Journal of Mathematical Research with Applications
May, 2022, Vol. 42, No. 3, pp. 261–278
DOI:10.3770/j.issn:2095-2651.2022.03.005
Http://jmre.dlut.edu.cn

Periodic Solutions on Generalized Abel's Differential Equation

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Abstract In this paper, we discuss the generalized Abelian differential equation. By using the fixed point theorem, we obtain sufficient conditions for the existence of two nonzero periodic solutions of the equation. We also discuss the case that there is no nonzero periodic solution and there is a unique nonzero periodic solution.

Keywords generalized Abel's differential equation; fixed point theory; periodic solutions

MR(2020) Subject Classification 34C27; 37C25

1. Introduction

The nonlinear Abel type first-order differential equation

$$\frac{dx}{dt} = a(t)x^3 + b(t)x^2 + c(t)x + d(t) \tag{1.1}$$

plays an important role in many physical and technical applications [1, 2]. The mathematical properties of Eq. (1.1) have been intensively investigated in the mathematical and physical literature [3–7]. Cima, Gasull and Manosas [8] gave the maximum number of polynomial solutions of some integrable Abel polynomial differential equations; Giné and Valls [9] studied the center problem for Abel polynomial differential equations of second kind; Huang and Liang [10] devoted to the investigation of Abel equation by means of Lagrange interpolation formula, they gave a criterion to estimate the number of limit cycles of the Abel's equations; Bülbül and Sezer [11] introduced a numerical power series algorithm which is based on the improved Taylor matrix method for the approximate solution of Abel-type differential equations; Ni et al. [12] discussed the existence and stability of the periodic solutions of Eq. (1.1), and obtained the sufficient conditions which guarnteed the existence and stability of the periodic solutions for Eq. (1.1) from a particular one.

Alwash [13] considered the class of equations:

$$\frac{dx}{dt} = x^n + \alpha(t)x^{n-1} + \beta(t)x^{n-2}, \quad n \in \mathbb{N}^+, \ n \geq 2 \tag{1.2}$$

as a generalized Abel's differential equation, where $\alpha(t), \beta(t)$ are ω-periodic continuous functions. They showed that if $\beta(t) \leq 0$, then this equation has at most two nonzero periodic solutions.

Received March 8, 2021; Accepted February 18, 2022
Supported by the Senior Personnel of Jiangsu University (Grant No. 14JDG176).
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They gave conditions on α and β that imply the equation has exactly two nonzero periodic solutions, one nonzero periodic solution, or no nonzero periodic solutions. Particular cases of this result, with $n = 4$ and $n = 5$, were given in [14, 15].

In this paper, we get the sufficient conditions of no nonzero periodic solutions for Eq. (1.2), and obtain the existence of two nonzero periodic solution of Eq. (1.2) by the fixed point theorem. And we give the range of periodic solutions, which is not obtained in [13]. In addition, we give conditions on α and β that imply the equation has exactly one nonzero periodic solution. The number of periodic solutions of the equation is closely related to the sign of the discriminant $\Delta = \alpha^2(t) - 4\beta(t)$. The conclusion of this paper is an important supplement to [13].

2. Some lemmas and abbreviations

In this section, we give some lemmas and definitions which will be used later.

Lemma 2.1 ([16]) *Consider the equation:*

$$\frac{dx}{dt} = a(t)x + b(t), \tag{2.1}$$

where $a(t), b(t)$ are ω-periodic continuous functions on $\mathbb{R}$. If $\int_0^\omega a(t)\mathrm{d}t \neq 0$, then Eq. (2.1) has a unique ω-periodic continuous solution $\eta(t)$, $\mathrm{mod}\,(\eta(t)) \subseteq \mathrm{mod}\,(a(t), b(t))$, and $\eta(t)$ can be written as follows:

$$\eta(t) = \begin{cases} \int_{-\infty}^t e^{\int_s^t a(\theta)d\theta} b(s)\mathrm{d}s, \int_0^\omega a(t)\mathrm{d}t < 0 \\ -\int_t^{+\infty} e^{\int_s^t a(\theta)d\theta} b(s)\mathrm{d}s, \int_0^\omega a(t)\mathrm{d}t > 0. \end{cases} \tag{2.2}$$

Lemma 2.2 ([17]) *Suppose that an ω-periodic function sequence $\{f_n(t)\}$ is convergent uniformly on any compact set of $\mathbb{R}$, $f(t)$ is an ω-periodic function, and $\mathrm{mod}\,(f_n) \subseteq \mathrm{mod}\,(f)$ $(n = 1, 2, \ldots)$, then $\{f_n(t)\}$ is convergent uniformly on $\mathbb{R}$.*

Lemma 2.3 ([18]) *Suppose $\mathbb{V}$ is a metric space, $\mathbb{C}$ is a convex closed set of $\mathbb{V}$, its boundary is $\partial\mathbb{C}$. If $T : \mathbb{V} \to \mathbb{V}$ is a continuous compact mapping, such that $T(\partial\mathbb{C}) \subseteq \mathbb{C}$, then T has a fixed point on $\mathbb{C}$.*

Definition 2.4 ([17]) *Suppose $f(t)$ is an ω-periodic continuous function on R, then*

$$a(f, \lambda) = \int_0^\omega f(t)e^{-i\lambda t}\mathrm{d}t \tag{2.3}$$

must exist, $a(f, \lambda)$ is called the Fourier coefficient of $f(t)$, the λ such that $a(f, \lambda) \neq 0$ is called the Fourier index of $f(t)$; There is a countable set Λ_f, when $\lambda \in \Lambda_f$, $a(f, \lambda) \neq 0$, as long as $\lambda \notin \Lambda_f$, there must be $a(f, \lambda) = 0$, Λ_f is called the exponential set of $f(t)$.

Definition 2.5 ([17]) *A set of real numbers composed of linear combinations of integer coefficients of elements in Λ_f is called a module or a frequency module of $f(t)$, which is denoted as $\mathrm{mod}\,(f)$, that is*

$$\mathrm{mod}\,(f) = \Big\{\mu | \mu = \sum_{j=1}^N n_j\lambda_j, n_j, N \in Z^+, N \geq 1, \lambda_j \in \Lambda_f\Big\}. \tag{2.4}$$

For the sake of convenience, suppose that $f(t)$ is an ω-periodic continuous function on $\mathbb{R}$, we denote

$$f_M = \sup_{t\in[0,\omega]} f(t), f_L = \inf_{t\in[0,\omega]} f(t). \tag{2.5}$$

3. No nonzero periodic solution

In this section, we give conditions on α and β that imply Eq. (1.2) has no nonzero periodic solution.

Theorem 3.1 *Consider Eq. (1.2), where n is an even number, and $\alpha(t), \beta(t)$ are ω-periodic continuous functions on $\mathbb{R}$. Suppose that the following condition holds:*

$$(H_1)\ \alpha^2(t) - 4\beta(t) < 0,$$

then Eq. (1.2) has no nonzero periodic continuous solution.

Proof The right end of Eq. (1.2) is a polynomial function with continuous coefficient functions, so Eq. (1.2) satisfies the condition of existence and uniqueness theorems for solutions of differential equations. If given initial condition $x(t_0) = 0$, then the unique solution $x(t)$ with initial condition $x(t_0) = 0$ satisfies

$$x(t) = 0.$$

If given initial condition $x(t_0) \neq 0$, then the unique solution $x(t)$ with initial condition $x(t_0) \neq 0$ satisfies

$$x(t) \neq 0.$$

Thus if initial condition $x(t_0) \neq 0$, integrating both sides of (1.2) from 0 to ω, and by (H_1), it follows

$$x(\omega) - x(0) = \int_0^\omega \mathrm{d}x = \int_0^\omega x^{n-2}(x^2 + \alpha(t)x + \beta(t))\mathrm{d}t > 0,$$

thus $x(t)$ cannot be a periodic solution of Eq. (1.2). This completes the proof. □

Theorem 3.2 *Consider Eq. (1.2), where n is an odd number, and $\alpha(t), \beta(t)$ are ω-periodic continuous functions on $\mathbb{R}$. Suppose that the following condition holds:*

$$(H_1)\ \alpha^2(t) - 4\beta(t) < 0,$$

then Eq. (1.2) has no nonzero ω-periodic continuous solution.

Proof The right end of Eq. (1.2) is a polynomial function with continuous coefficient functions, so Eq. (1.2) satisfies the condition of existence and uniqueness theorems for solutions of differential equations.

Consider Eq. (1.2), if given initial condition $x(t_0) = 0$, then the unique solution $x(t)$ of Eq. (1.2) with initial condition $x(t_0) = 0$ satisfies

$$x(t) = 0. \tag{3.1}$$

If given initial condition $x(t_0) \neq 0$, then the unique solution $x(t)$ of Eq. (1.2) with initial condition $x(t_0) \neq 0$ satisfies

$$x(t) \neq 0. \tag{3.2}$$

Thus if initial condition $x(t_0) \neq 0$, $n \neq 3$, then Eq. (1.2) satisfies

$$\frac{1}{3-n}\frac{dx^{3-n}}{dt} = x^2 + \alpha(t)x + \beta(t). \tag{3.3}$$

By (H_1), integrating both sides of (3.3) from 0 to ω gives

$$x^{3-n}(\omega) - x^{3-n}(0) = (3-n)\int_0^{\omega}(x^2 + \alpha(t)x + \beta(t))\mathrm{d}t > 0. \tag{3.4}$$

If initial condition $x(t_0) \neq 0$, $n = 3$, then Eq. (1.2) satisfies

$$\frac{d\ln|x|}{dt} = x^2 + \alpha(t)x + \beta(t). \tag{3.5}$$

By (H_1), integrating both sides of (3.5) from 0 to ω, we have

$$|x(\omega)| - |x(0)| = \int_0^{\omega}(x^2 + \alpha(t)x + \beta(t))\mathrm{d}t > 0. \tag{3.6}$$

Above two cases show that $x(t)$ cannot be a periodic solution of Eq. (1.2). This completes the proof. □

4. One nonzero constant periodic solution

In this section, we give conditions on α and β that imply Eq. (1.2) may have a unique nonzero constant periodic solution.

Theorem 4.1 *Consider Eq. (1.2), where $\alpha(t)$, $\beta(t)$ are ω-periodic continuous functions on $\mathbb{R}$. Suppose that the following condition holds:*

$$(H_1)\ \alpha^2(t) - 4\beta(t) = 0,$$

then Eq. (1.2) has no non-constant periodic solution.

Proof We divide the proof into two cases.

(i) If $\alpha(t) \not\equiv C$ $(C \neq 0)$.

Consider Eq. (1.2), if given initial condition $x(t_0) = 0$, then the unique solution $x(t)$ of Eq. (1.2) with initial condition $x(t_0) = 0$ satisfies

$$x(t) = 0. \tag{4.1}$$

If given initial condition $x(t_0) \neq 0$, then the unique solution $x(t)$ of Eq. (1.2) with initial condition $x(t_0) \neq 0$ satisfies

$$x(t) \neq 0. \tag{4.2}$$

Thus if initial condition $x(t_0) \neq 0$, $n \neq 3$, then Eq. (1.2) satisfies

$$\frac{1}{3-n}\frac{dx^{3-n}}{dt} = x^2 + \alpha(t)x + \beta(t). \tag{4.3}$$

By (H_1), integrating both sides of (4.3) from 0 to ω gives

$$x^{3-n}(\omega) - x^{3-n}(0) = (3-n)\int_0^\omega (x^2+\alpha(t)x+\beta(t))\mathrm{d}t = (3-n)\int_0^\omega (x+\frac{\alpha(t)}{2})^2\mathrm{d}t. \tag{4.4}$$

If initial condition $x(t_0)\neq 0$, $n=3$, then Eq. (1.2) satisfies

$$\frac{d\ln|x|}{dt} = x^2+\alpha(t)x+\beta(t). \tag{4.5}$$

By (H_1), integrating both sides of (4.5) from 0 to ω, we have

$$|x(\omega)| - |x(0)| = \int_0^\omega (x^2+\alpha(t)x+\beta(t))\mathrm{d}t = \int_0^\omega (x+\frac{\alpha(t)}{2})^2\mathrm{d}t. \tag{4.6}$$

If $x(t)\neq -\frac{\alpha(t)}{2}$, both (4.4) and (4.6) imply $x(\omega)\neq x(0)$, so, $x(t)$ cannot be an ω-periodic solution of Eq. (1.2). If $x(t) = -\frac{\alpha(t)}{2}$, then $x(t)$ satisfies neither (4.3) nor (4.5), thus $x(t)$ cannot be an ω-periodic solution of Eq. (1.2) either. Hence, Eq. (1.2) has no non-constant periodic solution.

(ii) If $\alpha(t)\equiv C$ $(C\neq 0)$.

By (H_1) and from (4.4) and (4.6), it is easy for us to see, $x(t) = -\frac{C}{2}$ is a nonzero constant periodic solution of (1.2).

Above two cases show that Eq. (1.2) has no non-constant solution. If Eq. (1.2) has a nonzero periodic solution, the nonzero periodic solution must be nonzero constant periodic solution. This completes the proof. □

5. Two nonzero periodic solutions

In this section, we give conditions on α and β that imply Eq. (1.2) has exactly two nonzero periodic solutions.

Theorem 5.1 *Consider Eq. (1.2), where n is an even number, and $\alpha(t)$, $\beta(t)$ are ω-periodic continuous functions on $\mathbb{R}$. Suppose that the following conditions hold:*

$$(H_1)\quad \alpha^2(t)-4\beta(t) > 0,$$
$$(H_2)\quad (-\alpha-\sqrt{\alpha^2-4\beta})_M < (-\alpha+\sqrt{\alpha^2-4\beta})_L,$$

then Eq. (1.2) has exactly two nonzero ω-periodic continuous solutions.

(1) One ω-periodic continuous solution is $\gamma_1(t)$, and

$$\frac{(-\alpha-\sqrt{\alpha^2-4\beta})_L}{2} \le \gamma_1(t) \le \frac{(-\alpha-\sqrt{\alpha^2-4\beta})_M}{2};$$

(2) Another ω-periodic continuous solution is $\gamma_2(t)$, and

$$\frac{(-\alpha+\sqrt{\alpha^2-4\beta})_L}{2} \le \gamma_2(t) \le \frac{(-\alpha+\sqrt{\alpha^2-4\beta})_M}{2}.$$

Proof By (H_1), Eq. (1.2) can be turned into

$$\frac{dx}{dt} = x^{n-2}(x+\frac{\alpha(t)+\sqrt{\alpha^2(t)-4\beta(t)}}{2})(x+\frac{\alpha(t)-\sqrt{\alpha^2(t)-4\beta(t)}}{2}). \tag{5.1}$$

Denote

$$\lambda_1(t) = \frac{-\alpha(t)-\sqrt{\alpha^2(t)-4\beta(t)}}{2},\ \lambda_2(t) = \frac{-\alpha(t)+\sqrt{\alpha^2(t)-4\beta(t)}}{2}. \tag{5.2}$$

It follows from (H_1) and (H_2) that

$$(\lambda_1)_L \le \lambda_1(t) \le (\lambda_1)_M < (\lambda_2)_L \le \lambda_2(t) \le (\lambda_2)_M. \tag{5.3}$$

It follows from (5.2) that Eq. (5.1) becomes

$$\frac{dx}{dt} = x^{n-2}(x - \lambda_1(t))(x - \lambda_2(t)). \tag{5.4}$$

Next, we divide the proof into three steps.

(1) We prove the existence of the periodic solution $\gamma_1(t)$ of Eq. (1.2).

Suppose

$$\mathbb{S} = \{\Phi(t) \in C(\mathbb{R}, \mathbb{R}) | \Phi(t + \omega) = \Phi(t)\}. \tag{5.5}$$

Given any $\Phi(t), \Psi(t) \in \mathbb{S}$, the distance is defined as follows:

$$\rho(\Phi, \Psi) = \sup_{t \in [0,\omega]} |\Phi(t) - \Psi(t)|, \tag{5.6}$$

thus $(\mathbb{S}, \rho)$ is a complete metric space.

Take a convex closed set $\mathbb{B}_1$ of $\mathbb{S}$ as follows:

$$\mathbb{B}_1 = \{\Phi(t) \in \mathbb{S} | (\lambda_1)_L \le \Phi(t) \le (\lambda_1)_M, \operatorname{mod}(\Phi) \subseteq \operatorname{mod}(\alpha, \beta)\}. \tag{5.7}$$

Given any $\Phi(t) \in \mathbb{B}_1$, consider the following equation:

$$\begin{aligned} \frac{dx}{dt} =& \Phi^{n-2}(t)(x - \lambda_1(t))(\Phi(t) - \lambda_2(t)) \\ =& \Phi^{n-2}(t)(\Phi(t) - \lambda_2(t))x - \Phi^{n-2}(t)(\Phi(t) - \lambda_2(t))\lambda_1(t). \end{aligned} \tag{5.8}$$

By (5.3), (5.7) and n is an even number, we get that

$$|\lambda_1|_M^{n-2}((\lambda_1)_L - (\lambda_2)_M) \le \Phi^{n-2}(t)(\Phi(t) - \lambda_2(t)) \le |\lambda_1|_L^{n-2}((\lambda_1)_M - (\lambda_2)_L) < 0, \tag{5.9}$$

hence we have

$$\int_0^\omega \Phi^{n-2}(t)(\Phi(t) - \lambda_2(t))\mathrm{d}t < 0. \tag{5.10}$$

Since $\Phi(t), \lambda_1(t)$ and $\lambda_2(t)$ are ω-periodic continuous functions, it follows

$$\Phi^{n-2}(t)(\Phi(t) - \lambda_2(t)), \Phi^{n-2}(t)(\Phi(t) - \lambda_2(t))\lambda_1(t)$$

are ω-periodic continuous functions, by (5.10), according to Lemma 2.1, Eq. (5.8) has a unique ω-periodic continuous solution as follows:

$$\eta(t) = -\int_{-\infty}^t e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta) - \lambda_2(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s, \tag{5.11}$$

and

$$\operatorname{mod}(\eta) \subseteq \operatorname{mod}(\Phi^{n-2}(t)(\Phi(t) - \lambda_2(t)), \Phi^{n-2}(t)(\Phi(t) - \lambda_2(t))\lambda_1(t)). \tag{5.12}$$

It follows from (5.2) and (5.7) that

$$\operatorname{mod}(\Phi^{n-2}(t)(\Phi(t) - \lambda_2(t))) \subseteq \operatorname{mod}(\alpha, \beta),$$

$$\operatorname{mod}(\Phi^{n-2}(t)(\Phi(t) - \lambda_2(t))\lambda_1(t)) \subseteq \operatorname{mod}(\alpha, \beta),$$

hence we have

$$\operatorname{mod}(\eta) \subseteq \operatorname{mod}(\alpha, \beta). \tag{5.13}$$

It follows from (5.7), (5.9) and (5.11) that

$$\begin{aligned}
\eta(t) \geq &-(\lambda_1)_L \int_{-\infty}^{t} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s)-\lambda_2(s))\mathrm{d}s \\
=&(\lambda_1)_L \int_{-\infty}^{t} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \mathrm{d}\Big(\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta\Big) \\
=&(\lambda_1)_L [e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}]_{-\infty}^{t} \\
=&(\lambda_1)_L [1-e^{\int_{-\infty}^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}] = (\lambda_1)_L,
\end{aligned}$$

and

$$\begin{aligned}
\eta(t) \leq &-(\lambda_M)_L \int_{-\infty}^{t} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s)-\lambda_2(s))\mathrm{d}s \\
=&(\lambda_1)_M \int_{-\infty}^{t} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \mathrm{d}\Big(\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta\Big) \\
=&(\lambda_1)_M [e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}]_{-\infty}^{t} \\
=&(\lambda_1)_M [1-e^{\int_{-\infty}^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}] = (\lambda_1)_M,
\end{aligned}$$

hence, $\eta(t) \in \mathbb{B}_1$.

Defining a mapping as follows

$$(T\Phi)(t) = -\int_{-\infty}^{t} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s)-\lambda_2(s))\lambda_1(s)\mathrm{d}s, \tag{5.14}$$

thus if given any $\Phi(t) \in \mathbb{B}_1$, then $(T\Phi)(t) \in \mathbb{B}_1$, hence $T : \mathbb{B}_1 \to \mathbb{B}_1$.

Now, we prove that the mapping T is a compact mapping. Consider any sequence $\{\Phi_k(t)\} \subseteq \mathbb{B}_1$ $(k = 1, 2, \ldots)$, then it follows

$$(\lambda_1)_L \leq \Phi_k(t) \leq (\lambda_1)_M, \operatorname{mod}(\Phi_k) \subseteq \operatorname{mod}(\alpha, \beta), \quad k = 1, 2, \ldots. \tag{5.15}$$

On the other hand, $(T\Phi_k)(t) = x_{\Phi_k}(t)$ satisfies

$$\frac{dx_{\Phi_k}(t)}{dt} = \Phi_k^{n-2}(t)(\Phi_k(t)-\lambda_2(t))x_{\Phi_k}(t) - \Phi_k^{n-2}(t)(\Phi_k(t)-\lambda_2(t))\lambda_1(t), \tag{5.16}$$

thus we have

$$|\frac{dx_{\Phi_k}(t)}{dt}| \leq 2|\lambda_1|_M^{n-2}|(\lambda_1)_L - (\lambda_2)_M||\lambda_1|_M, \operatorname{mod}(x_{\Phi_k}(t)) \subseteq \operatorname{mod}(\alpha, \beta), \tag{5.17}$$

hence $\{\frac{dx_{\Phi_k}(t)}{dt}\}$ is uniformly bounded, therefore, $\{x_{\Phi_k}(t)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$. By the theorem of Ascoli-arzela, for any sequence $\{x_{\Phi_k}(t)\} \subseteq \mathbb{B}_1$, there exists a subsequence (also denoted by $\{x_{\Phi_k}(t)\}$) such that $\{x_{\Phi_k}(t)\}$ is convergent uniformly on any compact set of $\mathbb{R}$. By (5.17), combined with Lemma 2.2, $\{x_{\Phi_k}(t)\}$ is convergent uniformly on $\mathbb{R}$, that is to say, T is relatively compact on $\mathbb{B}_1$.

Next, we prove that T is a continuous mapping.

Suppose $\{\Phi_k(t)\} \subseteq \mathbb{B}_1, \Phi(t) \in \mathbb{B}_1$, and

$$\Phi_k(t) \to \Phi(t), \quad k \to \infty. \tag{5.18}$$

It follows from (5.14) that

$$\begin{aligned}
&|(T\Phi_k)(t) - (T\Phi)(t)| \\
&= \Big| \int_{-\infty}^t e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \Phi_k^{n-2}(s)(\Phi_k(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s - \\
&\quad \int_{-\infty}^t e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s \Big| \\
&= \Big| \int_{-\infty}^t e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_2(\theta))\mathrm{d}\theta} [\Phi_k^{n-2}(s)(\Phi_k(s) - \lambda_2(s)) - \Phi^{n-2}(s)(\Phi(s) - \lambda_2(s))]\lambda_1(s)\mathrm{d}s + \\
&\quad \int_{-\infty}^t (e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_2(\theta))\mathrm{d}\theta} - e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta}) \Phi^{n-2}(s)(\Phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s \Big| \\
&= \Big| \int_{-\infty}^t e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_2(\theta))\mathrm{d}\theta} [\Phi_k^{n-2}(s) + \Phi_k^{n-3}(s)\Phi(s) + \cdots \Phi^{n-2}(s) - \\
&\quad \lambda_2(s)(\Phi_k^{n-3}(s) + \Phi_k^{n-4}(s)\Phi(s) + \cdots \Phi^{n-3}(s))](\Phi_k(s) - \Phi(s))\lambda_1(s)\mathrm{d}s + \\
&\quad \int_{-\infty}^t e^{\xi} \Big(\int_s^t [\Phi_k^{n-2}(\theta) + \Phi_k^{n-3}(\theta)\Phi(\theta) + \cdots \Phi^{n-2}(\theta) - \lambda_2(\theta)(\Phi_k^{n-3}(\theta) + \Phi_k^{n-4}(\theta)\Phi(\theta) + \\
&\quad \cdots + \Phi^{n-3}(\theta))](\Phi_k(\theta) - \Phi(\theta))\mathrm{d}\theta \Big) \Phi^{n-2}(s)(\Phi(s) - \lambda_2(s))\lambda_1(s)\mathrm{d}s \Big| \\
&\le \int_{-\infty}^t e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_2(\theta))\mathrm{d}\theta} |\Phi_k^{n-2}(s) + \Phi_k^{n-3}(s)\Phi(s) + \cdots \Phi^{n-2}(s) - \\
&\quad \lambda_2(s)(\Phi_k^{n-3}(s) + \Phi_k^{n-4}(s)\Phi(s) + \cdots \Phi^{n-3}(s))\lambda_1(s)|\mathrm{d}s + \\
&\quad \int_{-\infty}^t e^{\xi} \Big(\int_s^t |\Phi_k^{n-2}(\theta) + \Phi_k^{n-3}(\theta)\Phi(\theta) + \cdots \Phi^{n-2}(\theta) - \lambda_2(\theta)(\Phi_k^{n-3}(\theta) + \Phi_k^{n-4}(\theta)\Phi(\theta) + \\
&\quad \cdots \Phi^{n-3}(\theta))|\mathrm{d}\theta \Big) |\Phi^{n-2}(s)(\Phi(s) - \lambda_2(s))\lambda_1(s)|\mathrm{d}s \rho(\Phi_k, \Phi),
\end{aligned}$$

here, ξ is between $\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_2(\theta))\mathrm{d}\theta$ and $\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta$, thus ξ is between $|\lambda_1|_M^{n-2}((\lambda_1)_L - (\lambda_2)_M)(t-s)$ and $|\lambda_1|_L^{n-2}((\lambda_1)_M - (\lambda_2)_L)(t-s)$, hence we have

$$\begin{aligned}
&|(T\Phi_k)(t) - (T\Phi)(t)| \\
&\le \int_{-\infty}^t e^{|\lambda_1|_L^{n-2}((\lambda_1)_M-(\lambda_2)_L)(t-s)} (|\lambda_1|_M^{n-2} + |\lambda_1|_M^{n-3}|\lambda_1|_M + \cdots + |\lambda_1|_M^{n-2} + \\
&\quad |\lambda_2|_M(|\lambda_1|_M^{n-3} + |\lambda_1|_M^{n-4}|\lambda_1|_M + \cdots + |\lambda_1|_M^{n-3})|\lambda_1|_M)\mathrm{d}s + \\
&\quad \int_{-\infty}^t e^{|\lambda_1|_L^{n-2}((\lambda_1)_M-(\lambda_2)_L)(t-s)} \Big(\int_s^t (|\lambda_1|_M^{n-2} + |\lambda_1|_M^{n-3}|\lambda_1|_M + \cdots + |\lambda_1|_M^{n-2} + \\
&\quad |\lambda_2|_M(|\lambda_1|_M^{n-3} + |\lambda_1|_M^{n-4}|\lambda_1|_M + \cdots + |\lambda_1|_M^{n-3}))\mathrm{d}\theta \Big) \\
&\quad (|\lambda_1|_M^{n-2}|(\lambda_1)_L - (\lambda_2)_M||\lambda_1|_M)\mathrm{d}s \rho(\Phi_k, \Phi) \\
&= \int_{-\infty}^t e^{|\lambda_1|_L^{n-2}((\lambda_1)_M-(\lambda_2)_L)(t-s)} (|\lambda_1|_M^{n-2} + |\lambda_1|_M^{n-3}|\lambda_1|_M + \cdots + |\lambda_1|_M^{n-2} +
\end{aligned}$$

$$
\begin{aligned}
&|\lambda_2|_M(|\lambda_1|_M^{n-3}+|\lambda_1|_M^{n-4}|\lambda_1|_M+\cdots+|\lambda_1|_M^{n-3})|\lambda_1|_M)\mathrm{d}s+\\
&\int_{-\infty}^{t} e^{|\lambda_1|_L^{n-2}((\lambda_1)_M-(\lambda_2)_L)(t-s)}(t-s)(|\lambda_1|_M^{n-2}+|\lambda_1|_M^{n-3}|\lambda_1|_M+\cdots+|\lambda_1|_M^{n-2}+\\
&|\lambda_2|_M(|\lambda_1|_M^{n-3}+|\lambda_1|_M^{n-4}|\lambda_1|_M+\cdots+|\lambda_1|_M^{n-3}))\\
&(|\lambda_1|_M^{n-2}|(\lambda_1)_L-(\lambda_2)_M||\lambda_1|_M)\mathrm{d}s\rho(\Phi_k,\Phi)\\
=&(\frac{(n-1)|\lambda_1|_M^{n-2}+|\lambda_2|_M(n-2)|\lambda_1|_M^{n-2}}{|\lambda_1|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M)}+\\
&\frac{((n-1)||\lambda_1|_M^{n-2}+|\lambda_2|_M(n-2)|\lambda_1|_M^{n-3})(|\lambda_1|_M^{n-2}|(\lambda_1)_L-(\lambda_2)_M||\lambda_1|_M)}{(|\lambda_1|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M))^2})\rho(\Phi_k,\Phi).
\end{aligned}
$$

It follows from (5.18) and above inequality that

$$(T\Phi_k)(t)\to(T\Phi)(t),\quad k\to\infty, \tag{5.19}$$

therefore, T is continuous. By (5.14), it is easy to see that $T(\partial\mathbb{B}_1)\subseteq\mathbb{B}_1$. According to Lemma 2.3, T has at least a fixed point on $\mathbb{B}_1$, the fixed point is the ω-periodic continuous solution $\gamma_1(t)$ of Eq. (1.2), and

$$(\lambda_1)_L\le\gamma_1(t)\le(\lambda_1)_M. \tag{5.20}$$

(2) We prove the existence of the periodic solution $\gamma_2(t)$ of Eq. (1.2).

Suppose

$$\mathbb{S}=\{\Phi(t)\in C(\mathbb{R},\mathbb{R})|\Phi(t+\omega)=\Phi(t)\}. \tag{5.21}$$

Given any $\Phi(t),\Psi(t)\in\mathbb{S}$, the distance is defined as follows:

$$\rho(\Phi,\Psi)=\sup_{t\in[0,\omega]}|\Phi(t)-\Psi(t)|,$$

thus $(\mathbb{S},\rho)$ is a complete metric space.

Take a convex closed set $\mathbb{B}_2$ of $\mathbb{S}$ as follows:

$$\mathbb{B}_2=\{\Phi(t)\in\mathbb{S}|(\lambda_2)_L\le\Phi(t)\le(\lambda_2)_M,\mathrm{mod}(\Phi)\subseteq\mathrm{mod}(\alpha,\beta)\}. \tag{5.22}$$

Given any $\Phi(t)\in\mathbb{B}_2$, consider the following equation:

$$
\begin{aligned}
\frac{dx}{dt}=&\Phi^{n-2}(t)(\Phi(t)-\lambda_1(t))(x-\lambda_2(t))\\
=&\Phi^{n-2}(t)(\Phi(t)-\lambda_1(t))x-\Phi^{n-2}(t)(\Phi(t)-\lambda_1(t))\lambda_2(t).
\end{aligned}\tag{5.23}
$$

By (5.3), (5.22) and n is an even number, we get that

$$0<|\lambda_2|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M)\le\Phi^{n-2}(t)(\Phi(t)-\lambda_1(t))\le|\lambda_2|_M^{n-2}((\lambda_2)_M-(\lambda_1)_L), \tag{5.24}$$

hence we have

$$\int_0^\omega\Phi^{n-2}(t)(\Phi(t)-\lambda_1(t))\mathrm{d}t>0. \tag{5.25}$$

Since $\Phi(t),\lambda_1(t)$ and $\lambda_2(t)$ are ω-periodic continuous functions, it follows

$$\Phi^{n-2}(t)(\Phi(t)-\lambda_1(t)),\ \Phi^{n-2}(t)(\Phi(t)-\lambda_1(t))\lambda_2(t)$$

are ω-periodic continuous functions, by (5.25), according to Lemma 2.1, Eq. (5.23) has a unique ω-periodic continuous solution as follows:

$$\eta(t) = \int_t^{+\infty} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s \tag{5.26}$$

and

$$\mathrm{mod}(\eta) \subseteq \mathrm{mod}(\Phi^{n-2}(t)(\Phi(t) - \lambda_1(t)), \Phi^{n-2}(t)(\Phi(t) - \lambda_1(t))\lambda_2(t)). \tag{5.27}$$

It follows from (5.2), (5.22) that

$$\mathrm{mod}(\Phi^{n-2}(t)(\Phi(t) - \lambda_1(t))) \subseteq \mathrm{mod}(\alpha, \beta),$$

$$\mathrm{mod}(\Phi^{n-2}(t)(\Phi(t) - \lambda_1(t))\lambda_2(t)) \subseteq \mathrm{mod}(\alpha, \beta),$$

hence we have

$$\mathrm{mod}(\eta) \subseteq \mathrm{mod}(\alpha, \beta). \tag{5.28}$$

It follows from (5.22), (5.24) and (5.26) that

$$\begin{aligned}
\eta(t) \geq &(\lambda_2)_L \int_t^{+\infty} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))\mathrm{d}s \\
= &-(\lambda_2)_L \int_t^{+\infty} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \mathrm{d}\Big(\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta) - \lambda_1(\theta))\mathrm{d}\theta\Big) \\
= &-(\lambda_2)_L [e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta}]_t^{+\infty} \\
= &-(\lambda_2)_L [e^{\int_{+\infty}^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} - 1] = (\lambda_2)_L
\end{aligned}$$

and

$$\begin{aligned}
\eta(t) \leq &(\lambda_2)_M \int_t^{+\infty} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))\mathrm{d}s \\
= &-(\lambda_2)_M \int_t^{+\infty} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \mathrm{d}\Big(\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta) - \lambda_1(\theta))\mathrm{d}\theta\Big) \\
= &-(\lambda_2)_M [e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta}]_t^{+\infty} \\
= &-(\lambda_2)_M [e^{\int_{+\infty}^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_2(\theta))\mathrm{d}\theta} - 1] = (\lambda_2)_M,
\end{aligned}$$

hence, $\eta(t) \in \mathbb{B}_2$.

Defining a mapping as follows

$$(T\Phi)(t) = \int_t^{+\infty} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s, \tag{5.29}$$

thus if given any $\Phi(t) \in \mathbb{B}_2$, then $(T\Phi)(t) \in \mathbb{B}_2$, hence $T : \mathbb{B}_2 \to \mathbb{B}_2$. Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\Phi_k(t)\} \subseteq \mathbb{B}_2$ $(k = 1, 2, \ldots)$, then it follows

$$(\lambda_2)_L \leq \Phi_k(t) \leq (\lambda_2)_M, \mathrm{mod}(\Phi_k) \subseteq \mathrm{mod}(\alpha, \beta), \quad k = 1, 2, \ldots. \tag{5.30}$$

On the other hand, $(T\Phi_k)(t) = x_{\Phi_k}(t)$ satisfies

$$\frac{dx_{\Phi_k}(t)}{dt} = \Phi_k^{n-2}(t)(\Phi_k(t) - \lambda_1(t))x_{\Phi_k}(t) - \Phi_k^{n-2}(t)(\Phi_k(t) - \lambda_1(t))\lambda_2(t), \tag{5.31}$$

thus we have

$$|\frac{dx_{\Phi_k}(t)}{dt}| \leq 2|\lambda_2|_M^{n-2}((\lambda_2)_M - (\lambda_1)_L)|\lambda_2|_M, \mathrm{mod}(x_{\Phi_k}(t)) \subseteq \mathrm{mod}(\alpha, \beta), \tag{5.32}$$

hence $\{\frac{dx_{\Phi_k}(t)}{dt}\}$ is uniformly bounded, therefore, $\{x_{\Phi_k}(t)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$. By the theorem of Ascoli-arzela, for any sequence $\{x_{\Phi_k}(t)\} \subseteq \mathbb{B}_2$, there exists a subsequence (also denoted by $\{x_{\Phi_k}(t)\}$) such that $\{x_{\Phi_k}(t)\}$ is convergent uniformly on any compact set of $\mathbb{R}$. By (5.32), combined with Lemma 2.2, $\{x_{\Phi_k}(t)\}$ is convergent uniformly on $\mathbb{R}$, that is to say, T is relatively compact on $\mathbb{B}_2$.

Next, we prove that T is a continuous mapping. Suppose $\{\Phi_k(t)\} \subseteq \mathbb{B}_2, \Phi(t) \in \mathbb{B}_2$, and

$$\Phi_k(t) \to \Phi(t), \quad k \to \infty. \tag{5.33}$$

It follows from (5.29) that

$$\begin{aligned}
&|(T\Phi_k)(t) - (T\Phi)(t)| \\
&= \Big| \int_t^{+\infty} e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \Phi_k^{n-2}(s)(\Phi_k(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s - \\
&\quad \int_t^{+\infty} e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta} \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s \Big| \\
&= \Big| \int_t^{+\infty} e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_1(\theta))\mathrm{d}\theta} [\Phi_k^{n-2}(s)(\Phi_k(s) - \lambda_1(s)) - \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))]\lambda_2(s)\mathrm{d}s + \\
&\quad \int_t^{+\infty} (e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_1(\theta))\mathrm{d}\theta} - e^{\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta)-\lambda_1(\theta))\mathrm{d}\theta}) \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s \Big| \\
&= \Big| \int_t^{+\infty} e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_1(\theta))\mathrm{d}\theta} [\Phi_k^{n-2}(s) + \Phi_k^{n-3}(s)\Phi(s) + \cdots \Phi^{n-2}(s) - \\
&\quad \lambda_1(s)(\Phi_k^{n-3}(s) + \Phi_k^{n-4}(s)\Phi(s) + \cdots \Phi^{n-3}(s))](\Phi_k(s) - \Phi(s))\lambda_2(s)\mathrm{d}s + \\
&\quad \int_t^{+\infty} e^{\xi} \Big(\int_s^t [\Phi_k^{n-2}(\theta) + \Phi_k^{n-3}(\theta)\Phi(\theta) + \cdots \Phi^{n-2}(\theta) - \\
&\quad \lambda_1(\theta)(\Phi_k^{n-3}(\theta) + \Phi_k^{n-4}(\theta)\Phi(\theta) + \cdots \Phi^{n-3}(\theta))] \\
&\quad (\Phi_k(\theta) - \Phi(\theta))\mathrm{d}\theta \Big) \Phi^{n-2}(s)(\Phi(s) - \lambda_1(s))\lambda_2(s)\mathrm{d}s \Big| \\
&\leq \int_t^{+\infty} e^{\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta)-\lambda_2(\theta))\mathrm{d}\theta} |\Phi_k^{n-2}(s) + \Phi_k^{n-3}(s)\Phi(s) + \cdots \Phi^{n-2}(s) - \\
&\quad \lambda_1(s)(\Phi_k^{n-3}(s) + \Phi_k^{n-4}(s)\Phi(s) + \cdots \Phi^{n-3}(s))\lambda_2(s)|\mathrm{d}s + \\
&\quad \int_t^{+\infty} e^{\xi} \Big(\int_s^t |\Phi_k^{n-2}(\theta) + \Phi_k^{n-3}(\theta)\Phi(\theta) + \cdots \Phi^{n-2}(\theta) - \lambda_1(\theta)(\Phi_k^{n-3}(\theta) + \Phi_k^{n-4}(\theta)\Phi(\theta) + \\
&\quad \cdots \Phi^{n-3}(\theta))|\mathrm{d}\theta \Big) |\Phi^{n-2}(s)(\Phi(s) - \lambda_2(s))\lambda_2(s)|\mathrm{d}s \rho(\Phi_k, \Phi),
\end{aligned}$$

here, ξ is between $\int_s^t \Phi_k^{n-2}(\theta)(\Phi_k(\theta) - \lambda_1(\theta))\mathrm{d}\theta$ and $\int_s^t \Phi^{n-2}(\theta)(\Phi(\theta) - \lambda_1(\theta))\mathrm{d}\theta$, thus ξ is between $|\lambda_2|_M^{n-2}((\lambda_2)_M - (\lambda_1)_L)(t-s)$ and $|\lambda_2|_L^{n-2}((\lambda_2)_L - (\lambda_1)_M)(t-s)$, hence we have

$$\begin{aligned}
&|(T\Phi_k)(t) - (T\Phi)(t)| \\
&\quad \leq \int_t^{+\infty} e^{|\lambda_2|_L^{n-2}((\lambda_2)_L - (\lambda_1)_M)(t-s)} (|\lambda_2|_M^{n-2} + |\lambda_2|_M^{n-3}|\lambda_2|_M + \cdots + |\lambda_2|_M^{n-2} +
\end{aligned}$$

$$|\lambda_1|_M(|\lambda_2|_M^{n-3}+|\lambda_2|_M^{n-4}|\lambda_2|_M+\cdots+|\lambda_2|_M^{n-3})|\lambda_2|_M)\mathrm{d}s+$$
$$\int_t^{+\infty}e^{|\lambda_2|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M)(t-s)}\Big(\int_s^t(|\lambda_2|_M^{n-2}(\theta)+|\lambda_2|_M^{n-3}|\lambda_2|_M+\cdots+|\lambda_2|_M^{n-2}+$$
$$|\lambda_1|_M(|\lambda_2|_M^{n-3}+|\lambda_2|_M^{n-4}|\lambda_2|_M+\cdots+|\lambda_2|_M^{n-3}))\mathrm{d}\theta\Big)$$
$$(|\lambda_2|_M^{n-2}((\lambda_2)_M-(\lambda_1)_L)|\lambda_2|_M)\mathrm{d}s\rho(\Phi_k,\Phi)$$
$$=\int_t^{+\infty}e^{|\lambda_2|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M)(t-s)}(|\lambda_2|_M^{n-2}+|\lambda_2|_M^{n-3}|\lambda_2|_M+\cdots+|\lambda_2|_M^{n-2}+$$
$$|\lambda_1|_M(|\lambda_2|_M^{n-3}+|\lambda_2|_M^{n-4}|\lambda_2|_M+\cdots+|\lambda_2|_M^{n-3})|\lambda_2|_M)\mathrm{d}s+$$
$$\int_t^{+\infty}e^{|\lambda_2|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M)(t-s)}(s-t)(|\lambda_2|_M^{n-2}(\theta)+|\lambda_2|_M^{n-3}|\lambda_2|_M+\cdots+|\lambda_2|_M^{n-2}+$$
$$|\lambda_1|_M(|\lambda_2|_M^{n-3}+|\lambda_2|_M^{n-4}|\lambda_2|_M+\cdots+|\lambda_2|_M^{n-3}))$$
$$(|\lambda_2|_M^{n-2}((\lambda_2)_M-(\lambda_1)_L)|\lambda_2|_M)\mathrm{d}s\rho(\Phi_k,\Phi)$$
$$=\Big(\frac{(n-1)|\lambda_2|_M^{n-2}+|\lambda_1|_M(n-2)|\lambda_2|_M^{n-2}}{|\lambda_2|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M)}+$$
$$\frac{((n-1)|\lambda_2|_M^{n-2}+|\lambda_1|_M(n-2)|\lambda_2|_M^{n-3})(|\lambda_2|_M^{n-2}((\lambda_2)_M-(\lambda_1)_L)|\lambda_2|_M)}{(|\lambda_2|_L^{n-2}((\lambda_2)_L-(\lambda_1)_M))^2}\Big)\rho(\Phi_k,\Phi).$$

It follows from (5.33) and above inequality that

$$(T\Phi_k)(t)\to(T\Phi)(t),\quad k\to\infty,\tag{5.34}$$

therefore, T is continuous. By (5.29), it is easy to see that $T(\partial\mathbb{B}_2)\subseteq\mathbb{B}_2$. According to Lemma 2.3, T has at least a fixed point on $\mathbb{B}_2$, the fixed point is the ω-periodic continuous solution $\gamma_2(t)$ of Eq. (1.2), and

$$(\lambda_2)_L\le\gamma_2(t)\le(\lambda_2)_M.\tag{5.35}$$

(3) We prove that Eq. (1.2) has exactly two nonzero periodic solutions.

Without loss of generality, suppose $\gamma_1(t)<0<\gamma_2(t)$. Let us discuss the possible range of $x(t)$ of Eq. (1.2), we divide the initial condition $x(t_0)=x_0$ into the following parts:

$$x_0\in(-\infty,(\lambda_1)_L),[(\lambda_1)_L,(\lambda_1)_M],((\lambda_1)_M,0),x_0=0,(0,(\lambda_2)_L),$$
$$[(\lambda_2)_L,(\lambda_2)_M],((\lambda_2)_M,+\infty).$$

By (5.4), let

$$f(t,x)=x^n+\alpha(t)x^{n-1}+\beta(t)x^{n-2}=x^{n-2}(x-\lambda_1(t))(x-\lambda_2(t)),\tag{5.36}$$

then it follows

$$f'_x(t,x)=(n-2)x^{n-3}(x-\lambda_1(t))(x-\lambda_2(t))+x^{n-2}(x-\lambda_2(t))+x^{n-2}(x-\lambda_1(t)).\tag{5.37}$$

(I) If $x_0\in(-\infty,(\lambda_1)_L)$.

Consider Eq. (5.4), we have $\frac{dx}{dt}|_{(t_0,x_0)}=f(t_0,x_0)>0$, thus $x(t)$ may stay at $(-\infty,(\lambda_1)_L)$ or enter into $[(\lambda_1)_L,(\lambda_1)_M]$ at some time t ($t>t_0$). If $x(t)$ stays at $(-\infty,(\lambda_1)_L)$, then $\frac{dx}{dt}=f(t,x)>0$, thus $x(t)$ cannot be a periodic solution of Eq. (5.4). If $x(t)$ enters into $[(\lambda_1)_L,(\lambda_1)_M]$

at some time t ($t > t_0$), then there is not a t_1 ($t_1 > t_0$) such that $x(t_1) = x(t_0) = x_0$, thus $x(t)$ cannot be a periodic solution of Eq. (5.4) either. So, Eq. (5.4) has no nonzero periodic solution, that is, Eq. (1.2) has no nonzero periodic solution.

(II) If $x_0 \in [(\lambda_1)_L, (\lambda_1)_M]$, then Eq. (1.2) has an ω-periodic continuous solution $x(t) = \gamma_1(t)$ with initial value $x(t_0) = \gamma_1(t_0)$.

It follows from (5.37) that

$$f'_x(t, \lambda_1(t)) < 0, \tag{5.38}$$

$$f'_x(t, \lambda_2(t)) > 0. \tag{5.39}$$

Now, we prove

$$f'_x(t, \gamma_1(t)) < 0. \tag{5.40}$$

We use proof by contradiction. Suppose there is a t^* such that

$$f'_x(t^*, \gamma_1(t^*)) \geq 0. \tag{5.41}$$

We divide it into two cases:

(i) If

$$f'_x(t^*, \gamma_1(t^*)) > 0. \tag{5.42}$$

It follows from (5.38) that

$$f'_x(t^*, \lambda_1(t^*)) < 0. \tag{5.43}$$

From the continuity of $f'_x(t, x)$, we can see that there exists $\zeta_1^{(1)}(t^*)$ such that

$$f'_x(t^*, \zeta_1^{(1)}(t^*)) = 0, \quad \gamma_1(t^*) < \zeta_1^{(1)}(t^*) < \lambda_1(t^*), \tag{5.44}$$

here, $\zeta_1^{(1)}(t^*)$ is between $\lambda_1(t^*)$ and $\gamma_1(t^*)$, hence $\zeta_1^{(1)}(t^*)$ is between $(\lambda_1)_L$ and $(\lambda_1)_M$. Without loss of generality, suppose $\gamma_1(t^*) < \lambda_1(t^*)$, so we have $\gamma_1^{(1)}(t^*) < \zeta_1^{(1)}(t^*) < \lambda_1(t^*)$. And because

$$f(t, \lambda_1(t)) = f(t, 0) = f(t, \lambda_2(t)) = 0. \tag{5.45}$$

According to the differential mean value theorem, we can get that there exist $\zeta_2^{(1)}(t)$, $\zeta_3^{(1)}(t)$, such that

$$f'_x(t, \zeta_2^{(1)}(t)) = 0, \quad \lambda_1(t) < \zeta_2^{(1)}(t) < 0, \tag{5.46}$$

$$f'_x(t, \zeta_3^{(1)}(t)) = 0, \quad 0 < \zeta_3^{(1)}(t) < \lambda_2(t). \tag{5.47}$$

Obviously, we have

$$f'_x(t^*, \zeta_2^{(1)}(t^*)) = 0, \quad \lambda_1(t^*) < \zeta_2^{(1)}(t^*) < 0, \tag{5.48}$$

$$f'_x(t^*, \zeta_3^{(1)}(t^*)) = 0, \quad 0 < \zeta_3^{(1)}(t^*) < \lambda_2(t^*). \tag{5.49}$$

Noting that

$$f'_x(t, \zeta_1^{(1)}(t)) = f'_x(t, \zeta_2^{(1)}(t)) = f'_x(t, 0) = f'_x(t, \zeta_3^{(1)}(t)) = 0, \tag{5.50}$$

by the differential mean value theorem, we can get that there exist $\zeta_1^{(2)}(t)$, $\zeta_2^{(2)}(t)$, $\zeta_3^{(2)}(t)$, such that

$$f''_{xx}(t, \zeta_1^{(2)}(t)) = 0, \quad \zeta_1^{(1)}(t) < \zeta_1^{(2)}(t) < \zeta_2^{(1)}(t), \tag{5.51}$$

$$f''_{xx}(t,\zeta_2^{(2)}(t))=0,\quad \zeta_2^{(1)}(t)<\zeta_2^{(2)}(t)<0, \tag{5.52}$$

$$f''_{xx}(t,\zeta_3^{(2)}(t))=0,\quad 0<\zeta_3^{(2)}(t)<\zeta_3^{(1)}(t). \tag{5.53}$$

Obviously, we have

$$f''_{xx}(t^*,\zeta_1^{(2)}(t^*))=0,\quad \zeta_1^{(1)}(t^*)<\zeta_1^{(2)}(t^*)<\zeta_2^{(1)}(t^*), \tag{5.54}$$

$$f'_{xx'}(t^*,\zeta_2^{(2)}(t^*))=0,\quad \zeta_2^{(1)}<\zeta_2^{(2)}(t^*)<0, \tag{5.55}$$

$$f''_{xx}(t^*,\zeta_3^{(2)}(t^*))=0,\quad 0<\zeta_3^{(2)}(t^*)<\zeta_3^{(1)}(t^*). \tag{5.56}$$

Noting that

$$f''_{xx}(t,\zeta_1^{(2)}(t))=f''_{xx}(t,\zeta_2^{(2)}(t))=f''_{xx}(t,0)=f''_{xx}(t,\zeta_3^{(2)}(t))=0, \tag{5.57}$$

and using the differential mean value theorem, we can get that there exist $\zeta_1^{(3)}(t)$, $\zeta_2^{(3)}(t)$, $\zeta_3^{(3)}(t)$, such that

$$f'''_{xxx}(t,\zeta_1^{(3)}(t))=0,\quad \zeta_1^{(2)}(t)<\zeta_1^{(3)}(t)<\zeta_2^{(2)}(t), \tag{5.58}$$

$$f'''_{xxx}(t,\zeta_2^{(3)}(t))=0,\quad \zeta_2^{(2)}(t)<\zeta_2^{(3)}(t)<0, \tag{5.59}$$

$$f'''_{xxx}(t,\zeta_3^{(3)}(t))=0,\quad 0<\zeta_3^{(3)}(t)<\zeta_3^{(2)}(t). \tag{5.60}$$

Obviously, we have

$$f'''_{xxx}(t^*,\zeta_1^{(3)}(t^*))=0,\quad \zeta_1^{(2)}(t^*)<\zeta_1^{(3)}(t^*)<\zeta_2^{(2)}(t^*), \tag{5.61}$$

$$f'''_{xxx}(t^*,\zeta_2^{(3)}(t^*))=0,\quad \zeta_2^{(2)}(t^*)<\zeta_2^{(3)}(t^*)<0, \tag{5.62}$$

$$f'''_{xxx}(t^*,\zeta_3^{(3)}(t^*))=0,\quad 0<\zeta_3^{(3)}(t^*)<\zeta_3^{(2)}(t^*). \tag{5.63}$$

$$\cdots$$

Proceeding with the above arguments until the $(n-2)$th derivative of $f(t,x)$ with respect to x, we get

$$f^{(n-2)}_{x\cdots x}(t,\zeta_1^{(n-2)}(t))=f^{(n-2)}_{x\cdots x}(t,\zeta_2^{(n-2)}(t))=f^{(n-2)}_{x\cdots x}(t,\zeta_3^{(n-2)}(t))=0. \tag{5.64}$$

By the differential mean value theorem, we can get that there exist $\zeta_1^{(n-1)}(t)$, $\zeta_2^{(n-1)}(t)$ such that

$$f^{(n-1)}_{x\cdots x}(t,\zeta_1^{(n-1)}(t))=0,\quad \zeta_1^{(n-2)}(t)<\zeta_1^{(n-1)}(t)<\zeta_2^{(n-2)}(t), \tag{5.65}$$

$$f^{(n-1)}_{x\cdots x}(t,\zeta_2^{(n-1)}(t))=0,\quad \zeta_2^{(n-2)}(t)<\zeta_2^{(n-1)}(t)<\zeta_3^{(n-2)}(t). \tag{5.66}$$

Obviously, we have

$$f^{(n-1)}_{x\cdots x}(t^*,\zeta_1^{(n-1)}(t^*))=0,\quad \zeta_1^{(n-2)}(t^*)<\zeta_1^{(n-1)}(t^*)<\zeta_2^{(n-2)}(t^*), \tag{5.67}$$

$$f^{(n-1)}_{x\cdots x}(t^*,\zeta_2^{(n-1)}(t^*))=0,\quad \zeta_2^{(n-2)}(t^*)<\zeta_2^{(n-1)}(t^*)<\zeta_3^{(n-2)}(t^*). \tag{5.68}$$

Noting that

$$f^{(n-1)}_{x\cdots x}(t,\zeta_1^{(n-1)}(t))=f^{(n-1)}_{x\cdots x}(t,\zeta_2^{(n-1)}(t))=0, \tag{5.69}$$

and using the differential mean value theorem, we can get that there exists $\zeta_1^{(n)}(t)$ such that

$$f^{(n)}_{x\cdots x}(t,\zeta_1^{(n)}(t))=0. \tag{5.70}$$

Obviously, we have

$$f^{(n)}_{x\cdots x}(t^*,\zeta_1^{(n)}(t^*))=0. \tag{5.71}$$

But this is in contradiction with

$$f^{(n)}_{x\cdots x}(t^*,\zeta_1^{(n)}(t^*))=n_!. \tag{5.72}$$

(ii) If

$$f'_x(t^*,\gamma_1(t^*))=0, \tag{5.73}$$

then as long as we regard $\gamma_1(t^*)$ as $\zeta_1^{(1)}(t^*)$, we can get a contradiction with the same proof as (i).

Both (i) and (ii) show that (5.40) is true.

Remark 5.2 In the above proof, we assume that $\beta(t)\neq 0$. If $\beta(t)=0$, then $\lambda_1(t)=-\alpha(t)$, $\lambda_2(t)=0$, and $x=0$ is the $n-1$ multiple zero of $f(t,x)$. It can also be proved that (5.40) is true.

Now, suppose that there is another ω-periodic continuous solution $\Psi_1(t)$ of Eq. (1.2) which satisfies

$$(\lambda_1)_L\le\Psi_1(t)\le(\lambda_1)_M. \tag{5.74}$$

Because $f(t,x)$ is a polynomial function with continuous partial derivatives to x, Eq. (1.2) satisfies the existence and uniqueness of solutions to initial value problems of differential equations, thus

$$|\gamma_1(t)-\Psi_1(t)|>0,\quad \forall\, t\in R. \tag{5.75}$$

Similar to the analysis of (5.40), we can get

$$f'_x(t,\Psi_1(t))<0. \tag{5.76}$$

Consider the following equation:

$$\begin{aligned}\frac{d[\gamma_1(t)-\Psi_1(t)]}{dt}&=f(t,\gamma_1(t))-f(t,\Psi_1(t))\\ &=f'_x[t,\Psi_1(t)+\theta(\gamma_1(t)-\Psi_1(t))](\gamma_1(t)-\Psi_1(t)),\quad 0<\theta<1,\end{aligned} \tag{5.77}$$

thus we have

$$|\gamma_1(t)-\Psi_1(t)|=|\gamma_1(0)-\Psi_1(0)|e^{\int_0^t f'_x[s,\Psi_1(s)+\theta(\gamma_1(s)-\Psi_1(s))]\mathrm{d}s}. \tag{5.78}$$

It follows from (5.20) and (5.74) that

$$(\lambda_1)_L\le\Psi_1(t)+\theta(\gamma_1(t)-\Psi_1(t))\le(\lambda_1)_M. \tag{5.79}$$

Similar to the analysis of (5.40), we can get

$$f'_x[t,\Psi_1(t)+\theta(\gamma_1(t)-\Psi_1(t))]<0. \tag{5.80}$$

It follows from (5.78) and (5.80) that

$$|\gamma_1(t)-\Psi_1(t)|\to 0,\quad t\to+\infty. \tag{5.81}$$

It follows from (5.75) and (5.81) that this is a contradiction, thus $\Psi_1(t)$ cannot be a periodic solution of Eq. (1.2), that is to say, Eq. (1.2) has a unique ω-periodic continuous solution $\gamma_1(t)$ which satisfies $(\lambda_1)_L \le \gamma_1(t) \le (\lambda_1)_M$.

(III) If $x_0 \in ((\lambda_1)_M, 0)$.

Consider Eq. (5.4), we have $\frac{dx}{dt}|_{(t_0,x_0)} = f(t_0, x_0) < 0$, thus $x(t)$ may stay at $((\lambda_1)_M, 0)$ or enter into $[(\lambda_1)_L, (\lambda_1)_M]$ at some time t $(t > t_0)$. If $x(t)$ stays at $((\lambda_1)_M, 0)$, we have $\frac{dx}{dt} = f(t, x) < 0$, then $x(t)$ cannot be a periodic solution of Eq. (5.4). If $x(t)$ enters into $[(\lambda_1)_L, (\lambda_1)_M]$ at some time t $(t > t_0)$, then there is not a t_1 $(t_1 > t_0)$ such that $x(t_1) = x(t_0) = x_0$, thus $x(t)$ cannot be a periodic solution of Eq. (5.4) either. So, Eq. (5.4) has no nonzero periodic solution, that is, Eq. (1.2) has no nonzero periodic solution.

(IV) If $x_0 = 0$, then the unique solution of Eq. (1.2) with initial value $x_0 = 0$ is the constant periodic solution $x(t) = 0$.

(V) If $x_0 \in (0, (\lambda_2)_L)$.

Consider Eq. (5.4), we have $\frac{dx}{dt}|_{(t_0,x_0)} = f(t_0, x_0) < 0$, thus $x(t)$ $(t > t_0)$ stays at $(0, (\lambda_2)_L)$, and $\frac{dx}{dt} = f(t, x) < 0$, thus $x(t)$ cannot be a periodic solution of Eq. (5.4). So, Eq. (5.4) has no nonzero periodic solution, that is, Eq. (1.2) has no nonzero periodic solution.

Remark 5.3 When $x_0 \in (0, (\lambda_2)_L)$, we have $x(t) \ne 0$ $(t > t_0)$, by $\frac{dx}{dt}|_{(t_0,x_0)} = f(t_0, x_0) < 0$, thus $x(t)$ $(t > t_0)$ stays at $(0, (\lambda_2)_L)$.

(VI) If $x_0 \in [(\lambda_2)_L, (\lambda_2)_M]$.

Similarly to the case (II), Eq. (1.2) has an ω-periodic continuous solution $x(t) = \gamma_2(t)$ with initial value $x(t_0) = \gamma_2(t_0)$.

(VII) If $x_0 \in ((\lambda_2)_M, +\infty)$.

Consider Eq. (5.4), we have $\frac{dx}{dt}|_{(t_0,x_0)} = f(t_0, x_0) > 0$, thus $x(t)$ may stay at $\in ((\lambda_2)_M, +\infty)$ or $x(t) \to +\infty$ $(t \to +\infty)$. If $x(t)$ stays at $((\lambda_2)_M, +\infty)$, we have $\frac{dx}{dt} = f(t, x) > 0$, then $x(t)$ cannot be a periodic solution of Eq. (5.4). If $x(t) \to +\infty$ $(t \to +\infty)$, then $x(t)$ cannot be a periodic solution of Eq. (5.4) either. So, Eq. (5.4) has no nonzero periodic solution, that is, Eq. (1.2) has no nonzero periodic solution.

To sum up, Eq. (1.2) has exactly two nonzero ω-periodic continuous solutions $\gamma_i(t)$ $(i = 1, 2)$ which satisfy

$$(\lambda_i)_L \le \gamma_i(t) \le (\lambda_i)_M, \quad i = 1, 2. \tag{5.82}$$

This is the end of the proof of Theorem 5.1. □

Theorem 5.4 *Consider Eq. (1.2), where n is an odd number, and $\alpha(t), \beta(t)$ are ω-periodic continuous functions on $\mathbb{R}$. Suppose that the following conditions hold:*

$$(H_1)\ \ \alpha^2(t) - 4\beta(t) > 0,$$
$$(H_2)\ \ (-\alpha - \sqrt{\alpha^2 - 4\beta})_M < (-\alpha + \sqrt{\alpha^2 - 4\beta})_L,$$

then Eq. (1.2) has exactly two nonzero ω-periodic continuous solutions.

(1) One ω-periodic continuous solution is $\gamma_1(t)$, and

$$\frac{(-\alpha-\sqrt{\alpha^2-4\beta})_L}{2} \le \gamma_1(t) \le \frac{(-\alpha-\sqrt{\alpha^2-4\beta})_M}{2};$$

(2) Another ω-periodic continuous solution is $\gamma_2(t)$, and

$$\frac{(-\alpha+\sqrt{\alpha^2-4\beta})_L}{2} \le \gamma_2(t) \le \frac{(-\alpha+\sqrt{\alpha^2-4\beta})_M}{2}.$$

The proof of Theorem 5.4 is similar to that of Theorem 5.1, we omit it here.

6. Examples

The following examples show the feasibility of our main results.

Example 6.1 Consider the following equation:

$$\frac{dx}{dt} = x^3 + (\sin t - 6)x^2 + (2 - 3\sin t)x. \tag{6.1}$$

Here, $\alpha(t) = \sin t - 6$, $\beta(t) = 2 - 3\sin t$, and

$$\alpha^2(t) - 4\beta(t) = 28 + \sin^2 t > 0.$$

$$\frac{7-\sqrt{27}}{2} = (-\alpha-\sqrt{\alpha^2-4\beta})_M < (-\alpha+\sqrt{\alpha^2-4\beta})_L = \frac{5+\sqrt{29}}{2},$$

(H_1) and (H_2) of Theorem 5.1 are satisfied. It follows from Theorem 5.1 that Eq. (6.1) has two 2π-periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$, and

$$\frac{5-\sqrt{29}}{2} = \frac{(\alpha-\sqrt{\alpha^2-4\beta})_L}{2} \le \gamma_1(t) \le \frac{(\alpha-\sqrt{\alpha^2-4\beta})_M}{2} = \frac{7-\sqrt{29}}{2}$$

and

$$\frac{5+\sqrt{29}}{2} = \frac{(\alpha+\sqrt{\alpha^2-4\beta})_L}{2} \le \gamma_2(t) \le \frac{(\alpha+\sqrt{\alpha^2-4\beta})_M}{2} = \frac{7+\sqrt{29}}{2}.$$

7. Conclusions

In this paper, the existence of periodic solutions of Eq. (1.2) is discussed. In addition to having a zero periodic solution, the periodic solution of Eq. (1.2) has the following three cases:

(1) When $\alpha^2(t) - 4\beta(t) < 0$, there is no nonzero periodic solution of Eq. (1.2);

(2) When $\alpha^2(t) - 4\beta(t) = 0$ and $\alpha(t) \equiv C$ ($C \neq 0$), there is one nonzero constant periodic solution of Eq. (1.2);

(3) When $\alpha^2(t) - 4\beta(t) > 0$ and

$$(-\alpha-\sqrt{\alpha^2-4\beta})_M < (-\alpha+\sqrt{\alpha^2-4\beta})_L,$$

there are two nonzero periodic solutions of Eq. (1.2). The conclusion of this paper is an important supplement to [13].

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