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Nonlinear Maps Preserving the Mixed Triple Products between Factors

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Abstract Let $\mathcal{A}$ and $\mathcal{B}$ be two factors with $\dim \mathcal{A} > 4$. In this paper, it is proved that a bijective map $\phi : \mathcal{A} \to \mathcal{B}$ satisfies $\phi([A, B] \bullet C) = [\phi(A), \phi(B)] \bullet \phi(C)$ for all $A, B, C \in \mathcal{A}$ if and only if ϕ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

Keywords mixed triple product; isomorphism; factor

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a map. We consider that ϕ preserves the mixed triple product if $\phi([A, B] \bullet C) = [\phi(A), \phi(B)] \bullet \phi(C)$ for all $A, B, C \in \mathcal{A}$, where $[A, B] = AB - BA$ is the Lie product and $A \bullet B = AB + BA^*$ is the Jordan $*$-product of A and B. Recently, some authors have considered the mixture of (skew) Lie product and Jordan $*$-product [1–11]. For example, Yang and Zhang [1] proved the nonlinear maps preserving the mixed skew Lie triple product $[[A, B]_*, C]$ on factors. Zhao et al. [2] proved the nonlinear maps preserving mixed product $[A \bullet B, C]$ on von Neumann algebras. Yang and Zhang [3] proved the nonlinear maps preserving the second mixed Lie triple product $[[A, B], C]_*$ on factors. In this article, motivated by the above results, we will obtain the structure of the nonlinear maps preserving the mixed triple product $[A, B] \bullet C$ on factors.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. $\mathcal{A}$ is a factor means that its center only contains the scalar operators. It is well known that the factor $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$, if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

Choose an arbitrary nontrivial projection $P_1 \in \mathcal{A}$, write $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$, $i, j, = 1, 2$. Then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$, we can write it as $A = \sum_{i,j=1}^{2} A_{ij}$, where

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A_{ij} denotes an arbitrary element of $\mathcal{A}_{ij}$.

Lemma 1.1 ([6]) *Let $\mathcal{A}$ be a factor and $A \in \mathcal{A}$. Then $AB + BA^* = 0$ for all $B \in \mathcal{A}$ implies that $A \in \mathrm{i}\mathbb{R}I$ (i is the imaginary number unit).*

Lemma 1.2 ([9]) *Let $\mathcal{A}$ be a factor, for any $T_{ii} \in \mathcal{A}_{ii}$ with $i = 1, 2$, if $A_{21}T_{11} = T_{22}A_{21}$ for all $A_{21} \in \mathcal{A}_{21}$ or $T_{11}A_{12} = A_{12}T_{22}$ for all $A_{12} \in \mathcal{A}_{12}$, then $T_{11} + T_{22} \in \mathbb{C}I$.*

Lemma 1.3 ([12, Problem 230]) *Let $\mathcal{A}$ be a Banach algebra with the identity I. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ are such that $[A, B] = \lambda I$, where $[A, B] = AB - BA$, then $\lambda = 0$.*

2. Additivity

Our first theorem is as follows:

Theorem 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. Suppose that ϕ is a bijective map from $\mathcal{A}$ to $\mathcal{B}$ with $\phi([A, B] \bullet C) = [\phi(A), \phi(B)] \bullet \phi(C)$ for all $A, B, C \in \mathcal{A}$. Then ϕ is additive.*

Proof We will complete the proof by proving several claims.

Claim 1. $\phi(0) = 0$, $\phi(\mathbb{C}I) = \mathbb{C}I$.

Since ϕ is surjective, there exists $A \in \mathcal{A}$ such that $\phi(A) = 0$. Then we obtain

$$\phi(0) = \phi([0, 0] \bullet A) = [\phi(0), \phi(0)] \bullet \phi(A) = 0.$$

It is easy to verify that $0 = \phi([\lambda I, A] \bullet B) = [\phi(\lambda I), \phi(A)] \bullet \phi(B)$ for every $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. By applying the surjectivity of ϕ and Lemma 1.1, we obtain $[\phi(\lambda I), \phi(A)] \in \mathrm{i}\mathbb{R}I$. It follows from Lemma 1.3 that $[\phi(\lambda I), \phi(A)] = 0$. Thus $\phi(\lambda I) \in \mathbb{C}I$ for every $\lambda \in \mathbb{C}$. By considering ϕ^{-1}, we obtain that $\phi(\mathbb{C}I) = \mathbb{C}I$.

Claim 2. $\phi(A_{11} + A_{22}) = \phi(A_{11}) + \phi(A_{22})$ for all $A_{11} \in \mathcal{A}_{11}$ and $A_{22} \in \mathcal{A}_{22}$.

Choose $X = \sum_{i,j=1}^{2} X_{ij} \in \mathcal{A}$ such that $\phi(X) = \phi(A_{11} + A_{22}) - \phi(A_{11}) - \phi(A_{22})$. Since $[P_k, A_{ii}] \bullet P_k = 0, 1 \le k, i \le 2$, by applying Claim 1, we obtain

$$\begin{aligned}
&[\phi(P_k), \phi(A_{11} + A_{22})] \bullet \phi(P_k) = \phi([P_k, A_{11} + A_{22}] \bullet P_k) \\
&\quad = \phi([P_k, A_{11}] \bullet P_k) + \phi([P_k, A_{22}] \bullet P_k) \\
&\quad = [\phi(P_k), \phi(A_{11}) + \phi(A_{22})] \bullet \phi(P_k).
\end{aligned}$$

Thus $[\phi(P_k), \phi(X)] \bullet \phi(P_k) = \phi([P_k, X] \bullet P_k) = 0$. By the injectivity of ϕ, we obtain that $[P_k, X] \bullet P_k = 0$, and so $X_{12} = X_{21} = 0$.

For every $B_{kl} \in \mathcal{A}_{kl}$, since $[B_{kl}, P_l] \bullet A_{kk} = 0$, $k \neq l$, we obtain

$$\begin{aligned}
&[\phi(B_{kl}), \phi(P_l)] \bullet \phi(A_{11} + A_{22}) = \phi([B_{kl}, P_l] \bullet (A_{11} + A_{22})) \\
&\quad = \phi([B_{kl}, P_l] \bullet A_{ll}) = \phi([B_{kl}, P_l] \bullet A_{11}) + \phi([B_{kl}, P_l] \bullet A_{22}) \\
&\quad = [\phi(B_{kl}), \phi(P_l)] \bullet (\phi(A_{11}) + \phi(A_{22})).
\end{aligned}$$

Thus $[\phi(B_{kl}), \phi(P_l)] \bullet \phi(X) = \phi([B_{kl}, P_l] \bullet X) = 0$, and so $[B_{kl}, P_l] \bullet X = 0$, which indicates that $B_{kl}X_{ll} = 0$ for every $B_{kl} \in \mathcal{A}_{kl}$. Since $\mathcal{A}$ is prime, we have $X_{ll} = 0$, $l = 1, 2$, and so $X = 0$.

Consequently, $\phi(A_{11} + A_{22}) = \phi(A_{11}) + \phi(A_{22})$.

Claim 3. $\phi(A_{22} + A_{21} + A_{11}) - \phi(A_{22}) - \phi(A_{21}) - \phi(A_{11}) \in \mathbb{C}I$ and $\phi(A_{11} + A_{12} + A_{22}) - \phi(A_{11}) - \phi(A_{12}) - \phi(A_{22}) \in \mathbb{C}I$ for all $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$.

Choose $X = \sum_{i,j=1}^{2} X_{ij} \in \mathcal{A}$ such that $\phi(X) = \phi(A_{22}+A_{21}+A_{11})-\phi(A_{22})-\phi(A_{21})-\phi(A_{11})$. It follows from Claims 1 and 2 that

$$\begin{aligned}
&[\phi(P_k), \phi(A_{22} + A_{21} + A_{11})] \bullet \phi(P_k) = \phi([P_k, A_{22} + A_{21} + A_{11}] \bullet P_k) \\
&\quad = \phi([P_k, A_{21}] \bullet P_k) = [\phi(P_k), \phi(A_{22}) + \phi(A_{21}) + \phi(A_{11})] \bullet \phi(P_k),
\end{aligned}$$

which indicates that $[P_k, X] \bullet P_k = 0$. Thus we get $X_{12} = X_{21} = 0$.

For every $B_{21} \in \mathcal{A}_{21}$, since $[B_{21}, A_{21}] \bullet P_1 = 0$, from Claim 2, we obtain

$$\begin{aligned}
&[\phi(B_{21}), \phi(A_{22} + A_{21} + A_{11})] \bullet \phi(P_1) = \phi([B_{21}, A_{22} + A_{21} + A_{11}] \bullet P_1) \\
&\quad = \phi([B_{21}, A_{22} + A_{11}] \bullet P_1) + \phi([B_{21}, A_{21}] \bullet P_1) \\
&\quad = \phi([B_{21}, \phi(A_{22}) + \phi(A_{21}) + \phi(A_{11})] \bullet \phi(P_1).
\end{aligned}$$

Thus $[B_{21}, X] \bullet P_1 = 0$, this indicates that $B_{21}X_{11} = X_{22}B_{21}$ for every $B_{21} \in \mathcal{A}_{21}$. It follows from Lemma 1.2 that $X_{11} + X_{22} \in \mathbb{C}I$, and so $X \in \mathbb{C}I$. Since $\phi(\mathbb{C}I) = \mathbb{C}I$, we have $\phi(A_{22} + A_{21} + A_{11}) - \phi(A_{22}) - \phi(A_{21}) - \phi(A_{11}) \in \mathbb{C}I$. In the second case, we can similarly prove that the conclusion is valid.

Claim 4. $\phi(A_{11} + A_{12} + A_{21} + A_{22}) - \phi(A_{11}) - \phi(A_{12}) - \phi(A_{21}) - \phi(A_{22}) \in \mathbb{C}I$ for all $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$.

Let $X = \sum_{i,j=1}^{2} X_{ij} \in \mathcal{A}$ such that $\phi(X) = \phi(A_{11} + A_{12} + A_{21} + A_{22}) - \phi(A_{11}) - \phi(A_{12}) - \phi(A_{21}) - \phi(A_{22})$. For $k \neq l$, it follows from Claims 1 and 3 that

$$\begin{aligned}
&[\phi(P_k), \phi(A_{11} + A_{12} + A_{21} + A_{22})] \bullet \phi(P_k) \\
&\quad = \phi([P_k, A_{11} + A_{12} + A_{21} + A_{22}] \bullet P_k) = \phi([P_k, A_{lk}] \bullet P_k) \\
&\quad = [\phi(P_k), \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22})] \bullet \phi(P_k),
\end{aligned}$$

this indicates that $[P_k, X] \bullet P_k = 0$. Thus we get $X_{12} = X_{21} = 0$.

By Claim 3 again, we obtain

$$\begin{aligned}
&[\phi(B_{21}), \phi(A_{11} + A_{12} + A_{21} + A_{22})] \bullet \phi(P_1) \\
&\quad = \phi([B_{21}, A_{11} + A_{12} + A_{21} + A_{22}] \bullet P_1) \\
&\quad = \phi([B_{21}, A_{11} + A_{12} + A_{22}] \bullet P_1) + \phi([B_{21}, A_{21}] \bullet P_1) \\
&\quad = \phi([B_{21}, \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22})] \bullet \phi(P_1)
\end{aligned}$$

for every $B_{21} \in \mathcal{A}_{21}$. Thus $[B_{21}, X] \bullet P_1 = 0$, which indicates that $B_{21}X_{11} = X_{22}B_{21}$ for every $B_{21} \in \mathcal{A}_{21}$. It follows from Lemma 1.2 that $X_{11} + X_{22} \in \mathbb{C}I$, and so $X \in \mathbb{C}I$. By applying Claim 1, we obtain $\phi(A_{11} + A_{12} + A_{21} + A_{22}) - \phi(A_{11}) - \phi(A_{12}) - \phi(A_{21}) - \phi(A_{22}) \in \mathbb{C}I$.

Claim 5. Let $i, j \in \{1, 2\}$ with $i \neq j$. Then $\phi(A_{ij} + B_{ij}) - \phi(A_{ij}) - \phi(B_{ij}) \in \mathbb{C}I$ for all $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$.

Let $X=\sum_{i,j=1}^{2}X_{ij}\in\mathcal{A}$ such that $\phi(X)=\phi(A_{ij}+B_{ij})-\phi(A_{ij})-\phi(B_{ij})$. We obtain

$$\begin{aligned}&[\phi(P_i),\phi(A_{ij}+B_{ij})]\bullet\phi(P_i)=\phi([P_i,A_{ij}+B_{ij}]\bullet P_i)\\&\quad=\phi([P_i,A_{ij}]\bullet P_i)+\phi([P_i,B_{ij}]\bullet P_i)\\&\quad=[\phi(P_i),\phi(A_{ij})+\phi(B_{ij})]\bullet\phi(P_i),\end{aligned}$$

which indicates that $[P_i,X]\bullet P_i=0$. Thus we get $X_{ji}=0$.

By Claim 4, we obtain

$$\begin{aligned}&[\phi(A_{ij}+B_{ij}),\phi(P_j)]\bullet\phi(P_j)=\phi([A_{ij}+B_{ij},P_j]\bullet P_j)\\&\quad=\phi([P_i+A_{ij},P_j+B_{ij}]\bullet P_j)\\&\quad=[\phi(P_i)+\phi(A_{ij}),\phi(P_j)+\phi(B_{ij})]\bullet\phi(P_j)\\&\quad=[\phi(P_i),\phi(B_{ij})]\bullet\phi(P_j)+[\phi(A_{ij}),\phi(P_j)]\bullet\phi(P_j)\\&\quad=[\phi(A_{ij})+\phi(B_{ij}),\phi(P_j)]\bullet\phi(P_j),\end{aligned}$$

this indicates that $[X,P_j]\bullet P_j=0$, and so $X_{ij}=0$.

It is easy to verify that

$$\begin{aligned}&[\phi(C_{ij}),\phi(A_{ij}+B_{ij})]\bullet\phi(P_j)=\phi([C_{ij},A_{ij}+B_{ij}]\bullet P_j)\\&\quad=\phi([C_{ij},A_{ij}]\bullet P_j)+\phi([C_{ij},B_{ij}]\bullet P_j)\\&\quad=[\phi(C_{ij}),\phi(A_{ij})+\phi(B_{ij})]\bullet\phi(P_j)\end{aligned}$$

for every $C_{ij}\in\mathcal{A}_{ij}$. Thus $[C_{ij},X]\bullet P_j=0$, which indicates that $C_{ij}X_{jj}=X_{ii}C_{ij}$ for every $C_{ij}\in\mathcal{A}_{ij}$. It follows from Lemma 1.2 that $X_{ii}+X_{jj}\in\mathbb{C}I$, and so $X\in\mathbb{C}I$. By applying Claim 1, we obtain $\phi(A_{ij}+B_{ij})-\phi(A_{ij})-\phi(B_{ij})\in\mathbb{C}I$.

Claim 6. $\phi(A_{ii}+B_{ii})-\phi(A_{ii})-\phi(B_{ii})\in\mathbb{C}I$ for all $A_{ii},B_{ii}\in\mathcal{A}_{ii}$, $i=1,2$.

Choose $X=\sum_{i,j=1}^{2}X_{ij}\in\mathcal{A}$ such that $\phi(X)=\phi(A_{ii}+B_{ii})-\phi(A_{ii})-\phi(B_{ii})$. For $k=1,2$, we obtain

$$\begin{aligned}&[\phi(P_k),\phi(A_{ii}+B_{ii})]\bullet\phi(P_k)=\phi([P_k,A_{ii}+B_{ii}]\bullet P_k)\\&\quad=\phi([P_k,A_{ii}]\bullet P_k)+\phi([P_k,B_{ii}]\bullet P_k)\\&\quad=[\phi(P_k),\phi(A_{ii})+\phi(B_{ii})]\bullet\phi(P_k),\end{aligned}$$

this indicates that $[P_k,X]\bullet P_k=0$. Thus we get $X_{12}=X_{21}=0$.

It follows from Claims 4 and 5 that

$$\begin{aligned}&[\phi(A_{ii}+B_{ii})-\phi(A_{ii})-\phi(B_{ii}),\phi(C_{ij})]\bullet\phi(P_j)\\&\quad=\phi([A_{ii}+B_{ii},C_{ij}]\bullet P_j)-\phi([A_{ii},C_{ij}]\bullet P_j)-\phi([B_{ii},C_{ij}]\bullet P_j)\\&\quad=\phi(A_{ii}C_{ij}+B_{ii}C_{ij}+C_{ij}^*A_{ii}^*+C_{ij}^*B_{ii}^*)-\phi(A_{ii}C_{ij}+C_{ij}^*A_{ii}^*)-\phi(B_{ii}C_{ij}+C_{ij}^*B_{ii}^*)\in\mathbb{C}I\end{aligned}$$

for every $C_{ij}\in\mathcal{A}_{ij}$. This indicates that $[X,C_{ij}]\bullet P_j=X_{ii}C_{ij}-C_{ij}X_{jj}+C_{ij}^*X_{ii}^*-X_{jj}^*C_{ij}^*\in\mathbb{C}I$, and so $X_{ii}C_{ij}-C_{ij}X_{jj}=0$. By Lemma 1.2, we obtain $X_{ii}+X_{jj}\in\mathbb{C}I$. Thus $X\in\mathbb{C}I$. By applying Claim 1, we obtain $\phi(A_{ii}+B_{ii})-\phi(A_{ii})-\phi(B_{ii})\in\mathbb{C}I$.

Claim 7. ϕ is additive.

By applying Claims 4–6, for every $A, B \in \mathcal{A}$, we obtain $\phi(A+B) - \phi(A) - \phi(B) \in \mathbb{C}I$. Thus there exists a map $h : \mathcal{A} \times \mathcal{B} \to \mathbb{C}I$ such that $h(A, B) = \phi(A+B) - \phi(A) - \phi(B)$. For every $A, B, C, D \in \mathcal{A}$, we obtain

$$\begin{aligned}[\phi(C), \phi(D)] \bullet h(A, B) &= [\phi(C), \phi(D)] \bullet (\phi(A+B) - \phi(A) - \phi(B)) \\ &= \phi([C, D] \bullet (A+B)) - \phi([C, D] \bullet A) - \phi([C, D] \bullet B) \in \mathbb{C}I,\end{aligned}$$

which indicates that

$$[\phi(C), \phi(D)]h(A, B) + h(A, B)[\phi(C), \phi(D)]^* \in \mathbb{C}I.$$

Thus $h(A, B)[\phi(C), \phi(D)] \in \mathbb{C}I$ for every $A, B, C, D \in \mathcal{A}$. If there exist $A, B \in \mathcal{A}$ such that $h(A, B) \neq 0$, then $[\phi(C), \phi(D)] = 0$ for every $C, D \in \mathcal{A}$, which is a contradiction because $\mathcal{B}$ is not abelian, and so $h(A, B) = 0$ for every $A, B \in \mathcal{A}$. Thus ϕ is additive. □

3. Linearity

Our main theorem is as follows:

Theorem 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factors with $\dim \mathcal{A} > 4$. Suppose that ϕ is a bijective map from $\mathcal{A}$ to $\mathcal{B}$ with $\phi([A, B] \bullet C) = [\phi(A), \phi(B)] \bullet \phi(C)$ for all $A, B, C \in \mathcal{A}$. Then ϕ is a linear $*$-isomorphism, a conjugate linear $*$-isomorphism, the negative of a linear $*$-isomorphism, or the negative conjugate linear $*$-isomorphism.*

We will prove Theorem 3.1 by the following lemmas.

Lemma 3.2 *For any $A, B \in \mathcal{A}$, $[\phi(A), \phi(B)] = 0$ if and only if $[A, B] = 0$.*

Proof It is easy to verify that

$$0 = \phi([A, B] \bullet C) = [\phi(A), \phi(B)] \bullet \phi(C) \tag{3.1}$$

for every $A, B, C \in \mathcal{A}$ with $[A, B] = 0$. It follows from (3.1) and Lemma 1.1 that $[\phi(A), \phi(B)] \in \mathrm{i}\mathbb{R}I$, by Lemma 1.3, we obtain $[\phi(A), \phi(B)] = 0$. By considering ϕ^{-1}, we obtain that $[\phi(A), \phi(B)] = 0$ if and only if $[A, B] = 0$ for every $A, B \in \mathcal{A}$. □

Lemma 3.3 *For any $A, B \in \mathcal{A}$, we have*

(1) $\phi(\mathrm{i}A) - \theta(\mathrm{i}I)\phi(A) \in \mathbb{C}I$;

(2) $\phi([A, B]) = \epsilon[\phi(A), \phi(B)]$ and $\phi([A^*, B^*]) = \epsilon[\phi(A)^*, \phi(B)^*]$, where $\epsilon \in \{1, -1\}$.

Proof (1) From Lemma 3.2 and Theorem 2.1, ϕ is an additive bijection that preserves commutativity in both directions. It follows from [10, Theorem 3.1] that

$$\phi(A) = a\theta(A) + \xi(A)$$

for all $A \in \mathcal{A}$, where $a \in \mathbb{C}$ is a nonzero scalar, $\theta : \mathcal{A} \to \mathcal{B}$ is an additive Jordan isomorphism, and $\xi : \mathcal{A} \to \mathbb{C}I$ is an additive map with $\xi([A, B]) = 0$ for every $A, B \in \mathcal{A}$. It is easy to get that $\theta(\mathrm{i}I) = \pm \mathrm{i}I$.

For every $A \in \mathcal{A}$, we obtain

$$\begin{aligned}&\phi(\mathrm{i}A)-\theta(\mathrm{i}I)\phi(A)=a\theta(\mathrm{i}A)+\xi(\mathrm{i}A)-\theta(\mathrm{i}I)\phi(A)\\&\quad=a\theta(\mathrm{i}I)\theta(A)+\xi(\mathrm{i}A)-\theta(\mathrm{i}I)\phi(A)\\&\quad=\theta(\mathrm{i}I)(a\theta(A)+\xi(A))+\xi(\mathrm{i}A)-\theta(\mathrm{i}I)\xi(A)-\theta(\mathrm{i}I)\phi(A)\\&\quad=\xi(\mathrm{i}A)-\theta(\mathrm{i}I)\xi(A)\in\mathbb{C}I.\end{aligned}$$

(2) For every $A, B \in \mathcal{A}$, we obtain

$$\phi([A,B]-[A^*,B^*])=\phi([A,B]\bullet I)=\phi(I)([\phi(A),\phi(B)]-[\phi(A)^*,\phi(B)^*]). \tag{3.2}$$

By Lemma 3.3 (1), we obtain

$$\phi(\mathrm{i}[A,B])-\theta(\mathrm{i}I)\phi([A,B])=\xi([\mathrm{i}A,B])-\theta(\mathrm{i}I)\xi([A,B])=0$$

for every $A, B \in \mathcal{A}$. Replacing A by $\mathrm{i}A$ in (3.2), from the above equation and Lemma 3.3 (1), we obtain that

$$\begin{aligned}&\theta(\mathrm{i}I)\phi([A,B]+[A^*,B^*])=\phi([\mathrm{i}A,B]-[(\mathrm{i}A)^*,B^*])\\&\quad=\phi([\mathrm{i}A,B]\bullet I)=\phi(I)([\phi(\mathrm{i}A),\phi(B)]-[\phi(\mathrm{i}A)^*,\phi(B)^*])\\&\quad=\theta(\mathrm{i}I)\phi(I)([\phi(A),\phi(B)]+[\phi(A)^*,\phi(B)^*])\end{aligned}$$

for every $A, B \in \mathcal{A}$, this indicates

$$\phi([A,B]+[A^*,B^*])=\phi(I)([\phi(A),\phi(B)]+[\phi(A)^*,\phi(B)^*]). \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\phi([A,B])=\phi(I)[\phi(A),\phi(B)] \tag{3.4}$$

and

$$\phi([A^*,B^*])=\phi(I)[\phi(A)^*,\phi(B)^*] \tag{3.5}$$

for every $A, B \in \mathcal{A}$. By (3.4) and (3.5), we have

$$[\phi(A^*),\phi(B^*)]=[\phi(A)^*,\phi(B)^*] \tag{3.6}$$

for every $A, B \in \mathcal{A}$. From (3.4), we obtain

$$\phi([[A,B],C])=\phi(I)^2[[\phi(A),\phi(B)],\phi(C)] \tag{3.7}$$

for every $A, B, C \in \mathcal{A}$. By (3.7), we have

$$\begin{aligned}&\phi(C[A,B]-C[A^*,B^*])=\phi([A,B]\bullet C)-\phi([[A,B],C])\\&\quad=(1-\phi(I)^2)[\phi(A),\phi(B)]\phi(C)-\phi(C)[\phi(A)^*,\phi(B)^*]+\\&\qquad\phi(I)^2\phi(C)[\phi(A),\phi(B)]\end{aligned}$$

for every $A, B, C \in \mathcal{A}$. Replacing A by $\mathrm{i}A$ in the above equation, we obtain

$$\begin{aligned}&\phi(C[A,B]+C[A^*,B^*])-(1-\phi(I)^2)[\phi(A),\phi(B)]\phi(C)-\\&\quad\phi(C)[\phi(A)^*,\phi(B)^*]-\phi(I)^2\phi(C)[\phi(A),\phi(B)]\in\mathbb{C}I\end{aligned}$$

for every $A, B, C \in \mathcal{A}$. This indicates that

$$\phi(C[A,B]) - (1-\phi(I)^2)[\phi(A),\phi(B)]\phi(C) - \phi(I)^2\phi(C)[\phi(A),\phi(B)] \in \mathbb{C}I \quad (3.8)$$

and

$$\phi(C[A^*,B^*]) - \phi(C)[\phi(A)^*,\phi(B)^*] \in \mathbb{C}I \quad (3.9)$$

for every $A, B, C \in \mathcal{A}$. From (3.6), (3.8) and (3.9), we have

$$(1-\phi(I)^2)[[\phi(A),\phi(B)],\phi(C)] \in \mathbb{C}I \quad (3.10)$$

for every $A, B, C \in \mathcal{A}$. By (3.10), $\phi(\mathbb{C}I) = \mathbb{C}I$ and the surjectivity of ϕ, we obtain $\phi(I)^2 = I$. Hence there exists $\epsilon \in \{1,-1\}$ such that $\phi([A,B]) = \epsilon[\phi(A),\phi(B)]$ and $\phi([A^*,B^*]) = \epsilon[\phi(A)^*,\phi(B)^*]$. □

Remark 3.4 Let $\psi = \epsilon\phi, \epsilon \in \{1,-1\}$. From Theorem 2.1 and Lemma 3.3 (2), we have $\psi : \mathcal{A} \to \mathcal{B}$ is an additive bijective map preserving the mixed Lie triple product and satisfies

$$\psi([A,B]) = [\psi(A),\psi(B)]$$

and

$$[\psi(A^*),\psi(B^*)] = \psi([A^*,B^*]) = [\psi(A)^*,\psi(B)^*] \quad (3.11)$$

for every $A, B \in \mathcal{A}$. By [11, Theorem 2.1], there exists an additive map $f : \mathcal{A} \to \mathbb{C}I$ with $f([A,B]) = 0$ for all $A, B \in \mathcal{A}$ such that one of the following statements holds:

(i) $\psi(A) = \varphi(A) + f(A)$ for every $A \in \mathcal{A}$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive isomorphism;

(ii) $\psi(A) = -\varphi(A) + f(A)$ for every $A \in \mathcal{A}$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive anti-isomorphism.

Lemma 3.5 *The statement (ii) does not occur.*

Proof If $\psi = -\varphi + f$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive anti-isomorphism and $f : \mathcal{A} \to \mathbb{C}I$ is an additive map with $f([A,B]) = 0$ for every $A, B \in \mathcal{A}$, then

$$\psi([A,B] \bullet C) = [\psi(A),\psi(B)] \bullet \psi(C) = [\varphi(A),\varphi(B)] \bullet (-\varphi(C) + f(C))$$

for every $A, B, C \in \mathcal{A}$. On the other hand, from (3.11), we obtain

$$\begin{aligned}\psi([A,B] \bullet C) &= (-\varphi + f)([A,B] \bullet C)\\ &= -\varphi([A,B]C) + \varphi(C[A^*,B^*]) + f([A,B] \bullet C)\\ &= \varphi(C)[\varphi(A),\varphi(B)] - [\varphi(A)^*,\varphi(B)^*]\varphi(C) + f([A,B] \bullet C)\end{aligned}$$

for every $A, B, C \in \mathcal{A}$. Then

$$(\varphi(C) - f(C))([\varphi(A),\varphi(B)] + [\varphi(A),\varphi(B)]^*) + ([\varphi(A),\varphi(B)] + [\varphi(A),\varphi(B)]^*)\varphi(C) \in \mathbb{C}I \quad (3.12)$$

for every $A, B, C \in \mathcal{A}$. Replace $\varphi(A)$ by $\mathrm{i}\varphi(A)$ in (3.12) that

$$(\varphi(C) - f(C))([\varphi(A),\varphi(B)] - [\varphi(A),\varphi(B)]^*) + ([\varphi(A),\varphi(B)] - [\varphi(A),\varphi(B)]^*)\varphi(C) \in \mathbb{C}I \quad (3.13)$$

for every $A, B, C \in \mathcal{A}$. From (3.12) and (3.13), we obtain

$$(\varphi(C) - f(C))[\varphi(A),\varphi(B)] + [\varphi(A),\varphi(B)]\varphi(C) \in \mathbb{C}I \quad (3.14)$$

for every $A, B, C \in \mathcal{A}$. Let P be a non-trivial projection in $\mathcal{A}$. Then $\varphi(P)$ is a non-trivial idempotent in $\mathcal{B}$. Taking $C = P$ in (3.14), we have

$$(\varphi(P) - f(P))[\varphi(A), \varphi(B)] + [\varphi(A), \varphi(B)]\varphi(P) \in \mathbb{C}I. \tag{3.15}$$

Multiplying (3.15) on the right-hand side by $\varphi(P^\perp)$ and on the left-hand side by $\varphi(P)$, we obtain

$$(I - f(P))\varphi(P^\perp[B, A]P) = (I - f(P))\varphi(P)[\varphi(A), \varphi(B)]\varphi(P^\perp) = 0$$

for every $A, B \in \mathcal{A}$. Let $A \in \mathcal{A}$ be such that $P^\perp AP \neq 0$. Since $\mathcal{A}$ is prime and φ is a bijective map, we have

$$\varphi(P^\perp[P^\perp, A]P) = \varphi(P^\perp AP) \neq 0.$$

Thus, $f(P) = I$ for any non-trivial projection $P \in \mathcal{A}$, from (3.15), we have

$$\varphi(P)[\varphi(A), \varphi(B)]\varphi(P) \in \mathbb{C}\varphi(P)$$

for every $A, B \in \mathcal{A}$ and any non-trivial projection $P \in \mathcal{A}$. Hence

$$P[A, B]P \in \mathbb{C}P \text{ and } P^\perp[A, B]P^\perp \in \mathbb{C}P^\perp$$

for every $A, B \in \mathcal{A}$ and any non-trivial projection $P \in \mathcal{A}$. Then

$$[PAP, PBP] = 0 \text{ and } [P^\perp AP^\perp, P^\perp BP^\perp] = 0$$

for every $A, B \in \mathcal{A}$, from this, we obtain

$$P\mathcal{A}P = \mathbb{C}P \text{ and } P^\perp\mathcal{A}P^\perp = \mathbb{C}P^\perp.$$

This is a contradiction with $\dim \mathcal{A} > 4$. $\square$

Lemma 3.6 *ψ is an additive $*$-isomorphism.*

Proof From Remark 3.4 and Lemma 3.5, we obtain $\psi = \varphi + f$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is an additive isomorphism and $f : \mathcal{A} \to \mathbb{C}I$ is an additive map with $f[A, B] = 0$ for every $A, B \in \mathcal{A}$. Then

$$\psi([A, B] \bullet C) = [\psi(A), \psi(B)] \bullet \psi(C) = [\varphi(A), \varphi(B)] \bullet (\varphi(C) + f(C))$$

for every $A, B, C \in \mathcal{A}$. From (3.11), we obtain

$$\begin{aligned}\psi([A, B] \bullet C) &= \varphi([A, B]C) - \varphi(C[A^*, B^*]) + f([A, B] \bullet C)\\ &= [\varphi(A), \varphi(B)]\varphi(C) - \varphi(C)[\varphi(A)^*, \varphi(B)^*] + f([A, B] \bullet C)\\ &= [\varphi(A), \varphi(B)] \bullet \varphi(C) + f([A, B] \bullet C)\end{aligned}$$

for every $A, B, C \in \mathcal{A}$. From this, we have

$$f(C)([\varphi(A), \varphi(B)] + [\varphi(A), \varphi(B)]^*) \in \mathbb{C}I$$

for every $A, B, C \in \mathcal{A}$. This indicates that

$$f(C)[\varphi(A), \varphi(B)] = 0$$

for every $A, B, C \in \mathcal{A}$. From this, we get that $f(C) = 0$ for every $C \in \mathcal{A}$. Thus, $\psi = \varphi$ is an additive isomorphism. From (3.11), we obtain

$$\varphi([A,B]^*) = [\varphi(A), \varphi(B)]^* = \varphi([A,B])^* \tag{3.16}$$

for every $A, B \in \mathcal{A}$. From (3.16), we have

$$\varphi(A_{ij}^*) = \varphi(A_{ij})^*$$

for every $A_{ij} \in \mathcal{A}_{ij}, i \neq j$. From this and φ is an isomorphism, we obtain

$$\varphi(A_{ij})^*\varphi(A_{ii}^*) = \varphi((A_{ii}A_{ij})^*) = \varphi(A_{ij})^*\varphi(A_{ii})^*$$

and

$$\varphi(A_{ji})^*\varphi(A_{ii}^*) = \varphi((A_{ii}A_{ji})^*) = 0 = \varphi((A_{ii}A_{ji}))^* = \varphi(A_{ji})^*\varphi(A_{ii})^*$$

for every $A_{ii} \in \mathcal{A}_{ii}, A_{ji} \in \mathcal{A}_{ji}, 1 \leq i \neq j \leq 2$. This indicates that

$$A_{ij}^*T = 0 \text{ and } A_{ji}^*T = 0$$

for every $A_{ij} \in \mathcal{A}_{ij}, A_{ji} \in \mathcal{A}_{ji}, i \neq j$ and $T = \varphi^{-1}(\varphi(A_{ii}^*) - \varphi(A_{ii})^*)$. Then $P_iT = 0$ and $P_jT = 0$. Since $\mathcal{A}$ is prime, we have $T = 0$. Thus $\varphi(A_{ii}^*) = \varphi(A_{ii})^*$, and $\varphi(A^*) = \varphi(A)^*$ for every $A \in \mathcal{A}$. Therefore, $\psi = \varphi$ is an additive $*$-isomorphism. □

Proof of Theorem 3.1 From Remark 3.4 and Lemma 3.6, we obtain that $\phi = \epsilon\psi$ and $\psi : \mathcal{A} \to \mathcal{B}$ is an additive $*$-isomorphism. Then

$$\psi(\mathrm{i}I) = \pm\mathrm{i}I, \ \psi(qI) = qI$$

for any rational number q, and ψ is order preserving.

Let $\lambda \in \mathbb{R}$. Choose sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. From $a_nI \leq \lambda I \leq b_nI$, we obtain

$$a_nI \leq \phi(\lambda I) \leq b_nI.$$

Taking the limit, we have $\phi(\lambda I) = \lambda I$. This indicates for any $\alpha = a + \mathrm{i}b \in \mathbb{C}$,

$$\psi(\alpha I) = \psi(aI) + \psi(\mathrm{i}bI) = (a \pm \mathrm{i}b)I = \alpha I \text{ or } \overline{\alpha}I.$$

Then for every $A \in \mathcal{A}$, we obtain

$$\phi(\alpha A) = \phi((\alpha I)A) = \phi(\alpha I)\phi(A) = \alpha\phi(A) \text{ or } \overline{\alpha}\phi(A).$$

Therefore, ψ is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. □

By Theorem 3.1 and the fact that every ring isomorphism between type I factors is spatial, we obtain the following corollary.

Corollary 3.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be two type I factors acting on a complex Hilbert spaces $\mathcal{H}$ with* $\dim \mathcal{H} > 2$. *Then a bijective map $\phi : \mathcal{A} \to \mathcal{B}$ satisfies*

$$\phi([A,B] \bullet C) = [\phi(A), \phi(B)] \bullet \phi(C)$$

or all $A, B, C \in \mathcal{A}$ if and only if there exists $\epsilon \in \{1, -1\}$ such that $\phi(A) = \epsilon UAU^*$ for all $A \in \mathcal{A}$, where U is a unitary or conjugate unitary operator.

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