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A Note on a Problem of Sárközy and Sós

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Abstract Let $k, \ell \geq 2$ be positive integers. Let A be an infinite set of nonnegative integers. For $n \in \mathbb{N}$, let $r_{1,k,\ldots,k^{\ell-1}}(A,n)$ denote the number of solutions of $n = a_0 + ka_1 + \cdots + k^{\ell-1}a_{\ell-1}$, $a_0, \ldots, a_{\ell-1} \in A$. In this paper, we show that $r_{1,k,\ldots,k^{\ell-1}}(A,n) = 1$ for all $n \geq 0$ if and only if A is the set of all nonnegative integers such that all its digits in its k^ℓ-adic expansion are smaller than k. This result partially answers a question of Sárközy and Sós on representation for multivariate linear forms.

Keywords representation function; linear form

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. Let $\ell \geq 2$ be a fixed integer and $c_0, \ldots, c_{\ell-1}$ be positive integers. For $A \subseteq \mathbb{N}$, $n \in \mathbb{N}$, let

$$r_{c_0,\ldots,c_{\ell-1}}(A,n) = \sharp\{(a_0, \ldots, a_{\ell-1}) \in A^\ell : n = c_0a_0 + \cdots + c_{\ell-1}a_{\ell-1}\}.$$

In 1997, Sárközy and Sós [1] posed the following problem:

Problem 1.1 For which $(c_1, \ldots, c_k)$ can the representation function $r_{c_1,\ldots,c_k}(A,n)$, counting the number of solutions of $c_1a_1 + \cdots + c_ka_k = n$ $(a_1, \ldots, a_k \in A)$, be constant for $n > N_0$?

In 1962, for fixed positive integer $k \geq 2$, Moser [2] constructed a sequence A such that $r_{1,k}(A,n) = 1$ for all $n \geq 0$. In 2009, Cilleruelo and Rué [3] completely settled the problem of bivariate linear forms by showing that the only cases in which $r_{c_0,c_1}(A,n)$ may be constant are those considered by Moser. In 2009, the author of this paper [4] extended the Erdös-Fuchs theorem to $k > 2$, the author's result implied that if $(c_0, \ldots, c_{\ell-1}) = (1, \ldots, 1)$, then $r_{1,\ldots,1}(A,n)$ is not constant for n large enough. Recently, Rué and Spiegel [5] widely extended the previous results for multivariate linear forms. For example, they showed that for pairwise co-prime numbers $k_1, \ldots, k_d \geq 2$, there does not exist any infinite set of positive integers A such that $r_{k_1,\ldots,k_d}(A,n)$ becomes constant for n large enough. For other related problems we refer to [6,7].

In this paper, we generalize Moser's theorem and obtain the following result:

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Theorem 1.2 *Let k, $\ell \ge 2$ be positive integers. Let A be an infinite set of nonnegative integers. Then $r_{1,k,\ldots,k^{\ell-1}}(A,n) = 1$ for all $n \ge 0$ if and only if*

$$A = \Big\{ \sum_{j=0}^{\infty} r_j k^{\ell j} : r_j \in \mathbb{Z}, 0 \le r_j < k \Big\},$$

where in each sum there are only finitely many $r_j \ne 0$.

2. Proof of Theorem 1.2

Suppose that

$$A = \Big\{ \sum_{j=0}^{\infty} r_j k^{\ell j} : r_j \in \mathbb{Z}, 0 \le r_j < k \Big\},$$

where in each sum there are only finitely many $r_j \ne 0$.

For all $n \ge 0$, we know that n has a unique k^ℓ-adic representation in the form

$$n = \sum_{j=0}^{s} f_j k^{\ell j}, \quad 0 \le f_j < k^\ell, \ 0 \le j \le s. \tag{2.1}$$

For $j = 0, \ldots, s$, since $0 \le f_j < k^\ell$, there exist unique nonnegative integers $0 \le u_0^{(j)}, \ldots, u_{\ell-1}^{(j)} < k$ such that

$$f_j = u_0^{(j)} + u_1^{(j)} k + \cdots + u_{\ell-1}^{(j)} k^{\ell-1}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned} n &= \sum_{j=0}^{s} (u_0^{(j)} + u_1^{(j)} k + \cdots + u_{\ell-1}^{(j)} k^{\ell-1}) k^{\ell j} \\ &= \sum_{j=0}^{s} u_0^{(j)} k^{\ell j} + k \sum_{j=0}^{s} u_1^{(j)} k^{\ell j} + \cdots + k^{\ell-1} \sum_{j=0}^{s} u_{\ell-1}^{(j)} k^{\ell j}. \end{aligned} \tag{2.3}$$

Write

$$a_i = \sum_{j=0}^{s} u_i^{(j)} k^{\ell j}, \quad i = 0, \ldots, l-1.$$

Then $a_i \in A$ $(0 \le i \le \ell - 1)$. By (2.3), we have

$$r_{1,k,\ldots,k^{\ell-1}}(A,n) \ge 1$$

for all $n \ge 0$.

Assume that

$$\begin{aligned} n &= \sum_{j=0}^{s_0} u_0^{(j)} k^{\ell j} + k \sum_{j=0}^{s_1} u_1^{(j)} k^{\ell j} + \cdots + k^{\ell-1} \sum_{j=0}^{s_{\ell-1}} u_{\ell-1}^{(j)} k^{\ell j} \\ &= \sum_{j=0}^{t_0} v_0^{(j)} k^{\ell j} + k \sum_{j=0}^{t_1} v_1^{(j)} k^{\ell j} + \cdots + k^{\ell-1} \sum_{j=0}^{t_{\ell-1}} v_{\ell-1}^{(j)} k^{\ell j}, \end{aligned} \tag{2.4}$$

where $0 \le u_i^{(j)}$, $v_i^{(j)} < k$, $i = 0, \ldots, \ell - 1$. Write

$$s' = \max\{s_0, \ldots, s_{\ell-1}\},$$

$$t' = \max\{t_0, \dots, t_{\ell-1}\}.$$

For $0 \le i \le \ell - 1$, if $s_i < s'$, then let $u_i^{(j)} = 0$ for all $j = s_i + 1, \dots, s'$; if $t_i < t'$, then let $v_i^{(j)} = 0$ for all $j = t_i + 1, \dots, t'$.

By (2.4), we have

$$\begin{aligned} n &= \sum_{j=0}^{s'} (u_0^{(j)} + u_1^{(j)} k + \cdots + u_{\ell-1}^{(j)} k^{\ell-1}) k^{\ell j} \\ &= \sum_{j=0}^{t'} (v_0^{(j)} + v_1^{(j)} k + \cdots + v_{\ell-1}^{(j)} k^{\ell-1}) k^{\ell j}. \end{aligned} \tag{2.5}$$

Since n has a unique k^ℓ-adic representation, by (2.1) and (2.5), we have $s' = t' = s$ and for all $j = 0, \dots, s$, we have

$$\begin{aligned} f_j =& u_0^{(j)} + k u_1^{(j)} + \cdots + k^{\ell-1} u_{\ell-1}^{(j)} \\ =& v_0^{(j)} + k v_1^{(j)} + \cdots + k^{\ell-1} v_{\ell-1}^{(j)}. \end{aligned}$$

Noting that every f_j has a unique k-adic representation, we have $u_i^{(j)} = v_i^{(j)}$, $i = 0, \dots, \ell - 1$. Hence

$$r_{1,k,\dots,k^{\ell-1}}(A, n) = 1$$

for all $n \ge 0$.

On the other hand, for every set A of nonnegative integers, we write the formal power series $f_A(z)$ defined as

$$f_A(z) := f(z) = \sum_{a \in A} z^a.$$

Then

$$\sum_{n=0}^{\infty} r_{1,k,\dots,k^{\ell-1}}(A, n) z^n = \sum_{a_0,\dots,a_{\ell-1} \in A} z^{a_0 + k a_1 + \cdots + k^{l-1} a_{l-1}}.$$

If $r_{1,k,\dots,k^{\ell-1}}(A, n) = 1$ for all $n \ge 0$, then

$$\frac{1}{1-z} = f(z) f(z^k) \cdots f(z^{k^{\ell-1}}). \tag{2.6}$$

Change variable $z := z^k$, we have

$$\frac{1}{1-z^k} = f(z^k) f(z^{k^2}) \cdots f(z^{k^\ell}). \tag{2.7}$$

By (2.6) and (2.7), we have

$$\begin{aligned} f(z) &= \frac{1-z^k}{1-z} f(z^{k^\ell}) \\ &= (1 + z + z^2 + \cdots + z^{k-1}) f(z^{k^\ell}). \end{aligned}$$

By iterating we get

$$f(z) = \prod_{j=0}^{\infty} (1 + z^{(k^\ell)^j} + z^{2(k^\ell)^j} + \cdots + z^{(k-1)(k^\ell)^j}).$$

This product defines an analytic function at the origin, which can be written using its series expansion around $z = 0$. Moreover, by the unique k^ℓ-adic representation of an integer, the Taylor's coefficients of $f(z)$ are either 0 or 1. So the set A is the set of all nonnegative integers such that all its digits in its k^ℓ-adic expansion are smaller than k. $\square$

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