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Uniqueness of Meromorphic Solutions for a Class of Complex Linear Differential-Difference Equations

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Abstract In this paper, we mainly study the uniqueness of transcendental meromorphic solutions for a class of complex linear differential-difference equations. Specially, suppose that $f(z)$ is a finite order transcendental meromorphic solution of complex linear differential-difference equation: $W_1(z)f'(z+1)+W_2(z)f(z)=W_3(z)$, where $W_1(z)$, $W_2(z)$, $W_3(z)$ are nonzero meromorphic functions, with their orders of growth being less than one, such that $W_1(z)+W_2(z)\not\equiv 0$. If $f(z)$ and a meromorphic function $g(z)$ share 0, 1, ∞ CM, then either $f(z)\equiv g(z)$ or $f(z)+g(z)\equiv f(z)g(z)$ or $f^2(z)(g(z)-1)^2+g^2(z)(f(z)-1)^2\equiv f(z)g(z)(f(z)g(z)-1)$ or there exists a polynomial $\varphi(z)=az+b_0$ such that $f(z)=\frac{1-e^{\varphi(z)}}{e^{\varphi(z)}(e^{a_0-b_0}-1)}$, $g(z)=\frac{1-e^{\varphi(z)}}{1-e^{b_0-a_0}}$, where $a\,(\neq 0)$, a_0, b_0 are constants with $e^{a_0}\neq e^{b_0}$.

Keywords meromorphic solution; complex differential-difference equation; shared value; uniqueness; finite order

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1. Introduction

In this paper, we assume that the reader is familiar with the standard notations and fundamental results of Nevanlinna theory in [1–3]. A meromorphic function $f(z)$ usually refers to the meromorphic function in the whole complex plane $\mathbb{C}$. In a special case, if $T(r,a)=o\{T(r,f)\}$, then the meromorphic function $a(z)$ is called a small function of $f(z)$.

For meromorphic functions $f(z)$ and $g(z)$, $a\in\mathbb{C}\cup\{\infty\}$. If the zeros of $f(z)-a$ and $g(z)-a$ are the same with the same multiplicities (ignoring multiplicities), we often call that $f(z)$ and $g(z)$ share the value a CM (IM). Especially, $f(z)$ and $g(z)$ are said to share the value ∞ CM (IM) provided that the poles of $f(z)$ and $g(z)$ are the same with the same multiplicities (ignoring multiplicities).

The uniqueness theory of meromorphic functions plays a significant role in complex analysis. The research of meromorphic solutions of complex differential-difference equations has become a subject of great interest in the last decades, due to the application of Nevanlinna value distribution (difference analogue) theory in complex differential-difference equations. Moreover, the

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uniqueness theory of meromorphic functions is an important part of Nevanlinna theory. The following results are the classic five-value theorem and four-value theorem established by Nevanlinna.

Theorem 1.1 ([3,4]) *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions and let a_j $(j = 1, 2, 3, 4, 5)$ be five distinct complex numbers in the extended complex plane. If $f(z)$ and $g(z)$ share the values a_j $(j = 1, 2, 3, 4, 5)$ IM, then $f(z) \equiv g(z)$.*

Theorem 1.2 ([3,4]) *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions and let a_j $(j = 1, 2, 3, 4)$ be four distinct complex numbers in the extended complex plane. If $f(z)$ and $g(z)$ share the values a_j $(j = 1, 2, 3, 4)$ CM, then $f(z) \equiv g(z)$ or $f(z) \equiv T(g(z))$, where T is a Möbius transformation.*

In recent years, many scholars have devoted to studying whether the conditions for shared values can be relaxed, the number of shared values can be decreased, or shared values can be replaced by shared sets or small functions. In this direction, we recall two related situations. On the one hand, one study the case when a meromorphic function and its shift, or difference operator share some values such as [5–9]. On the other hand, with the rapid development of theoretical research on complex differential equations, difference equations and differential-difference equations, the uniqueness problems are often combined with meromorphic solutions of the equations as well as [5, 10–15]. Chen and Shon [16] considered the growth of meromorphic solutions of the following difference equation:

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = F(z), \tag{1.1}$$

where $F(z)$, $P_n(z)$, $P_0(z)$ are polynomials such that $F(z)P_n(z)P_0(z) \not\equiv 0$. They proved the following result.

Theorem 1.3 ([16]) *Let $F(z)$, $P_n(z)$, $P_0(z)$ be polynomials such that $F(z)P_n(z)P_0(z) \not\equiv 0$, satisfying $P_n + \cdots + P_0 \not\equiv 0$. If $f(z)$ is a finite order transcendental meromorphic solution of the equation (1.1), then it satisfies $\lambda(f) = \sigma(f) \geq 1$. Here, the order $\sigma(f)$ and the exponent $\lambda(f)$ of convergence of zeros of $f(z)$ are defined in turn as follows:*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f(z))}{\log r}, \quad \lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r, \frac{1}{f(z)})}{\log r}.$$

Later, Cui and Chen [11] investigated the uniqueness of meromorphic solutions of the equation (1.1) in the case of $n = 1$ and obtained the following conclusion.

Theorem 1.4 ([11]) *Suppose that $a_1(z)$, $a_0(z)$ are nonzero polynomials such that $a_1(z)+a_0(z) \not\equiv 0$. Let $f(z)$ be a finite order transcendental meromorphic solution of difference equation*

$$a_1(z)f(z+1) + a_0(z)f(z) = 0. \tag{1.2}$$

If $f(z)$ and a meromorphic function $g(z)$ share 0, 1, ∞ CM, then $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.

In [12], Cui and Chen generalized the homogeneous difference equation (1.2) to the non-homogeneous difference equation, and proved the following theorem.

Theorem 1.5 ([12]) *Let $a_1(z)$, $a_0(z)$, $F(z)$ be nonzero polynomials such that $a_1(z)+a_0(z)\not\equiv 0$. Let $f(z)$ be a finite order transcendental meromorphic solution of difference equation*

$$a_1(z)f(z+1)+a_0(z)f(z)=F(z). \tag{1.3}$$

If $f(z)$ and a meromorphic function $g(z)$ share 0, 1, ∞ CM, then one of the following cases holds:

(i) $f(z)\equiv g(z)$;

(ii) $f(z)+g(z)\equiv f(z)g(z)$;

(iii) There exists a polynomial $\beta(z)=az+b_0$ such that $f(z)=\frac{1-e^{\beta(z)}}{e^{\beta(z)}(e^{a_0-b_0}-1)}$, $g(z)=\frac{1-e^{\beta(z)}}{1-e^{b_0-a_0}}$, where a $(\neq 0)$, a_0, b_0 are constants with $e^{a_0}\neq e^{b_0}$.

Recently, Li and Chen [13] extended polynomial coefficients of the equation (1.3) to rational coefficients, and discussed the case that $f(z)$ and $g(z)$ are both meromorphic solutions of the same difference equation. In fact, they proved the following theorem.

Theorem 1.6 ([13]) *Let $R_1(z)\not\equiv 0$, $R_2(z)$, $R_3(z)\not\equiv 0$ be rational functions, and let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of difference equation*

$$R_1(z)f(z+1)+R_2(z)f(z)=R_3(z). \tag{1.4}$$

If $f(z)$ and $g(z)$ share 0, ∞ CM, then either $f(z)\equiv g(z)$ or $f(z)=\frac{R_3(z)}{2R_2(z)}(e^{a_1z+a_0}+1)$ and $g(z)=\frac{R_3(z)}{2R_2(z)}(e^{-a_1z-a_0}+1)$, where a_1, a_0 are constants such that $e^{-a_1}=e^{a_1}=-1$, and the coefficients of the equation (1.4) satisfy $R_1(z)R_3(z+1)=R_3(z)R_2(z+1)$.

Based on the above results, a natural inquisition would be to investigate the case when the difference equations (1.2)–(1.4) are replaced by a more general form of the differential-difference equation with meromorphic coefficients. In this paper, we shall investigate the whole situation, where the following main result is established.

Theorem 1.7 *Let $W_1(z)$, $W_2(z)$, $W_3(z)$ be nonzero meromorphic functions such that their orders of growth are less than one, satisfying $W_1(z)+W_2(z)\not\equiv 0$. Let $f(z)$ be a finite order transcendental meromorphic solution of differential-difference equation*

$$W_1(z)f'(z+1)+W_2(z)f(z)=W_3(z). \tag{1.5}$$

If $f(z)$ and a meromorphic function $g(z)$ share 0, 1, ∞ CM, then one of the following cases holds:

(i) $f(z)\equiv g(z)$;

(ii) $f(z)+g(z)\equiv f(z)g(z)$;

(iii) $f^2(z)(g(z)-1)^2+g^2(z)(f(z)-1)^2\equiv f(z)g(z)(f(z)g(z)-1)$;

(iv) There exists a polynomial $\varphi(z)=az+b_0$ such that $f(z)=\frac{1-e^{\varphi(z)}}{e^{\varphi(z)}(e^{a_0-b_0}-1)}$, $g(z)=\frac{1-e^{\varphi(z)}}{1-e^{b_0-a_0}}$, where a $(\neq 0)$, a_0, b_0 are constants with $e^{a_0}\neq e^{b_0}$.

Example 1.8 The finite order transcendental entire function $f(z)=e^z+1$ is a solution of differential-difference equation $f'(z+1)-ef(z)=-e$. For $g_1(z)=e^z+1$, obviously, $g_1(z)$ and $f(z)$ share 0, 1, ∞ CM, satisfying $f(z)\equiv g_1(z)$. For $g_2(z)=e^{-z}+1$, obviously, $g_2(z)$ and $f(z)$ share 0, 1, ∞ CM, satisfying $f(z)+g_2(z)\equiv f(z)g_2(z)$. For $g_3(z)=e^{2z}+1$, obviously, $g_3(z)$ and $f(z)$ share 1, ∞ CM. However, none of the four cases in Theorem 1.7 holds. This example illustrates that the cases (i), (ii) in Theorem 1.7 may occur and the number of sharing values cannot be less than 3.

Example 1.9 The finite order transcendental entire function $f(z) = 1 + e^{-z\ln 2} + e^{-z\ln 4}$ is a solution of differential-difference equation $f'(z+1) + \frac{\ln 2}{2} f(z) = \frac{\ln 2}{2}$. For $g(z) = 1 + e^{z\ln 2} + e^{z\ln 4}$, $g(z)$ and $f(z)$ share 0, 1, ∞ CM, satisfying $f^2(z)(g(z)-1)^2 + g^2(z)(f(z)-1)^2 \equiv f(z)g(z)(f(z)g(z)-1)$. This example illustrates that the case (iii) in Theorem 1.7 may occur.

Example 1.10 The finite order transcendental entire function $f(z) = \frac{1-e^{z+1}}{e^{z+1}(e-1)}$ is a solution of differential-difference equation $f'(z+1) + e^{-1}f(z) = -\frac{1}{e(e-1)}$. For $g(z) = \frac{1-e^{z+1}}{1-e^{-1}}$, it is evident that $g(z)$ and $f(z)$ share 0, 1, ∞ CM. This example illustrates that the case (iv) in Theorem 1.7 may occur.

Example 1.11 The finite order transcendental meromorphic function $f(z) = \frac{e^z+1}{z}$ is a solution of differential-difference equation $\frac{(z+1)^2}{ez} f'(z+1) - zf(z) = -\frac{1+ez}{ez}$. For $g(z) = \frac{e^z+1}{z}$, obviously, $g(z)$ and $f(z)$ share 0, 1, ∞ CM, satisfying $f(z) \equiv g(z)$. This example illustrates that the case (i) in Theorem 1.7 may occur when $f(z)$ is a transcendental meromorphic function.

Remark 1.12 The ideas for this work come from [11, 12].

2. Some lemmas

In order to prove our result, we need the following lemmas.

Lemma 2.1 ([12]) *Let $\psi(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and $\varphi(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0$ be polynomials, where $a_n\,(\neq 0), a_{n-1}, \ldots, a_0$, $b_n\,(\neq 0), b_{n-1}, \ldots, b_0$ are constants, and $n\,(\geq 1)$ is an integer. Then $\deg(\psi(z+1) - \psi(z)) = \deg(\varphi(z+1) - \varphi(z)) = n-1$ and $\deg(\psi(z+1) + \varphi(z) - \psi(z) - \varphi(z+1)) \leq n-1$.*

Lemma 2.2 ([17]) *Let $f(z)$ be a meromorphic function such that its order of growth $\sigma = \sigma(f)$ and $\sigma < +\infty$. Let η be a nonzero constant. Then for each $\varepsilon > 0$,*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.3 ([3]) *Let $f(z)$, $g(z)$ be nonconstant meromorphic functions, and let the order of growth of $f(z)$ be $\sigma(f)$, and the lower order of growth of $g(z)$ be $\mu(g)$. If $\sigma(f) < \mu(g)$, then*

$$T(r, f) = o\{T(r, g)\}, \quad r \to \infty.$$

Lemma 2.4 ([3]) *Let $f_j(z)\,(j = 1, 2, \ldots, n)$ be meromorphic functions, and $g_j(z)\,(j = 1, 2, \ldots, n)$ be entire functions, satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) *When $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not a constant;*

(iii) *When $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, \ r \notin E,$$

where $E \subset (1, \infty)$ is of finite logarithmic measure. Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.

3. Proof of Theorem 1.7

Now we give the proof of Theorem 1.7 in this section.

Proof Because of $f(z)$ and $g(z)$ sharing 0, 1, ∞ CM and by applying to the second main theorem for $g(z)$, we get

$$\begin{aligned} T(r,g) &\le N(r,g) + N(r,\frac{1}{g}) + N(r,\frac{1}{g-1}) + S(r,g) \\ &= N(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f-1}) + S(r,g) \\ &\le 3T(r,f) + S(r,g). \end{aligned}$$

Similarly, we obtain $T(r,f) \le 3T(r,g) + S(r,f)$. Then $S(r,g) = S(r,f)$ and $g(z)$ is also of finite order since $f(z)$ is of finite order.

From the condition that $f(z)$ shares 0, 1, ∞ CM with $g(z)$, we see that

$$\frac{g(z)}{f(z)} = e^{\psi(z)}, \tag{3.1}$$

$$\frac{g(z)-1}{f(z)-1} = e^{\delta(z)}, \tag{3.2}$$

where $\psi(z)$ and $\delta(z)$ are nonzero polynomials.

If $e^{\psi(z)} \equiv e^{\delta(z)}$, then from (3.1) and (3.2), we know $f(z) \equiv g(z)$. So the case (i) holds.

If $e^{\psi(z)} \not\equiv e^{\delta(z)}$, then by (3.1) and (3.2), we have

$$f(z) = \frac{1 - e^{\delta(z)}}{e^{\psi(z)} - e^{\delta(z)}}. \tag{3.3}$$

If $\psi(z)$ and $\delta(z)$ are nonzero constants such that $e^{\psi} \ne e^{\delta}$, then we see that $f(z)$ is a constant from (3.3). This contradicts that $f(z)$ is transcendental.

Next, suppose that one of $\psi(z)$ and $\delta(z)$ is not a constant at least. We discuss the following three cases.

Case 1. Assume that $\psi(z)$ is a constant and $\delta(z)$ is a nonconstant polynomial. Then we can let $e^{\psi(z)} = m(\ne 0)$ be a constant.

If $m = 1$, then from (3.1) we know $f(z) \equiv g(z)$. So the case (i) holds.

If $m \ne 1$, then we may rewrite (3.3) as

$$f(z) = \frac{1 - e^{\delta(z)}}{m - e^{\delta(z)}}. \tag{3.4}$$

Differentiating (3.4), we can easily obtain

$$\begin{aligned} f'(z) &= (\frac{1 - e^{\delta(z)}}{m - e^{\delta(z)}})' = \frac{-e^{\delta(z)}\delta'(z)(m - e^{\delta(z)}) - (1 - e^{\delta(z)})(-e^{\delta(z)})\delta'(z)}{(m - e^{\delta(z)})^2} \\ &= \frac{e^{\delta(z)}\delta'(z)(e^{\delta(z)} - m + 1 - e^{\delta(z)})}{(m - e^{\delta(z)})^2} = \frac{e^{\delta(z)}\delta'(z)(1-m)}{(m - e^{\delta(z)})^2}. \end{aligned} \tag{3.5}$$

Then substituting (3.4) and (3.5) into the equation (1.5), we know

$$U_{14}(z)e^{2\delta(z+1)+\delta(z)}+U_{13}(z)e^{2\delta(z)}+U_{12}(z)e^{\delta(z)}+U_{11}(z)e^{h_0(z)}=0, \tag{3.6}$$

where $h_0(z)\equiv 0$ and

$$\begin{cases} U_{14}(z)=W_2(z)-W_3(z),\\ U_{13}(z)=((1-m)W_1(z)\delta'(z+1)+2m(W_3(z)-W_2(z)))e^{\delta(z+1)-\delta(z)}+\\ \qquad (mW_3(z)-W_2(z))e^{2(\delta(z+1)-\delta(z))},\\ U_{12}(z)=((m^2-m)W_1(z)\delta'(z+1)+2m(W_2(z)-mW_3(z)))e^{\delta(z+1)-\delta(z)}+\\ \qquad m^2(W_2(z)-W_3(z)),\\ U_{11}(z)=m^2(mW_3(z)-W_2(z)).\end{cases}$$

From Lemma 2.1, we get $\deg(\delta(z+1)-\delta(z))=\deg\delta(z)-1$. Because of the order of $e^{\delta(z)}$ being of regular growth, then $T(r,e^{\delta(z+1)-\delta(z)})=o\{T(r,e^{\delta(z)})\}$.

According to Lemma 2.3, for $j=1,2,3,4$, we get

$$\begin{cases} T(r,U_{1j}(z))=o\{T(r,e^{\delta(z)})\},\\ T(r,U_{1j}(z))=o\{T(r,e^{2\delta(z+1)})\},\\ T(r,U_{1j}(z))=o\{T(r,e^{2\delta(z+1)-\delta(z)})\},\\ T(r,U_{1j}(z))=o\{T(r,e^{2\delta(z+1)+\delta(z)})\}.\end{cases}$$

Using Lemma 2.4 on (3.6), we have $U_{1j}(z)\equiv 0$ $(j=1,2,3,4)$. By $U_{11}(z)\equiv 0$ and $U_{14}(z)\equiv 0$, we obtain $m^2(m-1)W_3(z)\equiv 0$. So either $m\equiv 0$ or $m\equiv 1$ or $W_3(z)\equiv 0$, which is a contradiction.

Case 2. Assume that $\delta(z)$ is a constant and $\psi(z)$ is a nonconstant polynomial. Then we can let $e^{\delta(z)}=n$ $(\neq 0)$ be a constant.

If $n=1$, then from (3.2) we can get $f(z)\equiv g(z)$. So the case (i) holds.

If $n\neq 1$, then (3.3) can be rewritten as

$$f(z)=\frac{1-n}{e^{\psi(z)}-n}. \tag{3.7}$$

Taking the derivative in both sides of (3.7), we can easily have

$$f'(z)=(\frac{1-n}{e^{\psi(z)}-n})'=\frac{(n-1)\psi'(z)e^{\psi(z)}}{(e^{\psi(z)}-n)^2}. \tag{3.8}$$

Then substituting (3.7) and (3.8) into the equation (1.5), we obtain

$$U_{24}(z)e^{2\psi(z+1)+\psi(z)}+U_{23}(z)e^{2\psi(z)}+U_{22}(z)e^{\psi(z)}+U_{21}(z)e^{h_0(z)}=0, \tag{3.9}$$

where $h_0(z)\equiv 0$ and

$$\begin{cases} U_{24}(z)=W_3(z),\\ U_{23}(z)=((1-n)W_1(z)\psi'(z+1)-2nW_3(z))e^{\psi(z+1)-\psi(z)}+\\ \qquad ((n-1)W_2(z)-nW_3(z))e^{2(\psi(z+1)-\psi(z))},\\ U_{22}(z)=((n^2-n)(W_1(z)\psi'(z+1)-2W_2(z))+2n^2W_3(z))e^{\psi(z+1)-\psi(z)}+n^2W_3(z),\\ U_{21}(z)=n^2(nW_2(z)-W_2(z)-nW_3(z)).\end{cases}$$

From Lemma 2.1, we see $\deg(\psi(z+1)-\psi(z))=\deg\psi(z)-1$. Because of the order of $e^{\psi(z)}$ being of regular growth, then $T(r,e^{\psi(z+1)-\psi(z)})=o\{T(r,e^{\psi(z)})\}$.

According to Lemma 2.3, for $j=1,2,3,4$, we can know

$$\begin{cases} T(r,U_{2j}(z))=o\{T(r,e^{\psi(z)})\}, \\ T(r,U_{2j}(z))=o\{T(r,e^{2\psi(z+1)})\}, \\ T(r,U_{2j}(z))=o\{T(r,e^{2\psi(z+1)-\psi(z)})\}, \\ T(r,U_{2j}(z))=o\{T(r,e^{2\psi(z+1)+\psi(z)})\}. \end{cases}$$

Using Lemma 2.4 on (3.9), we have $U_{2j}(z)\equiv 0\,(j=1,2,3,4)$. From $U_{24}(z)\equiv 0$, we know $W_3(z)\equiv 0$, which is a contradiction.

Case 3. Assume that $\psi(z)$ and $\delta(z)$ are nonconstant polynomials satisfying $e^{\psi(z)}\not\equiv e^{\delta(z)}$. Taking the derivative in both sides of (3.3), we can easily have

$$\begin{aligned} f'(z)=&\Big(\frac{1-e^{\delta(z)}}{e^{\psi(z)}-e^{\delta(z)}}\Big)'=\frac{-e^{\delta(z)}\delta'(z)(e^{\psi(z)}-e^{\delta(z)})-(1-e^{\delta(z)})(\psi'(z)e^{\psi(z)}-\delta'(z)e^{\delta(z)})}{(e^{\psi(z)}-e^{\delta(z)})^2} \\ =&\frac{e^{\delta(z)}\delta'(z)(e^{\delta(z)}-e^{\psi(z)})+(e^{\delta(z)}-1)(\psi'(z)e^{\psi(z)}-\delta'(z)e^{\delta(z)})}{(e^{\psi(z)}-e^{\delta(z)})^2}. \end{aligned} \tag{3.10}$$

Then substituting (3.3) and (3.10) into the equation (1.5), we can get

$$\sum_{k=1}^{13} M_k(z)e^{N_k(z)}=0, \tag{3.11}$$

where

$$\begin{cases} M_{13}(z)=W_3(z), \\ M_{12}(z)=W_2(z)-W_3(z), \\ M_{11}(z)=W_3(z), \\ M_{10}(z)=W_2(z)-W_3(z), \\ M_9(z)=2(W_3(z)-W_2(z))+W_1(z)(\psi'(z+1)-\delta'(z+1)), \\ M_8(z)=W_1(z)(\delta'(z+1)-\psi'(z+1))-2W_3(z), \\ M_7(z)=W_1(z)\psi'(z+1), \\ M_6(z)=-W_1(z)\psi'(z+1), \\ M_5(z)=-W_1(z)\delta'(z+1), \\ M_4(z)=W_1(z)\delta'(z+1), \\ M_3(z)=-W_2(z), \\ M_2(z)=2W_2(z), \\ M_1(z)=-W_2(z), \end{cases}$$

and

$$\begin{cases} N_{13}(z)=2\psi(z+1)+\psi(z),\\ N_{12}(z)=2\psi(z+1)+\delta(z),\\ N_{11}(z)=2\delta(z+1)+\psi(z),\\ N_{10}(z)=2\delta(z+1)+\delta(z),\\ N_9(z)=\psi(z+1)+\delta(z+1)+\delta(z),\\ N_8(z)=\psi(z+1)+\delta(z+1)+\psi(z),\\ N_7(z)=\psi(z+1)+\psi(z),\\ N_6(z)=\psi(z+1)+\delta(z),\\ N_5(z)=\delta(z+1)+\psi(z),\\ N_4(z)=\delta(z+1)+\delta(z),\\ N_3(z)=2\psi(z+1),\\ N_2(z)=\psi(z+1)+\delta(z+1),\\ N_1(z)=2\delta(z+1). \end{cases}$$

We divide the degrees of the polynomials $\psi(z)$ and $\delta(z)$ into the following three cases.

Subcase 3.1. If $\deg\psi(z)>\deg\delta(z)\geq 1$, then by (3.11) we have

$$U_{34}(z)e^{2\psi(z+1)+\psi(z)}+U_{33}(z)e^{2\psi(z)}+U_{32}(z)e^{\psi(z)}+U_{31}(z)e^{h_0(z)}=0, \tag{3.12}$$

where $h_0(z)\equiv 0$ and

$$\begin{cases} U_{34}(z)=M_{13}(z),\\ U_{33}(z)=\sum_{k=1}^{4}M_{3,3,k}(z)e^{N_{3,3,k}(z)},\\ U_{32}(z)=\sum_{k=1}^{5}M_{3,2,k}(z)e^{N_{3,2,k}(z)},\\ U_{31}(z)=M_{10}(z)e^{2\delta(z+1)+\delta(z)}+M_4(z)e^{\delta(z+1)+\delta(z)}+M_1(z)e^{2\delta(z+1)}, \end{cases}$$

where

$$\begin{cases} M_{3,3,4}(z)=M_{12}(z),\\ M_{3,3,3}(z)=M_8(z),\\ M_{3,3,2}(z)=M_7(z),\\ M_{3,3,1}(z)=M_3(z), \end{cases}\quad \begin{cases} N_{3,3,4}(z)=2(\psi(z+1)-\psi(z))+\delta(z),\\ N_{3,3,3}(z)=\psi(z+1)-\psi(z)+\delta(z+1),\\ N_{3,3,2}(z)=\psi(z+1)-\psi(z),\\ N_{3,3,1}(z)=2(\psi(z+1)-\psi(z)), \end{cases}$$

and

$$\begin{cases} M_{3,2,5}(z)=M_{11}(z),\\ M_{3,2,4}(z)=M_9(z),\\ M_{3,2,3}(z)=M_6(z),\\ M_{3,2,2}(z)=M_5(z),\\ M_{3,2,1}(z)=M_2(z), \end{cases}\quad \begin{cases} N_{3,2,5}(z)=2\delta(z+1),\\ N_{3,2,4}(z)=\psi(z+1)-\psi(z)+\delta(z+1)+\delta(z),\\ N_{3,2,3}(z)=\psi(z+1)-\psi(z)+\delta(z),\\ N_{3,2,2}(z)=\delta(z+1),\\ N_{3,2,1}(z)=\psi(z+1)-\psi(z)+\delta(z+1). \end{cases}$$

From $\deg\psi(z)>\deg\delta(z)$ and $\deg(\psi(z+1)-\psi(z))=\deg\psi(z)-1$, we can obtain $\deg(\psi(z+1)-\psi(z)+\delta(z))<\deg\psi(z)$. Because of the order of $e^{\psi(z)}$ and $e^{\delta(z)}$ being of regular growth, then $T(r, e^{\delta(z)}) = o\{T(r, e^{\psi(z)})\}$, $T(r, e^{\psi(z+1)-\psi(z)}) = o\{T(r, e^{\psi(z)})\}$, $T(r, e^{\psi(z+1)-\psi(z)+\delta(z+j)}) = o\{T(r, e^{\psi(z)})\}$ and $T(r, e^{2\delta(z+1)+\delta(z)}) = o\{T(r, e^{\psi(z)})\}$ $(j = 0, 1)$.

By Lemma 2.3, we have

$$\begin{cases} T(r, U_{3j}(z)) = o\{T(r, e^{\psi(z)})\}, \\ T(r, U_{3j}(z)) = o\{T(r, e^{2\psi(z+1)})\}, \\ T(r, U_{3j}(z)) = o\{T(r, e^{2\psi(z+1)-\psi(z)})\}, \\ T(r, U_{3j}(z)) = o\{T(r, e^{2\psi(z+1)+\psi(z)})\}. \end{cases}$$

Using Lemma 2.4 on (3.12), we see $U_{3j}(z) \equiv 0$ $(j = 1, 2, 3, 4)$. From $U_{34}(z) \equiv 0$, we get $W_3(z) \equiv 0$, which is a contradiction.

Subcase 3.2. If $\deg \delta(z) > \deg \psi(z) \geq 1$, then (3.11) can be rewritten as

$$U_{44}(z)e^{2\delta(z+1)+\delta(z)} + U_{43}(z)e^{2\delta(z)} + U_{42}(z)e^{\delta(z)} + U_{41}(z)e^{h_0(z)} = 0, \tag{3.13}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} U_{44}(z) = M_{10}(z), \\ U_{43}(z) = \sum_{k=1}^{4} M_{4,3,k}(z)e^{N_{4,3,k}(z)}, \\ U_{42}(z) = \sum_{k=1}^{5} M_{4,2,k}(z)e^{N_{4,2,k}(z)}, \\ U_{41}(z) = M_{13}(z)e^{2\psi(z+1)+\psi(z)} + M_7(z)e^{\psi(z+1)+\psi(z)} + M_3(z)e^{2\psi(z+1)}, \end{cases}$$

where

$$\begin{cases} M_{4,3,4}(z) = M_{11}(z), \\ M_{4,3,3}(z) = M_9(z), \\ M_{4,3,2}(z) = M_4(z), \\ M_{4,3,1}(z) = M_1(z), \end{cases} \quad \begin{cases} N_{4,3,4}(z) = 2(\delta(z+1) - \delta(z)) + \psi(z), \\ N_{4,3,3}(z) = \delta(z+1) - \delta(z) + \psi(z+1), \\ N_{4,3,2}(z) = \delta(z+1) - \delta(z), \\ N_{4,3,1}(z) = 2(\delta(z+1) - \delta(z)), \end{cases}$$

and

$$\begin{cases} M_{4,2,5}(z) = M_{12}(z), \\ M_{4,2,4}(z) = M_8(z), \\ M_{4,2,3}(z) = M_6(z), \\ M_{4,2,2}(z) = M_5(z), \\ M_{4,2,1}(z) = M_2(z), \end{cases} \quad \begin{cases} N_{4,2,5}(z) = 2\psi(z+1), \\ N_{4,2,4}(z) = \delta(z+1) - \delta(z) + \psi(z+1) + \psi(z), \\ N_{4,2,3}(z) = \psi(z+1), \\ N_{4,2,2}(z) = \delta(z+1) - \delta(z) + \psi(z), \\ N_{4,2,1}(z) = \delta(z+1) - \delta(z) + \psi(z+1). \end{cases}$$

From $\deg \delta(z) > \deg \psi(z)$ and $\deg(\delta(z+1) - \delta(z)) = \deg \delta(z) - 1$, we obtain $\deg(\delta(z+1) - \delta(z) + \psi(z)) < \deg \delta(z)$. Because of the order of $e^{\psi(z)}$ and $e^{\delta(z)}$ being of regular growth, then $T(r, e^{\delta(z+1)-\delta(z)}) = o\{T(r, e^{\delta(z)})\}$, $T(r, e^{\delta(z+1)-\delta(z)+\psi(z+j)}) = o\{T(r, e^{\delta(z)})\}$, $T(r, e^{2\psi(z+1)+\psi(z)}) = o\{T(r, e^{\delta(z)})\}$ $(j = 0, 1)$ and $T(r, e^{\psi(z)}) = o\{T(r, e^{\delta(z)})\}$.

According to Lemma 2.3, for $j = 1, 2, 3, 4$, we know

$$\begin{cases} T(r, U_{4j}(z)) = o\{T(r, e^{\delta(z)})\}, \\ T(r, U_{4j}(z)) = o\{T(r, e^{2\delta(z+1)})\}, \\ T(r, U_{4j}(z)) = o\{T(r, e^{2\delta(z+1)-\delta(z)})\}, \\ T(r, U_{4j}(z)) = o\{T(r, e^{2\delta(z+1)+\delta(z)})\}. \end{cases}$$

Using Lemma 2.4 on (3.13), we get $U_{4j}(z) \equiv 0\,(j = 1, 2, 3, 4)$. From $U_{41}(z) \equiv 0$ we have

$$V_{42}(z)e^{2\psi(z+1)+\psi(z)} + V_{41}(z)e^{2\psi(z)} = 0, \tag{3.14}$$

where

$$\begin{cases} V_{42}(z) = W_3(z), \\ V_{41}(z) = W_1(z)\psi'(z+1)e^{\psi(z+1)-\psi(z)} - W_2(z)e^{2(\psi(z+1)-\psi(z))}. \end{cases}$$

Similar to the previous discussion, for $j = 1, 2$, we can see

$$T(r, V_{4j}(z)) = o\{T(r, e^{2\psi(z+1)-\psi(z)})\}.$$

Using Lemma 2.4 on (3.14), we get $V_{4j}(z) \equiv 0\,(j = 1, 2)$. By $V_{42}(z) \equiv 0$ we know $W_3(z) \equiv 0$, which is a contradiction.

Subcase 3.3. If $\deg \delta(z) = \deg \psi(z) = n > 1$, then we can let $\psi(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, and $\delta(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0$, where $a_n\ (\neq 0), a_{n-1}, \ldots, a_0, b_n\ (\neq 0), b_{n-1}, \ldots, b_0$ are constants.

Subcase 3.3.1. If $a_n \neq b_n$, $a_n \neq 2b_n$, $2a_n \neq b_n$, $3a_n \neq 2b_n$, and $2a_n \neq 3b_n$, then (3.11) can be rewritten as

$$\sum_{j=1}^{7} U_{5j}(z)e^{h_{5j}(z)} = 0, \tag{3.15}$$

where

$$\begin{cases} U_{57}(z) = M_{10}(z), \\ U_{56}(z) = M_{13}(z), \\ U_{55}(z) = M_{11}(z)e^{2(\delta(z+1)-\delta(z))} + \\ \qquad M_9(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{54}(z) = M_{12}(z)e^{2(\psi(z+1)-\psi(z))} + \\ \qquad M_8(z)^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{53}(z) = M_7(z)e^{\psi(z+1)-\psi(z)} + M_3(z)e^{2(\psi(z+1)-\psi(z))}, \\ U_{52}(z) = M_6(z)e^{\psi(z+1)-\psi(z)} + M_5(z)e^{\delta(z+1)-\delta(z)} + \\ \qquad M_2(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{51}(z) = M_4(z)e^{\delta(z+1)-\delta(z)} + M_1(z)e^{2(\delta(z+1)-\delta(z))}, \end{cases} \quad \begin{cases} h_{57}(z) = 2\delta(z+1) + \delta(z), \\ h_{56}(z) = 2\psi(z+1) + \psi(z), \\ h_{55}(z) = 2\delta(z) + \psi(z), \\ h_{54}(z) = 2\psi(z) + \delta(z), \\ h_{53}(z) = 2\psi(z), \\ h_{52}(z) = \psi(z) + \delta(z), \\ h_{51}(z) = 2\delta(z). \end{cases}$$

From $a_n \neq b_n$, $a_n \neq 2b_n$, $2a_n \neq b_n$, $3a_n \neq 2b_n$, and $2a_n \neq 3b_n$, we have

$$\begin{aligned} &\deg(2\delta(z+1) + \delta(z) - 2\psi(z+1) - \psi(z)) = \deg(2\delta(z+1) - \delta(z) - \psi(z)) \\ &\quad = \deg(\delta(z+1) - \psi(z)) = \deg(2\delta(z+1) + \delta(z) - 2\psi(z)) = \deg(2\delta(z+1) - \psi(z)) \\ &\quad = \deg(2\delta(z+1) - \delta(z)) = \deg(\psi(z+1) - \delta(z)) = \deg(2\psi(z+1) - \psi(z) - \delta(z)) \\ &\quad = \deg(2\psi(z+1) - \psi(z)) = \deg(2\psi(z+1) - \delta(z)) \\ &\quad = \deg(2\psi(z+1) + \psi(z) - 2\delta(z)) = \deg \delta(z) = \deg \psi(z) = n. \end{aligned}$$

According to Lemma 2.1, we get $\deg(\psi(z+1) - \psi(z)) = \deg(\delta(z+1) - \delta(z)) = n - 1$ and

$\deg(\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)) \le n-1$. By Lemma 2.3, for $j=1,2,3,4,5,6,7$, we obtain

$$T(r,U_{5j}(z)) = o\{T(r,e^{2\delta(z+1)+\delta(z)-2\psi(z+1)-\psi(z)})\}, \quad T(r,U_{5j}(z)) = o\{T(r,e^{\delta(z+1)-\psi(z)})\},$$
$$T(r,U_{5j}(z)) = o\{T(r,e^{2\delta(z+1)-\delta(z)-\psi(z)})\}, \quad T(r,U_{5j}(z)) = o\{T(r,e^{2\delta(z+1)-\delta(z)})\},$$
$$T(r,U_{5j}(z)) = o\{T(r,e^{2\delta(z+1)+\delta(z)-2\psi(z)})\}, \quad T(r,U_{5j}(z)) = o\{T(r,e^{2\delta(z+1)-\psi(z)})\},$$
$$T(r,U_{5j}(z)) = o\{T(r,e^{2\psi(z+1)-\psi(z)-\delta(z)})\}, \quad T(r,U_{5j}(z)) = o\{T(r,e^{\psi(z+1)-\delta(z)})\},$$
$$T(r,U_{5j}(z)) = o\{T(r,e^{2\psi(z+1)-\psi(z)})\}, \quad T(r,U_{5j}(z)) = o\{T(r,e^{2\psi(z+1)-\delta(z)})\},$$
$$T(r,U_{5j}(z)) = o\{T(r,e^{2\psi(z+1)+\psi(z)-2\delta(z)})\}, \quad T(r,U_{5j}(z)) = o\{T(r,e^{\delta(z)})\},$$
$$T(r,U_{5j}(z)) = o\{T(r,e^{\psi(z)})\}.$$

Using Lemma 2.4 on (3.15), we get $U_{5j}(z) \equiv 0\,(j=1,2,3,4,5,6,7)$. From $U_{56}(z) \equiv 0$ we see $W_3(z) \equiv 0$, which is a contradiction.

Subcase 3.3.2. If $a_n = 2b_n$, then (3.11) can be rewritten as

$$\sum_{j=1}^{5} U_{6j}(z)e^{h_{6j}(z)} = 0, \tag{3.16}$$

where

$$\begin{cases} U_{65}(z) = M_{13}(z), \\ U_{64}(z) = \sum_{k=1}^{4} M_{6,4,k}(z)e^{N_{6,4,k}(z)}, \\ U_{63}(z) = M_{12}(z)e^{2(\psi(z+1)-\psi(z))} + M_8(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{62}(z) = \sum_{k=1}^{4} M_{6,2,k}(z)e^{N_{6,2,k}(z)}, \\ U_{61}(z) = M_4(z)e^{\delta(z+1)-\delta(z)} + M_1(z)e^{2(\delta(z+1)-\delta(z))}, \end{cases}$$

and

$$\begin{cases} h_{65}(z) = 2\psi(z+1)+\psi(z), \\ h_{64}(z) = 2\delta(z)+\psi(z), \\ h_{63}(z) = 2\psi(z)+\delta(z), \\ h_{62}(z) = \psi(z)+\delta(z), \\ h_{61}(z) = 2\delta(z), \end{cases}$$

where

$$\begin{cases} M_{6,4,4}(z) = M_{11}(z), \\ M_{6,4,3}(z) = M_9(z), \\ M_{6,4,2}(z) = M_7(z), \\ M_{6,4,1}(z) = M_3(z), \end{cases} \qquad \begin{cases} N_{6,4,4}(z) = 2(\delta(z+1)-\delta(z)), \\ N_{6,4,3}(z) = \delta(z+1)-\delta(z)+\psi(z+1)-\psi(z), \\ N_{6,4,2}(z) = \psi(z+1)-2\delta(z), \\ N_{6,4,1}(z) = 2\psi(z+1)-\psi(z)-2\delta(z), \end{cases}$$

and

$$\begin{cases} M_{6,2,4}(z) = M_{10}(z), \\ M_{6,2,3}(z) = M_6(z), \\ M_{6,2,2}(z) = M_5(z), \\ M_{6,2,1}(z) = M_2(z), \end{cases} \qquad \begin{cases} N_{6,2,4}(z) = 2\delta(z+1)-\psi(z), \\ N_{6,2,3}(z) = \psi(z+1)-\psi(z), \\ N_{6,2,2}(z) = \delta(z+1)-\delta(z), \\ N_{6,2,1}(z) = \delta(z+1)-\delta(z)+\psi(z+1)-\psi(z). \end{cases}$$

From $a_n = 2b_n$, we know

$$\begin{aligned}&\deg(\psi(z+1)-\delta(z)) = \deg(2\psi(z+1)-\psi(z)-\delta(z)) = \deg(2\psi(z+1)-\delta(z))\\&\quad = \deg(2\psi(z+1)+\psi(z)-2\delta(z)) = \deg\delta(z) = \deg\psi(z) = n.\end{aligned}$$

According to Lemma 2.1, we get $\deg(\psi(z+1)-\psi(z)) = \deg(\delta(z+1)-\delta(z)) = \deg(\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)) = n-1$ and $\deg(\psi(z+1)-2\delta(z)) = \deg(2\psi(z+1)-\psi(z)-2\delta(z)) \le n-1$. By Lemma 2.3, for $j = 1,2,3,4,5$, we obtain

$$\begin{aligned}&T(r,U_{6j}(z)) = o\{T(r,e^{\psi(z+1)-\delta(z)})\}, && T(r,U_{6j}(z)) = o\{T(r,e^{2\psi(z+1)-\psi(z)-\delta(z)})\},\\&T(r,U_{6j}(z)) = o\{T(r,e^{2\psi(z+1)-\delta(z)})\}, && T(r,U_{6j}(z)) = o\{T(r,e^{2\psi(z+1)+\psi(z)-2\delta(z)})\},\\&T(r,U_{6j}(z)) = o\{T(r,e^{\delta(z)})\}, && T(r,U_{6j}(z)) = o\{T(r,e^{\psi(z)})\}.\end{aligned}$$

Using Lemma 2.4 on (3.16), we have $U_{6j}(z) \equiv 0\,(j = 1,2,3,4,5)$. Because of $U_{65}(z) \equiv 0$, we see $W_3(z) \equiv 0$, which is a contradiction.

Subcase 3.3.3. If $2a_n = b_n$, then we can rewrite (3.11) as

$$\sum_{j=1}^{5} U_{7j}(z)e^{h_{7j}(z)} = 0, \tag{3.17}$$

where

$$\begin{cases}U_{75}(z) = M_{10}(z),\\U_{74}(z) = M_{11}(z)e^{2(\delta(z+1)-\delta(z))}+\\\qquad\qquad M_9(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)},\\U_{73}(z) = M_7(z)e^{\psi(z+1)-\psi(z)} + M_3(z)e^{2(\psi(z+1)-\psi(z))},\\U_{72}(z) = \sum_{k=1}^{4} M_{7,2,k}(z)e^{N_{7,2,k}(z)},\\U_{71}(z) = \sum_{k=1}^{4} M_{7,1,k}(z)e^{N_{7,1,k}(z)},\end{cases}\qquad \begin{cases}h_{75}(z) = 2\delta(z+1)+\delta(z),\\h_{74}(z) = 2\delta(z)+\psi(z),\\h_{73}(z) = 2\psi(z),\\h_{72}(z) = \psi(z)+\delta(z),\\h_{71}(z) = 2\delta(z),\end{cases}$$

where

$$\begin{cases}M_{7,2,4}(z) = M_6(z),\\M_{7,2,3}(z) = M_5(z),\\M_{7,2,2}(z) = M_2(z),\\M_{7,2,1}(z) = M_{13}(z),\end{cases}\quad \begin{cases}N_{7,2,4}(z) = \psi(z+1)-\psi(z),\\N_{7,2,3}(z) = \delta(z+1)-\delta(z),\\N_{7,2,2}(z) = \psi(z+1)-\psi(z)+\delta(z+1)-\delta(z),\\N_{7,2,1}(z) = 2\psi(z+1)-\delta(z),\end{cases}$$

and

$$\begin{cases}M_{7,1,4}(z) = M_4(z),\\M_{7,1,3}(z) = M_1(z),\\M_{7,1,2}(z) = M_{12}(z),\\M_{7,1,1}(z) = M_8(z),\end{cases}\quad \begin{cases}N_{7,1,4}(z) = \delta(z+1)-\delta(z),\\N_{7,1,3}(z) = 2(\delta(z+1)-\delta(z)),\\N_{7,1,2}(z) = 2\psi(z+1)-\delta(z),\\N_{7,1,1}(z) = \psi(z+1)+\delta(z+1)+\psi(z)-2\delta(z).\end{cases}$$

From $2a_n = b_n$ we see

$$\begin{aligned}&\deg(2\delta(z+1)-\delta(z)-\psi(z)) = \deg(2\delta(z+1)+\delta(z)-2\psi(z)) = \deg(2\delta(z+1)-\psi(z))\\&\quad = \deg(2\delta(z+1)-\delta(z)) = \deg\delta(z) = \deg\psi(z) = \deg(\psi(z)-\delta(z)) = n.\end{aligned}$$

According to Lemma 2.1, we have $\deg(\psi(z+1)-\psi(z))=\deg(\delta(z+1)-\delta(z))=\deg(\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z))=n-1$ and $\deg(2\psi(z+1)-\delta(z))=\deg(\psi(z+1)+\psi(z)+\delta(z+1)-2\delta(z))\leq n-1$. By Lemma 2.3, for $j=1,2,3,4,5$, we obtain

$$T(r,U_{7j}(z))=o\{T(r,e^{2\delta(z+1)-\delta(z)-\psi(z)})\},\quad T(r,U_{7j}(z))=o\{T(r,e^{2\delta(z+1)+\delta(z)-2\psi(z)})\},$$
$$T(r,U_{7j}(z))=o\{T(r,e^{2\delta(z+1)-\psi(z)})\},\quad T(r,U_{7j}(z))=o\{T(r,e^{2\delta(z+1)-\delta(z)})\},$$
$$T(r,U_{7j}(z))=o\{T(r,e^{\delta(z)})\},\quad T(r,U_{7j}(z))=o\{T(r,e^{\psi(z)})\},$$
$$T(r,U_{7j}(z))=o\{T(r,e^{\psi(z)-\delta(z)})\}.$$

Using Lemma 2.4 on (3.17), we know $U_{7j}(z)\equiv 0\,(j=1,2,3,4,5)$. From $U_{73}(z)\equiv 0$, $U_{72}(z)\equiv 0$, and $U_{75}(z)\equiv 0$, we see

$$W_1(z)\psi'(z+1)e^{\psi(z+1)-\psi(z)}-W_2(z)e^{2(\psi(z+1)-\psi(z))}=0, \tag{3.18}$$

$$\sum_{k=1}^{4}M_{7,2,k}(z)e^{N_{7,2,k}(z)}=0, \tag{3.19}$$

$$W_2(z)-W_3(z)=0. \tag{3.20}$$

If $n\geq 2$, then $\psi(z+1)-\psi(z)=n-1\geq 1$. So $T(r,W_1(z)\psi'(z+1))=o\{T(r,e^{\psi(z+1)-\psi(z)})\}$ and $T(r,W_2(z))=o\{T(r,e^{\psi(z+1)-\psi(z)})\}$. Using Lemma 2.4 on (3.18), we obtain $W_2(z)\equiv 0$, which is a contradiction. Hence $n=1$.

We can let $\psi(z)=az+a_0$ and $\delta(z)=2az+b_0$, where $a\,(\neq 0)$, a_0, b_0 are constants. Now (3.18) and (3.19) can be rewritten as

$$W_1(z)ae^{a}-W_2(z)e^{2a}=0, \tag{3.21}$$

$$-W_1(z)ae^{a}-2W_1(z)ae^{2a}+2W_2(z)e^{3a}+W_3(z)e^{2a+2a_0-b_0}=0. \tag{3.22}$$

This together with (3.20) gives that

$$e^{2a}W_2(z)(e^{2a_0-b_0}-1)=0. \tag{3.23}$$

If $e^{2a_0-b_0}=1$, then $e^{\delta(z)}=e^{2\psi(z)}$. Thus combining (3.1) with (3.2), we have $\frac{g^2(z)}{f^2(z)}=\frac{g(z)-1}{f(z)-1}$, that is,

$$(f(z)g(z)-(f(z)+g(z)))(f(z)-g(z))=0.$$

So either $f(z)\equiv g(z)$ or $f(z)+g(z)\equiv f(z)g(z)$. This proves that either the case (i) or the case (ii) holds.

If $e^{2a_0-b_0}\neq 1$, then from (3.23), we know $W_2(z)\equiv 0$, which is a contradiction.

Subcase 3.3.4. If $2a_n=3b_n$, then (3.11) can be rewritten as

$$\sum_{j=1}^{6}U_{8j}(z)e^{h_{8j}(z)}=0, \tag{3.24}$$

where

$$\begin{cases} U_{86}(z) = M_{13}(z), \\ U_{85}(z) = M_{11}(z)e^{2(\delta(z+1)-\delta(z))} + \\ \qquad M_9(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{84}(z) = M_{12}(z)e^{2(\psi(z+1)-\psi(z))} + \\ \qquad M_8(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{83}(z) = M_7(z)e^{\psi(z+1)-\psi(z)} + M_3(z)e^{2(\psi(z+1)-\psi(z))} + \\ \qquad M_{10}(z)e^{2\delta(z+1)+\delta(z)-2\psi(z)}, \\ U_{82}(z) = M_6(z)e^{\psi(z+1)-\psi(z)} + M_5(z)e^{\delta(z+1)-\delta(z)} + \\ \qquad M_2(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{81}(z) = M_4(z)e^{\delta(z+1)-\delta(z)} + M_1(z)e^{2(\delta(z+1)-\delta(z))}, \end{cases} \quad \begin{cases} h_{86}(z) = 2\psi(z+1) + \psi(z), \\ h_{85}(z) = 2\delta(z) + \psi(z), \\ h_{84}(z) = 2\psi(z) + \delta(z), \\ h_{83}(z) = 2\psi(z), \\ h_{82}(z) = \psi(z) + \delta(z), \\ h_{81}(z) = 2\delta(z). \end{cases}$$

From $2a_n = 3b_n$, we know

$$\begin{aligned} &\deg(\psi(z+1) - \delta(z)) = \deg(2\psi(z+1) - \psi(z) - \delta(z)) = \deg(2\psi(z+1) - \psi(z)) \\ &\quad = \deg(2\psi(z+1) - \delta(z)) = \deg(2\psi(z+1) + \psi(z) - 2\delta(z)) = \deg(2\delta(z) - \psi(z)) \\ &\quad = \deg \delta(z) = \deg \psi(z) = n. \end{aligned}$$

According to Lemma 2.1, we get $\deg(\psi(z+1) - \psi(z)) = \deg(\delta(z+1) - \delta(z)) = \deg(\psi(z+1) - \psi(z) + \delta(z+1) - \delta(z)) = n-1$ and $\deg(2\delta(z+1) + \delta(z) - 2\psi(z)) \le n-1$. From Lemma 2.3, for $j = 1,2,3,4,5,6$, we have

$$\begin{aligned} &T(r, U_{8j}(z)) = o\{T(r, e^{\psi(z+1)-\delta(z)})\}, && T(r, U_{8j}(z)) = o\{T(r, e^{2\psi(z+1)-\psi(z)-\delta(z)})\}, \\ &T(r, U_{8j}(z)) = o\{T(r, e^{2\psi(z+1)-\psi(z)})\}, && T(r, U_{8j}(z)) = o\{T(r, e^{2\psi(z+1)-\delta(z)})\}, \\ &T(r, U_{8j}(z)) = o\{T(r, e^{2\psi(z+1)+\psi(z)-2\delta(z)})\}, && T(r, U_{8j}(z)) = o\{T(r, e^{\delta(z)})\}, \\ &T(r, U_{8j}(z)) = o\{T(r, e^{2\delta(z)-\psi(z)})\}, && T(r, U_{8j}(z)) = o\{T(r, e^{\psi(z)})\}. \end{aligned}$$

Using Lemma 2.4 on (3.24), we obtain $U_{8j}(z) \equiv 0\,(j = 1,2,3,4,5,6)$. From $U_{86}(z) \equiv 0$ we see $W_3(z) \equiv 0$, which is a contradiction.

Subcase 3.3.5. If $3a_n = 2b_n$, then (3.11) can be rewritten as

$$\sum_{j=1}^{6} U_{9j}(z)e^{h_{9j}(z)} = 0, \tag{3.25}$$

where

$$\begin{cases} U_{96}(z) = M_{10}(z), \\ U_{95}(z) = M_{11}(z)e^{2(\delta(z+1)-\delta(z))} + M_9(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{94}(z) = M_{12}(z)e^{2(\psi(z+1)-\psi(z))} + M_8(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{93}(z) = M_7(z)e^{\psi(z+1)-\psi(z)} + M_3(z)e^{2(\psi(z+1)-\psi(z))}, \\ U_{92}(z) = M_6(z)e^{\psi(z+1)-\psi(z)} + M_5(z)e^{\delta(z+1)-\delta(z)} + M_2(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)}, \\ U_{91}(z) = M_4(z)e^{\delta(z+1)-\delta(z)} + M_1(z)e^{2(\delta(z+1)-\delta(z))} + M_{13}(z)e^{2\psi(z+1)+\psi(z)-2\delta(z)}, \end{cases}$$

and

$$\begin{cases} h_{96}(z) = 2\delta(z+1) + \delta(z), \\ h_{95}(z) = 2\delta(z) + \psi(z), \\ h_{94}(z) = 2\psi(z) + \delta(z), \\ h_{93}(z) = 2\psi(z), \\ h_{92}(z) = \psi(z) + \delta(z), \\ h_{91}(z) = 2\delta(z). \end{cases}$$

Because of $3a_n = 2b_n$, we see

$$\begin{aligned} &\deg(2\delta(z+1) - \delta(z) - \psi(z)) = \deg(\delta(z+1) - \psi(z)) = \deg(2\delta(z+1) + \delta(z) - 2\psi(z)) \\ &\quad = \deg(2\delta(z+1) - \psi(z)) = \deg(2\delta(z+1) - \delta(z)) = \deg \delta(z) = \deg \psi(z) \\ &\quad = \deg(2\psi(z) - \delta(z)) = n. \end{aligned}$$

According to Lemma 2.1, we get $\deg(\psi(z+1) - \psi(z)) = \deg(\delta(z+1) - \delta(z)) = \deg(\psi(z+1) - \psi(z) + \delta(z+1) - \delta(z)) = n-1$ and $\deg(2\psi(z+1) + \psi(z) - 2\delta(z)) \le n-1$. By Lemma 2.3, for $j = 1, 2, 3, 4, 5, 6$, we obtain

$$\begin{aligned} &T(r, U_{9j}(z)) = o\{T(r, e^{2\delta(z+1)-\delta(z)-\psi(z)})\}, &&T(r, U_{9j}(z)) = o\{T(r, e^{\delta(z+1)-\psi(z)})\}, \\ &T(r, U_{9j}(z)) = o\{T(r, e^{2\delta(z+1)+\delta(z)-2\psi(z)})\}, &&T(r, U_{9j}(z)) = o\{T(r, e^{2\delta(z+1)-\psi(z)})\}, \\ &T(r, U_{9j}(z)) = o\{T(r, e^{2\delta(z+1)-\delta(z)})\}, &&T(r, U_{9j}(z)) = o\{T(r, e^{\delta(z)})\}, \\ &T(r, U_{9j}(z)) = o\{T(r, e^{\psi(z)})\}, &&T(r, U_{9j}(z)) = o\{T(r, e^{2\psi(z)-\delta(z)})\}. \end{aligned}$$

Using Lemma 2.4 on (3.25), we have $U_{9j}(z) \equiv 0\,(j = 1, 2, 3, 4, 5, 6)$. From $U_{93}(z) \equiv 0$, $U_{91}(z) \equiv 0$, $U_{95}(z) \equiv 0$, and $U_{96}(z) \equiv 0$, we see

$$W_1(z)\psi'(z+1)e^{\psi(z+1)-\psi(z)} - W_2(z)e^{2(\psi(z+1)-\psi(z))} = 0, \tag{3.26}$$

$$W_1(z)\delta'(z+1)e^{\delta(z+1)-\delta(z)} - W_2(z)e^{2(\delta(z+1)-\delta(z))} + W_3(z)e^{2\psi(z+1)+\psi(z)-2\delta(z)} = 0, \tag{3.27}$$

$$M_{11}(z)e^{2(\delta(z+1)-\delta(z))} + M_9(z)e^{\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z)} = 0, \tag{3.28}$$

$$W_2(z) - W_3(z) = 0. \tag{3.29}$$

If $n \ge 2$, then $\psi(z+1) - \psi(z) = n - 1 \ge 1$. So $T(r, W_1(z)\psi'(z+1)) = o\{T(r, e^{\psi(z+1)-\psi(z)})\}$ and $T(r, W_2(z)) = o\{T(r, e^{\psi(z+1)-\psi(z)})\}$. Using Lemma 2.4 on (3.26), we get $W_2(z) \equiv 0$, which is a contradiction. Hence $n = 1$.

We may let $\psi(z) = \frac{2}{3}az + a_0$ and $\delta(z) = az + b_0$, where $a\,(\neq 0)$, a_0, b_0 are constants. Now we rewrite (3.26)–(3.28) as

$$W_1(z)\frac{2}{3}ae^{\frac{2}{3}a} - W_2(z)e^{\frac{4}{3}a} = 0, \tag{3.30}$$

$$W_1(z)ae^{a} - W_2(z)e^{2a} + W_3(z)e^{\frac{4}{3}a+3a_0-2b_0} = 0, \tag{3.31}$$

$$W_3(z)e^{2a} + (2W_3(z) - W_1(z)a + W_1(z)\frac{2}{3}a - 2W_2(z))e^{\frac{5}{3}a} = 0. \tag{3.32}$$

By (3.29), (3.30) and (3.32), we get

$$e^{a} = 8. \tag{3.33}$$

From (3.29)–(3.31), we have

$$e^{\frac{4}{3}a}W_2(z)(\frac{3}{2}e^{\frac{1}{3}a}-e^{\frac{2}{3}a}+e^{3a_0-2b_0})=0. \tag{3.34}$$

Substituting (3.33) into (3.34), we obtain

$$2^4W_2(z)(-1+e^{3a_0-2b_0})=0. \tag{3.35}$$

If $e^{3a_0-2b_0}=1$, then $e^{2\delta(z)}=e^{3\psi(z)}$. Thus combining (3.1) with (3.2), we get $\frac{g^3(z)}{f^3(z)}=\frac{(g(z)-1)^2}{(f(z)-1)^2}$. So

$$[f^2(z)(g(z)-1)^2+g^2(z)(f(z)-1)^2-f(z)g(z)(f(z)g(z)-1)](f(z)-g(z))=0,$$

that is, $f(z)\equiv g(z)$ or $f^2(z)(g(z)-1)^2+g^2(z)(f(z)-1)^2\equiv f(z)g(z)(f(z)g(z)-1)$. Therefore, either the case (i) or the case (iii) holds.

If $e^{3a_0-2b_0}\neq 1$, then from (3.35) we obtain $W_2(z)\equiv 0$, which is a contradiction.

Subcase 3.3.6. If $a_n=b_n$, then (3.11) can be rewritten as

$$U_{10,2}(z)e^{2\delta(z+1)+\delta(z)}+U_{10,1}(z)e^{\delta(z)+\psi(z+1)}=0, \tag{3.36}$$

where

$$\begin{cases} U_{10,2}(z)=\sum_{k=1}^{6}M_{10,2,k}(z)e^{N_{10,2,k}(z)},\\ U_{10,1}(z)=\sum_{k=1}^{7}M_{10,1,k}(z)e^{N_{10,1,k}(z)},\end{cases}$$

where

$$\begin{cases} M_{10,2,6}(z)=M_{13}(z),\\ M_{10,2,5}(z)=M_{12}(z),\\ M_{10,2,4}(z)=M_{11}(z),\\ M_{10,2,3}(z)=M_{10}(z),\\ M_{10,2,2}(z)=M_{9}(z),\\ M_{10,2,1}(z)=M_{8}(z),\end{cases}\qquad \begin{cases} N_{10,2,6}(z)=2(\psi(z+1)-\delta(z+1))+\psi(z)-\delta(z),\\ N_{10,2,5}(z)=2(\psi(z+1)-\delta(z+1)),\\ N_{10,2,4}(z)=\psi(z)-\delta(z),\\ N_{10,2,3}(z)=0,\\ N_{10,2,2}(z)=\psi(z+1)-\delta(z+1),\\ N_{10,2,1}(z)=\psi(z+1)-\delta(z+1)+\psi(z)-\delta(z),\end{cases}$$

and

$$\begin{cases} M_{10,1,7}(z)=M_{7}(z),\\ M_{10,1,6}(z)=M_{6}(z),\\ M_{10,1,5}(z)=M_{5}(z),\\ M_{10,1,4}(z)=M_{4}(z),\\ M_{10,1,3}(z)=M_{3}(z),\\ M_{10,1,2}(z)=M_{2}(z),\\ M_{10,1,1}(z)=M_{1}(z),\end{cases}\qquad \begin{cases} N_{10,1,7}(z)=\psi(z)-\delta(z),\\ N_{10,1,6}(z)=0,\\ N_{10,1,5}(z)=\psi(z)-\psi(z+1)+\delta(z+1)-\delta(z),\\ N_{10,1,4}(z)=\delta(z+1)-\psi(z+1),\\ N_{10,1,3}(z)=\psi(z+1)-\delta(z),\\ N_{10,1,2}(z)=\delta(z+1)-\delta(z),\\ N_{10,1,1}(z)=2\delta(z+1)-\delta(z)-\psi(z+1).\end{cases}$$

According to Lemma 2.1 and $a_n=b_n$, we know $\deg(\psi(z+1)-\psi(z)+\delta(z+1)-\delta(z))=n-1$, $\deg(2\delta(z+1)-\psi(z+1))=n$, $\deg(\psi(z+i)-\delta(z+j))\leq n-1\,(i,j=0,1)$, $\deg(2\delta(z+1)-\delta(z)-\psi(z+1))\leq n-1$ and $\deg(\psi(z+1)-\delta(z+1)-\psi(z)+\delta(z))\leq n-1$. From Lemma 2.3, for $j=1,2$, we have

$$T(r,U_{10,j}(z))=o\{T(r,e^{2\delta(z+1)-\psi(z+1)})\}.$$

Using Lemma 2.4 on (3.36), we get $U_{10,j}(z) \equiv 0\,(j=1,2)$. Because of $U_{10,1}(z) \equiv 0$, we know

$$\sum_{k=1}^{7} M_{10,1,k}(z)e^{N_{10,1,k}(z)} = 0. \tag{3.37}$$

If $\deg \psi(z) = n \geq 2$, then $\deg(\psi(z+1) - \psi(z)) = n-1 \geq 1$.

If $\deg(\psi(z+i) - \delta(z+j)) \leq n-2\,(i,j=0,1)$, then by Lemma 2.3, $T(r, e^{\psi(z+i)-\delta(z+j)}) = o\{T(r, e^{\delta(z+1)-\delta(z)})\}\,(i,j=0,1)$, $T(r, e^{\delta(z+1)-\delta(z)-\psi(z+1)+\psi(z)}) = o\{T(r, e^{\delta(z+1)-\delta(z)})\}$. Moreover, we can rewrite (3.37) as

$$V_{10,2}(z)e^{\delta(z+1)-\delta(z)} + V_{10,1}(z)e^{h_0(z)} = 0, \tag{3.38}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} V_{10,2}(z) = M_1(z)e^{\delta(z+1)-\psi(z+1)} + M_2(z), \\ V_{10,1}(z) = \sum_{k=1}^{5} K_{10,1,k}(z)e^{L_{10,1,k}(z)}, \end{cases}$$

where

$$\begin{cases} K_{10,1,5}(z) = M_7(z), \\ K_{10,1,4}(z) = M_6(z), \\ K_{10,1,3}(z) = M_5(z), \\ K_{10,1,2}(z) = M_4(z), \\ K_{10,1,1}(z) = M_3(z), \end{cases} \qquad \begin{cases} L_{10,1,5}(z) = \psi(z) - \delta(z), \\ L_{10,1,4}(z) = 0, \\ L_{10,1,3}(z) = \delta(z+1) - \delta(z) + \psi(z) - \psi(z+1), \\ L_{10,1,2}(z) = \delta(z+1) - \psi(z+1), \\ L_{10,1,1}(z) = \psi(z+1) - \delta(z). \end{cases}$$

According to Lemma 2.4 and (3.38), we know $V_{10,j}(z) \equiv 0\,(j=1,2)$. By $V_{10,2}(z) \equiv 0$, we get

$$(-W_2(z)e^{\delta(z)-\psi(z+1)})e^{\delta(z+1)-\delta(z)} + 2W_2(z) = 0. \tag{3.39}$$

Similarly to the previous discussion, using Lemma 2.4 on (3.39), we have $W_2(z) \equiv 0$, which is a contradiction.

If $\deg(\psi(z+i) - \delta(z+j)) = n-1\,(i,j=0,1)$, then from (3.37) and Lemma 2.4, we deduce $W_2(z) \equiv 0$, which is a contradiction. Hence $n = 1$.

We may let $\psi(z) = az + a_0$ and $\delta(z) = az + b_0$, where $a\,(\neq 0)$, a_0 and b_0 are constants such that $e^{a_0 - b_0} \neq 1$. By (3.3), we know

$$f(z) = \frac{1 - e^{\delta(z)}}{e^{\psi(z)} - e^{\delta(z)}} = \frac{1 - e^{\delta(z)}}{e^{\delta(z)}(e^{\psi(z)-\delta(z)} - 1)} = \frac{1 - e^{\delta(z)}}{e^{\delta(z)}(e^{a_0 - b_0} - 1)}.$$

From (3.1), we see

$$g(z) = e^{\psi(z)}f(z) = \frac{e^{\psi(z)}(1 - e^{\delta(z)})}{e^{\psi(z)} - e^{\delta(z)}} = \frac{1 - e^{\delta(z)}}{1 - e^{\delta(z)-\psi(z)}} = \frac{1 - e^{\delta(z)}}{1 - e^{b_0 - a_0}}.$$

So the case (iv) holds.

This completes the proof of Theorem 1.7. □

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References

[1] W. K. HAYMAN. *Meromorphic Functions*. Clarendon Press, Oxford, 1964.

[2] I. LAINE. *Nevanlinna Theory and Complex Differential Equations*. De Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.

[3] Chungchun YANG, Hongxun YI. *Uniqueness Theory of Meromorphic Functions*. Kluwer Academic Publishers, Dordrecht, 2003.

[4] R. NEVANLINNA. *Le Théorème De Picard-Borel et La Théorie Des Fonctions Méromorphes*. Gauthier-Villars, Paris, 1929.

[5] Zongxuan CHEN. *Complex Differences and Difference Equations*. Science Press, Beijing, 2014.

[6] Zongxuan CHEN, Hongxun YI. *On sharing values of meromophic functions and their differences*. Results Math., 2013, **63**: 557–565.

[7] J. HEITTOKANGAS, R. KORHONEN, I. LAINE, et al. *Uniqueness of meromorphic functions sharing values with their shifts*. Complex Var. Elliptic Equ., 2011, **56**(1-4): 81–92.

[8] J. HEITTOKANGAS, R. KORHONEN, I. LAINE, et al. *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*. J. Math. Anal. Appl., 2009, **355**(1): 352–363.

[9] Xiaomin LI, Hongxun YI. *Meromorphic functions sharing four values with their difference operators or shifts*. Bull. Korean Math. Soc., 2016, **53**(4): 1213–1235.

[10] Baoqin CHEN, Sheng LI. *Uniqueness of meromorphic solutions sharing values with a meromorphic function to* $w(z+1)w(z-1)=h(z)w^m(z)$. Adv. Difference Equ., 2019, **372**: 1–9.

[11] Ning CUI, Zongxuan CHEN. *Unicity for meromorphic solutions of some difference equations sharing three values with any meromorphic functions*. J. South China Normal Univ. Nat. Sci., 2016, **48**: 83–87. (in Chinese)

[12] Ning CUI, Zongxuan CHEN. *Uniqueness for meromorphic solutions sharing three values with a meromorphic function to some linear difference equations*. Chin. Ann. Math. Ser. A, 2017, **38**: 13–22. (in Chinese)

[13] Sheng LI, Baoqin CHEN. *Uniqueness of meromorphic solutions of the difference equation* $R_1(z)f(z+1)+R_2(z)f(z)=R_3(z)$. Adv. Difference Equ., 2019, **250**: 1–11.

[14] Feng LÜ, Qi HAN, Weiran LÜ. *On unicity of meromorphic solutions to difference equations of Malmquist type*. Bull. Aust. Math. Soc., 2016, **93**(1): 92–98.

[15] Xiaoguang QI, Nan LI, Lianzhong YANG. *Uniqueness of meromorphic functions concerning their differences and solutions of difference Painlevé equations*. Comput. Methods Funct. Theory, 2018, **18**(4): 567–582.

[16] Zongxuan CHEN, K. H. SHON. *On growth of meromorphic solutions for linear difference equations*. Abstr. Appl. Anal. 2013, Art. ID 619296, 6 pp.

[17] Y. M. CHIANG, Shaoji FENG. *On the Nevanlinna characteristic of* $f(z+\eta)$ *and difference equations in the complex plane*. Ramanujan J., 2008, **16**(1): 105–129.