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On the Stability of Orthogonal Additivity in β-Homogeneous F-Spaces

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Abstract In this paper, we study the stability of the orthogonal equation, which is closely related to the results by W. Fechner and J. Sikorska in 2010. There are some differences that we consider the target space with the β-homogeneous norm and quasi-norm. Overcoming the β-homogeneous norm and quasi-norm bottlenecks, we get some new results.

Keywords Hyers-Ulam stability; β-homogeneous F-spaces; quasi-Banach spaces; orthogonal additivity

MR(2020) Subject Classification 39B55; 39B82

1. Introduction

The stability problem of functional equations originated from a problem raised by Ulam [1] on the stability of group homomorphisms in 1940:

Given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$?

Hyers [2] gave a first affirmative partial answer to the question of Ulam for additive mappings on Banach spaces. Firstly, for the additive mappings, the generalized form of Hyers theorem was solved by Aoki [3], and further, for linear mapping, it was generalized by Rassias [4] taking an unbounded Cauchy difference in consideration. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference with a general control function. Because of their breakthrough achievements, the stability of functional equations has been widely studied by mathematicians.

Although various studies on stability have been successfully conducted, there are not many corresponding stability results due to the non-linear structure of the infinite-dimensional F-space. The nonlinear structure of F-space plays an important role in functional analysis and other mathematical fields. The $L^p([0,1])$ for $0 < p < 1$ equipped with the metric $d(f,g) = \int |f(x) - g(x)|^p \mathrm{d}x$ is an example of an F-space but not a Banach space. Besides these, we refer

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to [6,7] for F-spaces.

Definition 1.1 *Suppose X is a linear space. A non-negative valued function $\|\cdot\|$ achieves an F-norm if the following conditions are satisfied:*

(1) $\|x\| = 0$ *if and only if* $x = 0$;

(2) $\|\lambda x\| = \|x\|$ *for all* λ, $|\lambda| = 1$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$;

(4) $\|\lambda_n x\| \to 0$ *provided* $\lambda_n \to 0$;

(5) $\|\lambda x_n\| \to 0$ *provided* $x_n \to 0$;

(6) $\|\lambda_n x_n\| \to 0$ *provided* $\lambda_n \to 0, x_n \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathcal{C}$ (see [8,9]).

If a quasi-norm is p-subadditive, then it is called p-norm $(0 < p < 1)$. In other words, if it satisfies

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad x, y \in X.$$

We note that the p-subadditive quasi-norm $\|\cdot\|$ induces an F norm. We refer the reader to [6] and [10] for background on it.

Definition 1.2 ([11]) *A quasi-norm on $\|\cdot\|$ on vector space X over a field K $(\mathbb{R})$ is a map $X \longrightarrow [0, \infty)$ with the following properties:*

(1) $\|x\| = 0$ *if and only if* $x = 0$;

(2) $\|ax\| = |a|\|x\|$, $a \in \mathbb{R}$, $x \in X$;

(3) $\|x + y\| \leq C(\|x\| + \|y\|)$, $x, y \in X$. *Where $C \geq 1$ is a constant independent of $x, y \in X$. The smallest C for which (3) holds is called the quasi-norm constant of $(X, \|\cdot\|)$.*

It is vital to emphasize the well-known theorem in nonlocally convex theory, that is, Aoki-Rolewicz theorem [8], which asserts that for some $0 < p \leq 1$, every quasi-norm admits an equivalent p-norm.

Various more results for the stability of functional equations in quasi-Banach spaces can be seen in [12,13]. However, the results are more interesting and meaningful when orthogonality is taken into account.

The notion of orthogonality goes a long way back in time and various extensions have been introduced over the last decades. In particular, proposing the notion of orthogonality in normed linear spaces has been the object of extensive efforts of many mathematicians.

We recall two orthogonality types introduced in normed linear spaces. In 1945, James [14] introduced the so-called isosceles orthogonality as follows:

$$x \perp_I y \text{ if and only if } \|x + y\| = \|x - y\|.$$

Taking into account the classical Pythagorean theorem, one can define the orthogonal relation in a normed space $(X, \|\cdot\|)$:

$$x \perp_P y \text{ if and only if } \|x-y\|^2 = \|x\|^2 + \|y\|^2.$$

Some other known orthogonalities in normed linear spaces can be found in [15–18] and references therein.

When orthogonality is the general orthogonality on the inner product space, Pinsker [19] characterized the orthogonality additive functional on the inner product space. Sundaresan [20] extended this result to arbitrary Banach Spaces with Birkhoff-James orthogonality. The following orthogonal Cauchy functional equation was first investigated by Gudder and Strawther [21]

$$f(x+y) = f(x) + f(y), \quad x \perp y,$$

where $\perp$ is an abstract orthogonality relation and it will paly a crucial role in orthogonal stability.

In addition to the different definitions of orthogonality in normed space, we can also give some axiomatic definitions of such relations in linear space. We show the following standard definition by Rätz [22]:

Definition 1.3 *Let X be a real linear space with $\dim X \geq 2$ and let $\perp$ be a binary relation on X such that*

(1) $x \perp 0$ and $0 \perp x$ for all $x \in X$;

(2) If $x, y \in X \backslash \{0\}$ and $x \perp y$, then x and y are linearly independent;

(3) If $x, y \in X$ and $x \perp y$, then for all $\alpha, \beta \in \mathbb{R}$ we have $\alpha x \perp \beta y$;

(4) For any two-dimensional subspace P of X and for every $x \in P$, $\lambda \in [0, \infty)$, there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

An ordered pair $(X, \perp)$ is called an orthogonality space.

In 2010, Fechner and Sikorska [23] studied the stability of orthogonality and proposed the definition of orthogonality as follows.

Definition 1.4 *Let X be an Abelian group and let $\perp$ be a binary relation defined on X with the properties:*

(1) If $x, y \in X$ and $x \perp y$, then $x \perp -y, -x \perp y$ and $2x \perp 2y$;

(2) For every $x \in X$, there exists a $y \in X$ such that $x \perp y$ and $x + y \perp x - y$.

It is worth noting that every orthogonal space satisfies these conditions as well as any normed linear space with the isosceles orthogonality, but Pythagorean orthogonality no longer satisfies these conditions.

Considering the current gaps, in this paper, we have made an attempt to prove the stability of orthogonal additivity in β-homogeneous F-spaces and quasi-Banach spaces.

During the entire course of this work, β_2 are considered as positive real numbers with $\beta_2 \geq 0$. Furthermore, X is assumed as an Abelian group while Y is a β_2-homogeneous F-space.

2. Stability of the orthogonally additive functional equation

In this section, following some ideas from [23], we deal with the stability problem for the orthogonally additive functional equation in β-homogeneous F-spaces. Our main theorem is the following.

Theorem 2.1 *Let X be an Abelian group, and Y be a β_2-homogeneous F-space. For $\varepsilon \geq 0$, assume $f : X \to Y$ be a mapping such that for all $x, y \in X$ one has*

$$x \perp y \text{ implies } \|f(x+y) - f(x) - f(y)\| \leq \varepsilon. \tag{2.1}$$

Then there exists a mapping $g : X \to Y$ such that

$$x \perp y \text{ implies } g(x+y) = g(x) + g(y) \tag{2.2}$$

and

$$\|f(x) - g(x)\| \leq b\varepsilon \tag{2.3}$$

with

$$b = \frac{2^{\beta_2+2} + 3^{\beta_2+1} + 3}{8^{\beta_2}}\Big(1 + \sum_{n=2}^{\infty}((\frac{2^{n-1}+1}{2\cdot 4^{n-1}})^{\beta_2} + (\frac{2^{n-1}-1}{2\cdot 4^{n-1}})^{\beta_2})\Big)$$

for all $x \in 2X = \{2x : x \in X\}$. Moreover, the mapping g is unique on the set $2X$.

Proof By Definition 1.4, we get immediately there exists a $y \in X$ such that $x \perp y$ and $x+y \perp x-y$. Moreover, we also have $\pm 2x \perp \pm 2y, \pm(x+y) \perp \pm(x-y)$. Thus we conclude that

$$\begin{aligned}
\|3f(4x) - 8f(2x) - f(-4x)\| \leq\ & 3^{\beta_2}\|f(4x) - f(2x+2y) - f(2x-2y)\| + \\
& \|f(-2x+2y) + f(-2x-2y) - f(-4x)\| + 3^{\beta_2}\|f(2x+2y) - f(2x) - f(2y)\| + \\
& 3^{\beta_2}\|f(2x-2y) - f(2x) - f(-2y)\| + \|f(-2x) + f(2y) - f(-2x+2y)\| + \\
& \|f(-2x) + f(-2y) - f(-2x-2y)\| + 2^{\beta_2}\|f(2y) - f(y-x) - f(y+x)\| + \\
& 2^{\beta_2}\|f(-2y) - f(-y-x) - f(-y+x)\| + 2^{\beta_2}\|f(y-x) + f(-y-x) - f(-2x)\| + \\
& 2^{\beta_2}\|f(y+x) + f(-y+x) - f(2x)\| \\
\leq\ & (2^{\beta_2+2} + 3^{\beta_2+1} + 3)\varepsilon.
\end{aligned}$$

This proves that

$$\|f(2x) - \frac{3}{8}f(4x) + \frac{1}{8}f(-4x)\| \leq \frac{2^{\beta_2+2} + 3^{\beta_2+1} + 3}{8^{\beta_2}}\varepsilon, \quad x \in X. \tag{2.4}$$

From now on, we set $a = \frac{2^{\beta_2+2}+3^{\beta_2+1}+3}{8^{\beta_2}}$ for convenience, then we have

$$\|f(2x) - \frac{3}{8}f(4x) + \frac{1}{8}f(-4x)\| \leq a\varepsilon, \quad x \in X. \tag{2.5}$$

Now we prove that for all $n \in \mathbb{N}$,

$$\|f(2x) - \frac{2^n+1}{2\cdot 4^n}f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n}f(-2^{n+1}x)\| \leq b\varepsilon, \quad x \in X, \tag{2.6}$$

with $b = a(1 + \sum_{n=2}^{\infty}((\frac{2^{n-1}+1}{2\cdot 4^{n-1}})^{\beta_2} + (\frac{2^{n-1}-1}{2\cdot 4^{n-1}})^{\beta_2}))$.

First, using (2.5), through a simple estimate we obtain

$$\|f(2x) - \frac{2^{n+1}+1}{2\cdot 4^{n+1}}f(2^{n+2}x) + \frac{2^{n+1}-1}{2\cdot 4^{n+1}}f(-2^{n+2}x)\|$$

$$
\begin{aligned}
&\le \|f(2x) - \frac{2^n+1}{2\cdot 4^n} f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n} f(-2^{n+1}x)\| + \\
&(\frac{2^n+1}{2\cdot 4^n})^{\beta_2} \|f(2^{n+1}x) - \frac{3}{8} f(2^{n+2}x) + \frac{1}{8} f(-2^{n+2}x)\| + \\
&(\frac{2^n-1}{2\cdot 4^n})^{\beta_2} \|f(-2^{n+1}x) - \frac{3}{8} f(-2^{n+2}x) + \frac{1}{8} f(2^{n+2}x)\| \\
&\le \|f(2x) - \frac{2^n+1}{2\cdot 4^n} f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n} f(-2^{n+1}x)\| + \\
&((\frac{2^n+1}{2\cdot 4^n})^{\beta_2} + (\frac{2^n-1}{2\cdot 4^n})^{\beta_2}) a\varepsilon,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\|f(2x) - \frac{2^{n+1}+1}{2\cdot 4^{n+1}} f(2^{n+2}x) + \frac{2^{n+1}-1}{2\cdot 4^{n+1}} f(-2^{n+2}x)\| - \\
&\quad \|f(2x) - \frac{2^n+1}{2\cdot 4^n} f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n} f(-2^{n+1}x)\| \\
&\le ((\frac{2^n+1}{2\cdot 4^n})^{\beta_2} + (\frac{2^n-1}{2\cdot 4^n})^{\beta_2}) a\varepsilon.
\end{aligned}
$$

Now we let

$$
h(x,n) = \|f(2x) - \frac{2^n+1}{2\cdot 4^n} f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n} f(-2^{n+1}x)\|,
$$

so we have that

$$
h(x,n+1) - h(x,n) \le ((\frac{2^n+1}{2\cdot 4^n})^{\beta_2} + (\frac{2^n-1}{2\cdot 4^n})^{\beta_2}) a\varepsilon,
$$

and then

$$
\begin{aligned}
h(x,n) &= \sum_{i=2}^{n} (h(x,i) - h(x,i-1)) + h(x,1) \\
&\le \sum_{i=2}^{n} ((\frac{2^{i-1}+1}{2\cdot 4^{i-1}})^{\beta_2} + (\frac{2^{i-1}-1}{2\cdot 4^{i-1}})^{\beta_2}) a\varepsilon + a\varepsilon \le b\varepsilon,
\end{aligned}
$$

with

$$
b = \sum_{i=2}^{n} ((\frac{2^{i-1}+1}{2\cdot 4^{i-1}})^{\beta_2} + (\frac{2^{i-1}-1}{2\cdot 4^{i-1}})^{\beta_2} + 1) a.
$$

This means that

$$
\|f(2x) - \frac{2^n+1}{2\cdot 4^n} f(2^{n+1}x) + \frac{2^n-1}{2\cdot 4^n} f(-2^{n+1}x)\| \le b\varepsilon, \quad x \in X.
$$

The next step is to prove that for each $x \in X$ the sequence

$$
g_n(x) := \frac{2^n+1}{2\cdot 4^n} f(2^n x) - \frac{2^n-1}{2\cdot 4^n} f(-2^n x), \quad n \in \mathbb{N}
$$

is convergent in Y. Since Y is complete, it suffices to show that $(g_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$. Applying estimate (2.5) twice, then we have

$$
\begin{aligned}
\|g_n(x) - g_{n+1}(x)\| =& \|\frac{2^n+1}{2\cdot 4^n} (f(2^n x) - \frac{3}{8} f(2^{n+1}x) + \frac{1}{8} f(-2^{n+1}x)) - \\
& \frac{2^n-1}{2\cdot 4^n} (f(-2^n x) - \frac{3}{8} f(-2^{n+1}x) + \frac{1}{8} f(2^{n+1}x))\| \\
\le& ((\frac{2^n+1}{2\cdot 4^n})^{\beta_2} + (\frac{2^n-1}{2\cdot 4^n})^{\beta_2}) a\varepsilon,
\end{aligned}
$$

for each $n \in \mathbb{N}$, which gives us that $(g_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence.

Hence, the mapping $g : X \to Y$ can be defined as:

$$g(x) := \lim_{n\to\infty} g_n(x),$$

for all $x \in X$. Combining with (2.6), we have

$$\|f(2x) - g(2x)\| \le b\varepsilon, \quad x \in X.$$

In order to prove that g is orthogonally additive, observe first that for $x, y \in X$ such that $x \perp y$ and $n \in \mathbb{N}, n > 1$ we have

$$\begin{aligned}
&\|g_n(x+y) - g_n(x) - g_n(y)\| \\
&= \|\frac{2^n+1}{2\cdot 4^n} f(2^n(x+y)) - \frac{2^n-1}{2\cdot 4^n} f(-2^n(x+y)) - \\
&\quad \frac{2^n+1}{2\cdot 4^n} f(2^n x) + \frac{2^n-1}{2\cdot 4^n} f(-2^n x) - \frac{2^n+1}{2\cdot 4^n} f(2^n y) + \frac{2^n-1}{2\cdot 4^n} f(-2^n y)\| \\
&= \|\frac{2^n+1}{2\cdot 4^n} [f(2^n(x+y)) - f(2^n x) - f(2^n y)] - \\
&\quad \frac{2^n-1}{2\cdot 4^n} [f(2^n(-x-y)) - f(-2^n x) - f(-2^n y)]\| \\
&\le (\frac{2^n+1}{2\cdot 4^n})^{\beta_2} \|f(2^n(x+y)) - f(2^n x) - f(2^n y)\| + \\
&\quad (\frac{2^n-1}{2\cdot 4^n})^{\beta_2} \|f(2^n(-x-y)) - f(-2^n x) - f(-2^n y)\| \\
&\le ((\frac{2^n-1}{2\cdot 4^n})^{\beta_2} + (\frac{2^n-1}{2\cdot 4^n})^{\beta_2})\varepsilon.
\end{aligned}$$

Moreover, letting $n \to \infty$, we get (2.2).

Now, we show the uniqueness of g. Assume g' as another mapping satisfying (2.2) and (2.3) that yields:

$$\|g(x) - g'(x)\| \le \|g(x) - f(x)\| + \|g'(x) - f(x)\| \le 2b\varepsilon,$$

for all $x \in 2X$.

On the other hand, the mapping $g - g'$ satisfies (2.2) and thus, in particular, (2.1) with $\varepsilon = 0$. By applying (2.6) to $g - g'$ we see that

$$\begin{aligned}
g(2x) - g'(2x) =& \frac{2^n+1}{2\cdot 4^n} [g(2^{n+1}x) - g'(2^{n+1}x)] - \\
& \frac{2^n-1}{2\cdot 4^n} [g(-2^{n+1}x) - g'(-2^{n+1}x)]
\end{aligned}$$

and therefore

$$\begin{aligned}
\|g(2x) - g'(2x)\| \le& (\frac{2^n+1}{2\cdot 4^n})^{\beta_2} \|g(2^{n+1}x) - g'(2^{n+1}x)\| + \\
& (\frac{2^n-1}{2\cdot 4^n})^{\beta_2} \|g(-2^{n+1}x) - g'(-2^{n+1}x)\| \\
\le& ((\frac{2^n-1}{2\cdot 4^n})^{\beta_2} + (\frac{2^n-1}{2\cdot 4^n})^{\beta_2}) 2b\varepsilon
\end{aligned}$$

for $x \in X$.

Combining the both inequalities, we can easily get the conclusion. □

Remark 2.2 Compare with the theorem of [23], Y is an F-space with β_2-homogeneousness instead of the homogeneousness of [23], so the estimate combined with β_2-homogeneousness of (2.6) here would be more difficult and complicated. And moreover (2.6) is small enough so that we can get the convergence of the function sequence $(g_n(x))_{n\in\mathbb{N}}$, that is the main point here.

By the same method, we can also obtain the stability result for different target spaces, where the space Y is equipped with quasi-norm. From now on, in corollaries, assume that X is an Abelian group and Y is a quasi-Banach space.

Corollary 2.3 *For $\varepsilon \geq 0$, let $f : X \to Y$ be a mapping such that for all $x, y \in X$ one has*

$$x \perp y \text{ implies } \|f(x+y) - f(x) - f(y)\| \leq \varepsilon.$$

Then there exists a mapping $g : X \to Y$ such that

$$x \perp y \text{ implies } g(x+y) = g(x) + g(y),$$

and

$$\|f(x) - g(x)\| \leq b^{\frac{1}{p}}\varepsilon$$

with

$$b = \frac{2^{p+2} + 3^{p+1} + 3}{8^p}\Big(1 + \sum_{n=2}^{\infty}\Big(\Big(\frac{2^{n-1}+1}{2\cdot 4^{n-1}}\Big)^p + \Big(\frac{2^{n-1}-1}{2\cdot 4^{n-1}}\Big)^p\Big)\Big),$$

for all $x \in 2X = \{2x : x \in X\}$. Moreover, the mapping g is unique on the set $2X$.

Proof Let $\|\cdot\|_p = \|\cdot\|^p$. Then it is obvious that $(Y, \|\cdot\|_p)$ is p-homogeneous, and we obtain

$$x \perp y \text{ implies } \|f(x+y) - f(x) - f(y)\|_p \leq \varepsilon^p.$$

According to Theorem 2.1, we obtain that there exists a mapping $g : X \to Y$ such that

$$x \perp y \text{ implies } g(x+y) = g(x) + g(y),$$

and

$$\|f(x) - g(x)\|_p \leq b\varepsilon^p$$

with

$$b = \frac{2^{p+2} + 3^{p+1} + 3}{8^p}\Big(1 + \sum_{n=2}^{\infty}\Big(\Big(\frac{2^{n-1}+1}{2\cdot 4^{n-1}}\Big)^p + \Big(\frac{2^{n-1}-1}{2\cdot 4^{n-1}}\Big)^p\Big)\Big)$$

for all $x \in 2X = \{2x : x \in X\}$. Moreover, the mapping g is unique on the set $2X$ and the claim follows. $\square$

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