger}_{M_3,N_3} \\ -D^{\dagger}_{M_3,N_3}CG^{\dagger}_{M_1,N_1} & D^{\dagger}_{M_3,N_3} + D^{\dagger}_{M_3,N_3}CG^{\dagger}_{M_1,N_1}BD^{\dagger}_{M_3,N_3} \end{pmatrix}$$

*if and only if*

$$BD^{\dagger}_{M_3,N_3}D = B,\ DD^{\dagger}_{M_3,N_3}C = C,\ CG^{\dagger}_{M_1,N_1}G = C,\ GG^{\dagger}_{M_1,N_1}B = B.$$

For an upper triangular operator matrix, we can get the following concise theorem.

**Theorem 3.6** *Let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2). Assume that $H = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ with $\mathcal{R}(A)$ and $\mathcal{R}(D)$ being closed such that*

$$M_2DD^{\dagger}_{M_3,N_3} = (M_2^*AA^{\dagger}_{M_1,N_1})^*,\ N_2D^{\dagger}_{M_3,N_3}D = (N_2^*A^{\dagger}_{M_1,N_1}A)^*.$$

*Then*

$$H^{\dagger}_{M,N} = \begin{pmatrix} A^{\dagger}_{M_1,N_1} & -A^{\dagger}_{M_1,N_1}BD^{\dagger}_{M_3,N_3} \\ 0 & D^{\dagger}_{M_3,N_3} \end{pmatrix}$$

*if and only if*

$$AA^{\dagger}_{M_1,N_1}B = B,\quad BD^{\dagger}_{M_3,N_3}D = B.$$

**Proof** It can be easily verified according to the proof of Lemma 3.1. Note that the condition (3.3) of Lemma 3.1 is not required in Theorem 3.6. □

Secondly, using the properties of generalized inverses, weighted generalized inverses and generalized Schur complement, we study the weighted-EP properties of block operator matrices on Hilbert spaces.

For the general weights $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ of the form (2.2), we have only the following sufficient condition.

**Theorem 3.7** *Let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2) and let $H$ be an operator matrix of the form (2.1). Assume that $A$ and $S$ are weighted-EP operators such that*

$$CA^{\dagger}_{M_1,N_1}A = C,\ AA^{\dagger}_{M_1,N_1}B = B,\ BS^{\dagger}_{M_3,N_3}S = B,\ SS^{\dagger}_{M_3,N_3}C = C,$$

$$M_2SS^{\dagger}_{M_3,N_3} = (M_2^*AA^{\dagger}_{M_1,N_1})^*,\ N_2S^{\dagger}_{M_3,N_3}S = (N_2^*A^{\dagger}_{M_1,N_1}A)^*.$$

*Then $H$ is a weighted-EP operator matrix.*

**Proof** According to the assumption, by the Lemma 3.2, $H^{\dagger}_{M,N}$ exists and $H^{\dagger}_{M,N}$ is given by

$$H^{\dagger}_{M,N} = \begin{pmatrix} A^{\dagger}_{M_1,N_1} + A^{\dagger}_{M_1,N_1}BS^{\dagger}_{M_3,N_3}CA^{\dagger}_{M_1,N_1} & -A^{\dagger}_{M_1,N_1}BS^{\dagger}_{M_3,N_3} \\ -S^{\dagger}_{M_3,N_3}CA^{\dagger}_{M_1,N_1} & S^{\dagger}_{M_3,N_3} \end{pmatrix}.$$

By $AA^{\dagger}_{M_1,N_1}B=B$ and $SS^{\dagger}_{M_3,N_3}C=C$, $HH^{\dagger}_{M,N}$ is described as the form

$$HH^{\dagger}_{M,N}=\begin{pmatrix} AA^{\dagger}_{M_1,N_1} & 0 \\ 0 & SS^{\dagger}_{M_3,N_3} \end{pmatrix}.$$

Similarly, by $CA^{\dagger}_{M_1,N_1}A=C$ and $BS^{\dagger}_{M_3,N_3}S=B$, $H^{\dagger}_{M,N}H$ is given by

$$H^{\dagger}_{M,N}H=\begin{pmatrix} A^{\dagger}_{M_1,N_1}A & 0 \\ 0 & S^{\dagger}_{M_3,N_3}S \end{pmatrix}.$$

Since $A$ and $S$ are weighted-EP, $AA^{\dagger}_{M_1,N_1}=A^{\dagger}_{M_1,N_1}A$ and $SS^{\dagger}_{M_3,N_3}=S^{\dagger}_{M_3,N_3}S$. Thus

$$HH^{\dagger}_{M,N}=H^{\dagger}_{M,N}H.$$

Therefore, by Proposition 2.8, $H$ is weighted-EP. $\square$

Using the generalized Schur complement $G=A-BD^{\dagger}_{M_3,N_3}C$, similar to Theorem 3.7, one can get the following corollary.

**Corollary 3.8** *Let $M,N\in\mathcal{B}^{+}(\mathcal{H}\oplus\mathcal{K})$ be two operator matrices of the form (2.2) and let $H$ be an operator matrix of the form (2.1). Assume that $D$ and $G$ are weighted-EP operators such that*

$$BD^{\dagger}_{M_3,N_3}D=B,\ DD^{\dagger}_{M_3,N_3}C=C,\ CG^{\dagger}_{M_1,N_1}G=C,\ GG^{\dagger}_{M_1,N_1}B=B,$$

$$M_2^{*}GG^{\dagger}_{M_1,N}=(M_2DD^{\dagger}_{M_3,N_3})^{*},\ N_2^{*}G^{\dagger}_{M_1,N_1}G=(N_2D^{\dagger}_{M_3,N_3}D)^{*}.$$

*Then $H$ is a weighted-EP operator matrix.*

When considering diagonal weights $M$ and $N$, we have the following concise result providing a necessary and sufficient condition.

**Theorem 3.9** *Let $M=\left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_3 \end{smallmatrix}\right)\in\mathcal{B}^{+}(\mathcal{H}\oplus\mathcal{K})$, $N=\left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_3 \end{smallmatrix}\right)\in\mathcal{B}^{+}(\mathcal{H}\oplus\mathcal{K})$. Assume that $H$ is an operator matrix of the form (2.1) with $\mathcal{R}(A)$ and $\mathcal{R}(S)$ being closed such that*

$$CA^{\dagger}_{M_1,N_1}A=C,\ AA^{\dagger}_{M_1,N_1}B=B,\ BS^{\dagger}_{M_3,N_3}S=B,\ SS^{\dagger}_{M_3,N_3}C=C.$$

*Then the following conditions are equivalent:*

*(i) $H$ is a weighted-EP operator matrix;*

*(ii) $A$ and $S$ are weighted-EP operators.*

**Proof** Suppose $H$ is weighted-EP, i.e., $MH$ and $NH$ are EP operators. To prove that $A$ and $S$ are weighted-EP, we just need to show that $M_1A$, $N_1A$, $M_3S$ and $N_3S$ are EP. Since $\mathcal{R}(A)$ and $\mathcal{R}(S)$ are closed, let us define the operator matrices

$$L_1:=\begin{pmatrix} I & 0 \\ CA^{\dagger}_{M_1,N_1} & I \end{pmatrix},\ R_1:=\begin{pmatrix} I & BS^{\dagger}_{M_3,N_3} \\ 0 & I \end{pmatrix},\ P:=\begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix},$$

$$L_2:=\begin{pmatrix} I & 0 \\ B^{*}(A^{\dagger})^{*}_{N_1,M_1} & I \end{pmatrix},\ R_2:=\begin{pmatrix} I & C^{*}(S^{\dagger})^{*}_{N_3,M_3} \\ 0 & I \end{pmatrix},\ P^{*}:=\begin{pmatrix} A^{*} & 0 \\ 0 & S^{*} \end{pmatrix}.$$

Obviously, $L_1$, $L_2$, $R_1$ and $R_2$ are invertible. By assumption $CA^{\dagger}_{M_1,N_1}A = C$, $AA^{\dagger}_{M_1,N_1}B = B$, $BS^{\dagger}_{M_3,N_3}S = B$ and $SS^{\dagger}_{M_3,N_3}C = C$, it is clear that $MH$ and $(MH)^*$ can be decomposed as

$$MH = ML_1R_1P, \quad (MH)^* = H^*M = L_2R_2P^*M. \tag{3.4}$$

Since $MH$ is EP, we have

$$\mathcal{N}(MH) = \mathcal{N}((MH)^*) = \mathcal{N}(H^*M).$$

From (3.4) it follows that $\mathcal{N}(P) = \mathcal{N}(P^*M)$. By Lemma 2.4 it is immediate that

$$\mathcal{N}(P) \subseteq \mathcal{N}(P^*M) \Leftrightarrow P^*M = (P^*M)P^-P \tag{3.5}$$

and

$$\mathcal{N}(P) \supseteq \mathcal{N}(P^*M) \Leftrightarrow P = P(P^*M)^-(P^*M), \tag{3.6}$$

hold for inner inverse $P^-$ and $(P^*M)^-$ of $P$ and $P^*M$, respectively. In particular, $P^-$ and $(P^*M)^-$ are given by

$$P^- = \begin{pmatrix} A^- & 0 \\ 0 & S^- \end{pmatrix}, \quad (P^*M)^- = \begin{pmatrix} (A^*M_1)^- & 0 \\ 0 & (S^*M_3)^- \end{pmatrix}.$$

From (3.5) and (3.6), we have

$$\begin{aligned} P^*M = \begin{pmatrix} A^*M_1 & 0 \\ 0 & S^*M_3 \end{pmatrix} &= \begin{pmatrix} A^*M_1 & 0 \\ 0 & S^*M_3 \end{pmatrix}\begin{pmatrix} A^-A & 0 \\ 0 & S^-S \end{pmatrix} \\ &= \begin{pmatrix} (A^*M_1)A^-A & 0 \\ 0 & (S^*M_3)S^-S \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} P = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} &= \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} (A^*M_1)^- & 0 \\ 0 & (S^*M_3)^- \end{pmatrix}\begin{pmatrix} A^*M_1 & 0 \\ 0 & S^*M_3 \end{pmatrix} \\ &= \begin{pmatrix} A(A^*M_1)^-(A^*M_1) & 0 \\ 0 & S(S^*M_3)^-(S^*M_3) \end{pmatrix}. \end{aligned}$$

Hence $A^*M_1 = (A^*M_1)A^-A$ implies $\mathcal{N}(M_1A) = \mathcal{N}(A) \subseteq \mathcal{N}(A^*M_1) = \mathcal{N}((M_1A)^*)$ and $A = A(A^*M_1)^-(A^*M_1)$ implies $\mathcal{N}((M_1A)^*) = \mathcal{N}(A^*M_1) \subseteq \mathcal{N}(A) = \mathcal{N}(M_1A)$. This shows that $\mathcal{N}(M_1A) = \mathcal{N}((M_1A)^*)$. Then $M_1A$ is an EP operator. Similarly, $S^*M_3 = (S^*M_3)S^-S$ implies $\mathcal{N}(M_3S) = \mathcal{N}(S) \subseteq \mathcal{N}(S^*M_3) = \mathcal{N}((M_3S)^*)$ and $S = S(S^*M_3)^-(S^*M_3)$ implies $\mathcal{N}((M_3S)^*) = \mathcal{N}(S^*M_3) \subseteq \mathcal{N}(S) = \mathcal{N}(M_3S)$. This shows that $\mathcal{N}(M_3S) = \mathcal{N}((M_3S)^*)$. Then $M_3S$ is an EP operator.

Similarly, replacing positive definite operator $M$ with $N$ in the above proof, it is effortless to prove that $N_1A$ and $N_3S$ are EP operators. Therefore, by Proposition 2.8, $A$ and $S$ are weighted-EP operators.

Conversely, by Theorem 3.4, $H^{\dagger}_{M,N}$ exists and $H^{\dagger}_{M,N}$ is given by

$$H^{\dagger}_{M,N} = \begin{pmatrix} A^{\dagger}_{M_1,N_1} + A^{\dagger}_{M_1,N_1}BS^{\dagger}_{M_3,N_3}CA^{\dagger}_{M_1,N_1} & -A^{\dagger}_{M_1,N_1}BS^{\dagger}_{M_3,N_3} \\ -S^{\dagger}_{M_3,N_3}CA^{\dagger}_{M_1,N_1} & S^{\dagger}_{M_3,N_3} \end{pmatrix}.$$

Similarly to the proof of Theorem 3.7, it is easy to verify that $H$ is weighted-EP. $\square$

Using the weighted generalized Schur complement $G = A - BD^{\dagger}_{M_3,N_3}C$, similar to Theorem 3.9, one can get the following corollary.

**Corollary 3.10** *Let $M = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_3 \end{smallmatrix}\right) \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$, $N = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_3 \end{smallmatrix}\right) \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$. Assume that $H$ is an operator matrix of the form (2.1) with $\mathcal{R}(D)$ and $\mathcal{R}(G)$ being closed such that*

$$BD^{\dagger}_{M_3,N_3}D = B,\ DD^{\dagger}_{M_3,N_3}C = C,\ CG^{\dagger}_{M_1,N_1}G = C,\ GG^{\dagger}_{M_1,N_1}B = B.$$

*Then the following conditions are equivalent:*

*(i) $H$ is a weighted-EP operator;*

*(ii) $D$ and $G$ are weighted-EP operators.*

**Remark 3.11** In the previous results, we often directly assume the closedness of $\mathcal{R}(S)$ or $\mathcal{R}(G)$. Actually, it is possible to give some condition on the operator entries such that $\mathcal{R}(S)$ or $\mathcal{R}(G)$ is closed. For example, letting $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2) and $H$ an operator matrix of the form (2.1), we have:

(i) If $\mathcal{R}(A)$ is closed such that

$$CA^{\dagger}_{M_1,N_1}A = C,\ AA^{\dagger}_{M_1,N_1}B = B,$$

then $\mathcal{R}(H)$ is closed if and only if $\mathcal{R}(S)$ is closed;

(ii) If $\mathcal{R}(D)$ is closed such that

$$BD^{\dagger}_{M_1,N_1}D = B,\ DD^{\dagger}_{M_1,N_1}C = C,$$

then $\mathcal{R}(H)$ is closed if and only if $\mathcal{R}(G)$ is closed.

In fact, since $\mathcal{R}(A)$ is closed, $A^{\dagger}_{M_1,N_1}$ exists. Let us define the operator matrices

$$L = \begin{pmatrix} I & 0 \\ CA^{\dagger}_{M_1,N_1} & I \end{pmatrix},\ R := \begin{pmatrix} I & A^{\dagger}_{M_1,N_1}B \\ 0 & I \end{pmatrix},\ H' := \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}.$$

By assumption $CA^{\dagger}_{M_1,N_1}A = C$ and $AA^{\dagger}_{M_1,N_1}B = B$, it is clear that $H$ can be decomposed as $H = LH'R$. In view of the invertibility of $L$ and $R$, $\mathcal{R}(H)$ is closed if and only if $\mathcal{R}(H')$ is closed. Since $\mathcal{R}(A)$ is closed, $\mathcal{R}(H)$ is closed if and only if $\mathcal{R}(S)$ is closed, which proves (i). The proof of (ii) is analogous.

Finally, using the properties of generalized inverses and weighted generalized inverses, in view of Theorem 3.7, we have the following conclusions.

**Theorem 3.12** *Let $H = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ and let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2). Assume that $A$ and $D$ are weighted-EP operators such that*

$$AA^{\dagger}_{M_1,N_1}B = B,\ BD^{\dagger}_{M_3,N_3}D = B,$$
$$M_2DD^{\dagger}_{M_3,N_3} = (M_2^*AA^{\dagger}_{M_1,N_1})^*,\ N_2D^{\dagger}_{M_3,N_3}D = (N_2^*A^{\dagger}_{M_1,N_1}A)^*.$$

*Then $H$ is a weighted-EP operator matrix.*

**Proof** The proof is analogous to that of Theorem 3.7. $\square$

**Corollary 3.13** *Let $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $H = \left(\begin{smallmatrix} A & AXD \\ 0 & D \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ and let $M, N \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ be two operator matrices of the form (2.2). Assume that $A$ and $D$ are weighted-EP operators such that*

$$M_2 DD^\dagger_{M_3,N_3} = (M_2^* AA^\dagger_{M_1,N_1})^*, \ \ N_2 D^\dagger_{M_3,N_3} D = (N_2^* A^\dagger_{M_1,N_1} A)^*.$$

*Then $H$ is a weighted-EP operator matrix.*

**Corollary 3.14** *Let $H = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and let $M = \left(\begin{smallmatrix} M_1 & M_1 \\ M_1 & M_1 \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} N_1 & N_1 \\ N_1 & N_1 \end{smallmatrix}\right) \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{H})$. If $A$ is a weighted-EP operator, then $H$ is a weighted-EP operator matrix.*

When considering diagonal weights $M$ and $N$, we have the following concise result providing a necessary and sufficient condition.

**Theorem 3.15** *Let $H = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ with $\mathcal{R}(A)$ and $\mathcal{R}(D)$ being closed. Assume that $M = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_3 \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_3 \end{smallmatrix}\right) \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$ such that*

$$AA^\dagger_{M_1,N_1} B = B, \ \ BD^\dagger_{M_3,N_3} D = B.$$

*Then the following conditions are equivalent:*

*(i) $H$ is a weighted-EP operator matrix;*

*(ii) $A$ and $D$ are weighted-EP operators.*

**Proof** It can be easily verified according to the proofs of Theorems 3.7 and 3.9. □

**Corollary 3.16** *Let $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $H = \left(\begin{smallmatrix} A & AXD \\ 0 & D \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ with $\mathcal{R}(A)$ and $\mathcal{R}(D)$ being closed. Assume that $M = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_3 \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_3 \end{smallmatrix}\right) \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{K})$. Then the following conditions are equivalent:*

*(i) $H$ is a weighted-EP operator matrix;*

*(ii) $A$ and $D$ are weighted-EP operators.*

**Remark 3.17** The factorization $B = AXD$ of Corollaries 3.12 and 3.16 have important applications in operator equations [27–29].

**Corollary 3.18** *Let $H = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ with $\mathcal{R}(A)$ being closed. Assume that $M = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & M_1 \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & N_1 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ are two positive definite operator matrices. Then the following conditions are equivalent:*

*(i) $H$ is a weighted-EP operator matrix;*

*(ii) $A$ is a weighted-EP operator.*

## 4. The Application of weighted EP operators

In this section, let $\mathcal{H}, \mathcal{K}$ and $\mathcal{G}$ be separable Hilbert spaces. We establish the solvability conditions and the general expression for the weighted EP solution to the operator equations

$$AX = C, \quad XB = D, \tag{4.1}$$

where $A, C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B, D \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and $X \in \mathcal{B}(\mathcal{H})$.

For each $T \in \mathcal{B}(\mathcal{H})$ and let $M \in \mathcal{B}^+(\mathcal{H})$, then the weighted adjoint operator

$$T^\sharp = M^{-1}T^*M \in \mathcal{B}(\mathcal{H}).$$

If $T = T^\sharp$, then $T \in \mathcal{B}(\mathcal{H})$ is called the weighted self-adjoint operator with respect to $M \in \mathcal{B}^+(\mathcal{H})$. The set of weighted self-adjoint operators of $\mathcal{B}(\mathcal{H})$ with respect to $M \in \mathcal{B}^+(\mathcal{H})$ will be denoted by $\mathfrak{S}_M(\mathcal{B}(\mathcal{H}))$.

Before considering the mentioned property, some preparation is needed.

**Lemma 4.1** ([17]) *Let $T \in \mathcal{B}(\mathcal{H})$ and consider two positive definite operators $M$, $N \in \mathcal{B}^+(\mathcal{H})$. Then, the following statements are equivalent:*

*(i) $T$ is weighted EP with weights $M$ and $N$;*

*(ii) There exist two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $J \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H})$ isomorphism such that $T = J(T_1 \oplus 0)J^{-1}, T^\dagger_{M,N} = J(T_1^{-1} \oplus 0)J^{-1}, J(I_{\mathcal{H}_1} \oplus 0)J^{-1} \in \mathfrak{S}_M(\mathcal{B}(\mathcal{H}))$ and $J(0 \oplus I_{\mathcal{H}_2})J^{-1} \in \mathfrak{S}_N(\mathcal{B}(\mathcal{H}))$, where $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$.*

**Lemma 4.2** ([30]) *Let $A, C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B, D \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. Suppose that $A$ and $B$ have closed ranges. Then the equation (4.1) has a common solution $X \in \mathcal{B}(\mathcal{H})$ if and only if*

$$\mathcal{N}(A^*) \subseteq \mathcal{N}(C^*),\ \mathcal{N}(B) \subseteq \mathcal{N}(D),\ AD = CB.$$

*In which case, the general common solution is given by*

$$X = A^-C + DB^- - A^-ADB^- + (I_\mathcal{H} - A^-A)Y(I_\mathcal{H} - BB^-),$$

*where $Y \in \mathcal{B}(\mathcal{H})$ is arbitrary.*

Now we consider the weighted EP solution to the equation (4.1). By the Lemma 4.1, for the isomorphism operator $J \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H})$, the solution has the following factorization:

$$X = J\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}J^{-1},\ X^\dagger_{M,N} = J\begin{pmatrix} X_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}J^{-1},$$

where $X_1 \in \mathcal{B}(\mathcal{H}_1)$. Let $\mathcal{R}(A)$, $\mathcal{R}(B)$ be closed, and

$$AJ = \begin{pmatrix} A_1 & A_2 \end{pmatrix},\ CJ = \begin{pmatrix} C_1 & C_2 \end{pmatrix},\ J^{-1}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ J^{-1}D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},$$

where $A_1, C_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{K})$, $A_2, C_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{K})$, $B_1, D_1 \in \mathcal{B}(\mathcal{G}_1, \mathcal{H})$, $B_2, D_2 \in \mathcal{B}(\mathcal{G}_2, \mathcal{H})$, and $\mathcal{R}(A_1)$, $\mathcal{R}(B_1)$ are closed. Then the equation (4.1) has weighted EP solution if and only if operator equations

$$A_1X_1 = C_1,\ X_1B_1 = D_1,\ C_2 = 0,\ D_2 = 0$$

have a common solution. By Lemmas 4.1 and 4.2, we have the following theorem.

**Theorem 4.3** *Let $A, C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B, D \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. Let $\mathcal{R}(A)$, $\mathcal{R}(B)$ be closed and consider two positive definite operators $M$, $N \in \mathcal{B}^+(\mathcal{H})$. Suppose that $J \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H})$ is isomorphism such that*

$$J(I_{\mathcal{H}_1} \oplus 0)J^{-1} \in \mathfrak{S}_M(\mathcal{B}(\mathcal{H})),\ J(0 \oplus I_{\mathcal{H}_2})J^{-1} \in \mathfrak{S}_N(\mathcal{B}(\mathcal{H})),$$

$$AJ = \begin{pmatrix} A_1 & A_2 \end{pmatrix},\ CJ = \begin{pmatrix} C_1 & C_2 \end{pmatrix},\ J^{-1}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ J^{-1}D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},$$

*where $A_1, C_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{K})$, $A_2, C_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{K})$, $B_1, D_1 \in \mathcal{B}(\mathcal{G}_1, \mathcal{H})$, $B_2, D_2 \in \mathcal{B}(\mathcal{G}_2, \mathcal{H})$, and $\mathcal{R}(A_1)$, $\mathcal{R}(B_1)$ are closed. Then the equation (4.1) has a weighted EP solution $X \in \mathcal{B}(\mathcal{H})$ if and only if*

$$\mathcal{N}(A_1^*) \subseteq \mathcal{N}(C_1^*),\ \mathcal{N}(B_1) \subseteq \mathcal{N}(D_1),\ A_1D_1 = C_1B_1,\ C_2 = D_2 = 0.$$

*In this case, the general weighted EP solution of equation (4.1) is given by*

$$X = J \begin{pmatrix} A_1^- C_1 + D_1B_1^- - A_1^-A_1D_1B_1^- + (I_{\mathcal{H}_1} - A_1^-A_1)Y_1(I_{\mathcal{H}_1} - B_1B_1^-) & 0 \\ 0 & 0 \end{pmatrix} J^{-1},$$

*where $Y_1 \in \mathcal{B}(\mathcal{H}_1)$ is arbitrary.*

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