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Uniqueness and Iterative Schemes of Positive Solutions for Conformable Fractional Differential Equations via Sum-Type Operator Method

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Abstract We are concerned with two points boundary value problems for a kind of conformable fractional differential equations in this paper. By employing the fixed point theorems for a class of sum-type operator defined on a cone, the existence-uniqueness and iterative schemes converging to unique positive solution are established. As applications, two examples are presented to illustrate our main results.

Keywords positive solutions; existence-uniqueness; conformable fractional derivatives; sum-type operator; boundary value problems

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1. Introduction

In recent decades, fractional calculus has become one of the most powerful mathematical tools to model various complex problems in many science fields, including economy, chemistry, biology and engineering [1–6], the study on all kinds of boundary value problems of fractional differential equations received a great attention. Furthermore, based on the nonlinear operator theory in abstract Banach space, a large number of results concerning the existence and uniqueness of solutions had arisen in many literatures [7–11].

Zhai [12] investigated the existence and uniqueness of positive solutions for the following fractional boundary value problems given by

$$\begin{cases} -D^{\alpha}_{0+}u(t) = f(t,u(t)) + g(t,u(t)); \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. Existence and uniqueness of positive solution were obtained by employing the fixed point theorem for a class of sum-type operator. Liu and Zhang [13] studied the existence and uniqueness of the positive solutions for

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the following singular fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}x(t)+p(t)f(t,x(t),x(t))+q(t)g(t,x(t))=0; \\ x(0)=x'(0)=\cdots=x^{(n-1)}(0)=0; \\ x(1)=\int_0^1 k(s)x(s)\mathrm{d}A(s), \end{cases}$$

where $n-1<\alpha\le n$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. By employing a kind of mixed monotone operator fixed point theorem, existence and uniqueness of positive solutions were obtained for above integral boundary value problems. In [14], Qiao investigated the positive solutions of fractional differential equations with infinite-point boundary value condition as follows:

$$\begin{cases} D_{0+}^{\alpha}x(t)+q(t)f(t,x(t))=0; \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0; \\ D_{0+}^{\beta}x(1)=\sum_{i=1}^{\infty}\alpha_i x(\xi_i), \end{cases}$$

where $n-1<\alpha\le n$, $\beta\in[1,\alpha-1]$ is a fixed number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Existence of positive solutions was obtained by using upper and lower solution method.

In recent years, Khalil [15] defined a new fractional derivative called conformable fractional derivative, compared with Riemann-Liouville and Caputo fractional derivative definitions, the conformable fractional derivative is well-behaved and it just depends on the basic limit definition as follows

$$T_\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},$$

where $0<\alpha<1$ and ε is a small enough variable. The new definition seems to be a natural extension of the usual integer derivative, and it satisfies the major properties of the integer derivative. Based on the definition of conformable fractional calculus, Bayour [16] investigated the initial value problems of conformable fractional differential equation as follows:

$$\begin{cases} x^{(\alpha)}(t)=f(t,x(t)); \\ x(a)=x_0, \end{cases}$$

where $f:[a,b]\times R\to R$ is a continuous function, $x^{(\alpha)}(t)=T_\alpha x(t)$ denotes the conformable fractional derivative of x at t of order α. In [17], He et al. studied a class of nonlinear Zoomeron equation with conformable time-fractional derivative, by employing the $\exp(-\varphi(\varepsilon))$-expansion method and the first integral method, various exact analytical traveling wave solutions to the Zoomeron equation were obtained, such as solitary wave, breaking wave and periodic wave.

In this paper, we investigate the existence and uniqueness of positive solution for high-order conformable fractional differential equations with sum-type nonlinear terms as follows:

$$\begin{cases} -T_{\alpha}^{0+}u(t)=f(t,u(t),u(t))+g(t,u(t)),\ 0\le t\le 1; \\ u^{(i)}(0)=0,\ i=0,1,2,3,...,n-2; \\ [T_{\beta}^{0^+}u(t)]_{t=1}=0,\ m-1\le\beta\le m, \end{cases}\tag{1.1}$$

where $n-1<\alpha\le n$, $m=1,2,\ldots,n-1$, $T_{\alpha}^{0^+}$ is conformable fractional derivative.

The remainder of this article is organized as follows. In Section 2, we recall some concepts relative to the conformable fractional calculus and give some lemmas with respect to sum-type operators. In Section 3, we prove the main results about the existence and uniqueness of positive solution for BVP (1.1). In Section 4, we give some examples to verify our main results.

2. Preliminaries

For the convenience, in this section, we give some definitions and lemmas on conformable fractional derivative and integral.

Definition 2.1 ([15]) *The conformable fractional derivative staring from a of a function $f : [a,\infty) \to R$ of order $0<\alpha<1$ is defined by*

$$(T_\alpha^a f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\epsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon},$$

when $a=0$, we write T_α. If $(T_\alpha^a f)(t)$ exists on $[a,b]$, then

$$(T_\alpha^a f)(a) = \lim_{t\to a^+} (T_\alpha^a f)(t).$$

The conformable fractional integral staring from a of a function $f : [a,\infty) \to R$ is defined by

$$(I_\alpha^a f)(t) = I_1^a((t-a)^{\alpha-1} f(t)) = \int_a^t \frac{f(x)}{(x-a)^{1-\alpha}} \mathrm{d}x,$$

where $\alpha \in (0,1)$.

Definition 2.2 ([15]) *Let $\alpha\in(n,n+1)$. The conformable fractional derivative staring from a of a function $f : [a,+\infty) \to R$ of order α, where $f^{(n)}(t)$ exists, is defined by*

$$(T_\alpha^a f)(t) = (T_{\alpha-n}^a f^{(n)}(t)).$$

Let $\alpha\in(n,n+1)$. The conformable fractional integral of order α staring at a is defined by

$$\begin{aligned}(I_\alpha^a f)(t) &= I_{n+1}^a((t-a)^{\alpha-n-1} f(t)) \\ &= \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\alpha-n-1} f(x)\mathrm{d}x.\end{aligned}$$

Lemma 2.3 ([15]) *Let $\alpha\in(n,n+1]$ and $f : [a,+\infty)$ be $(n+1)$ times differentiable. For $t>a$ we have*

$$I_\alpha^a T_\alpha^a f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

Lemma 2.4 *Let $g\in C[0,1]$ be given and T_α^{0+} denote the conformable fractional derivative. Then the following boundary value problem for fractional differential equation*

$$\begin{cases} -T_\alpha^{0+} u(t) = g(t),\ n-1<\alpha\le n; \\ u^{(i)}(0)=0,\quad 0\le i\le n-2; \\ [T_\beta^{0+} u(t)]_{t=1} = 0,\ m-1<\beta\le m,\ 1\le m\le n-1 \end{cases} \tag{2.1}$$

has a unique positive solution

$$u(t) = \int_0^1 G(t,s)g(s)\mathrm{d}s, \tag{2.2}$$

where

$$G(t,s)=\frac{1}{\Gamma(n)}\begin{cases} s^{\alpha-n}[(1-s)^{n-m-1}t^{n-1}-(t-s)^{n-1}], & 0\le s\le t\le 1;\\ (1-s)^{n-m-1}s^{\alpha-n}t^{n-1}, & 0\le t\le s\le 1\end{cases} \tag{2.3}$$

is the Green function.

Proof In view of Lemma 2.3 and Definition 2.2, we can deduce that

$$\begin{aligned} I_\alpha^{0+}T_\alpha^{0+}u(t)&=u(t)-u(0)-u'(0)t-\frac{u''(0)}{2!}t^2-\cdots-\frac{u^{(n-2)}(0)}{(n-2)!}t^{n-2}-\frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1}\\ &=u(t)-\frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1}\end{aligned}$$

and

$$I_\alpha^{0+}g(t)=\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\mathrm{d}s.$$

From $u^{(i)}(0)=0$, it is easy to know that

$$u(t)=\frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1}-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\mathrm{d}s. \tag{2.4}$$

If $1<\beta\le 2$, taking β order conformable fractional derivative on both sides of equation (2.4), we have

$$\begin{aligned} T_\beta^{0+}u(t)&=T_{\beta-1}^{0+}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\mathrm{d}s\Big]\\ &=T_{\beta-1}^{0+}\Big[\frac{u^{(n-1)}(0)}{(n-2)!}t^{n-2}-\frac{1}{(n-2)!}\int_0^t(t-s)^{n-2}s^{\alpha-n}g(s)\mathrm{d}s\Big]\\ &=t^{2-\beta}\frac{u^{(n-1)}(0)}{(n-3)!}t^{n-3}-\frac{t^{2-\beta}}{(n-3)!}\int_0^t(t-s)^{n-3}s^{\alpha-n}g(s)\mathrm{d}s\\ &=\frac{u^{(n-1)}(0)}{(n-3)!}t^{n-\beta-1}-\frac{t^{2-\beta}}{(n-3)!}\int_0^t(t-s)^{n-3}s^{\alpha-n}g(s)\mathrm{d}s.\end{aligned} \tag{2.5}$$

Setting $t=1$, from $[T_\beta^{0+}u(t)]_{t=1}=0$, it is evident that

$$u^{(n-1)}(0)=\int_0^1(1-s)^{n-3}s^{\alpha-n}g(s)\mathrm{d}s. \tag{2.6}$$

Now, combining (2.6) with (2.4), we obtain that

$$u(t)=\frac{1}{(n-1)!}\int_0^1(1-s)^{n-3}s^{\alpha-n}t^{n-1}g(s)\mathrm{d}s-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\mathrm{d}s.$$

If $2<\beta\le 3$, we can deduce that

$$u(t)=\frac{1}{(n-1)!}\int_0^1(1-s)^{n-4}s^{\alpha-n}t^{n-1}g(s)\mathrm{d}s-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\mathrm{d}s.$$

Evidently, if $m-1<\beta\le m$, we have

$$\begin{aligned}u(t)=&\frac{1}{(n-1)!}\int_0^1(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}g(s)\mathrm{d}s-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\mathrm{d}s\\ =&\frac{1}{(n-1)!}\int_0^t(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}g(s)\mathrm{d}s+\frac{1}{(n-1)!}\int_t^1(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}g(s)\mathrm{d}s-\\ &\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\mathrm{d}s\\ =&\int_0^1G(t,s)g(s)\mathrm{d}s,\end{aligned}$$

where

$$G(t,s)=\frac{1}{\Gamma(n)}\begin{cases}s^{\alpha-n}[(1-s)^{n-m-1}t^{n-1}-(t-s)^{n-1}],\ 0\le s\le t\le 1;\\(1-s)^{n-m-1}s^{\alpha-n}t^{n-1},\ 0\le t\le s\le 1\end{cases}$$

is Green function. $\square$

Lemma 2.5 *For $\forall(t,s)\in(0,1)\times(0,1)$, the Green function (2.3) has the following properties:*

(i) $\frac{(1-s)^{n-m-1}[1-(1-s)^m]s^{\alpha-n}t^{n-1}}{\Gamma(n)}\le G(t,s)\le\frac{(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}}{\Gamma(n)}$, *where $n=[\alpha]+1$;*

(ii) $G(t,s)$ is a continuous function and $G(t,s)\ge 0$.

Proof It is evident that the right inequality of (i) holds. So, we only need to prove the left inequality holds. For convenience, set

$$G_1(t,s)=\frac{1}{\Gamma(n)}[s^{\alpha-n}(1-s)^{n-m-1}t^{n-1}-s^{\alpha-n}(t-s)^{n-1}]$$

and

$$G_2(t,s)=\frac{1}{\Gamma(n)}s^{\alpha-n}(1-s)^{n-m-1}t^{n-1},\quad 0\le t\le s\le 1.$$

If $0\le s\le t\le 1$, we have

$$0\le t-s\le t-ts=(1-s)t,$$

and thus

$$(t-s)^{n-1}\le(1-s)^{n-1}t^{n-1}.$$

Then we get

$$\begin{aligned}\Gamma(n)G_1(t,s)&=s^{\alpha-n}[(1-s)^{n-m-1}t^{n-1}-(t-s)^{n-1}]\\&\ge s^{\alpha-n}[(1-s)^{n-m-1}t^{n-1}-(1-s)^{n-1}t^{n-1}]\\&=(1-s)^{n-m-1}[1-(1-s)^m]s^{\alpha-n}t^{n-1}.\end{aligned}$$

If $0\le t\le s\le 1$, we can deduce that

$$\begin{aligned}\Gamma(n)G_2(t,s)&=s^{\alpha-n}(1-s)^{n-m-1}t^{n-1}\\&\ge s^{\alpha-n}[(1-s)^{n-m-1}-(1-s)^{n-1}]t^{n-1}\\&=(1-s)^{n-m-1}[1-(1-s)^m]s^{\alpha-n}t^{n-1}.\end{aligned}$$

It is obvious that $G_1(t,s)$ and $G_2(t,s)$ are continuous on their domains and $G_1(s,s)=G_2(s,s)$. In addition, for $\forall s,t\in[0,1]$, from (i), we know that

$$G(t,s)\geq\frac{1}{\Gamma(n)}(1-s)^{n-m-1}[1-(1-s)^m]s^{\alpha-n}t^{n-1}\geq 0. \quad\square$$

Lemma 2.6 ([18]) *Let $\alpha\in(0,1)$, $A:P\times P\to P$ be a mixed monotone operator satisfying*

$$A(tx,t^{-1}y)\geq t^{\alpha}A(x,y),\quad t\in(0,1),\ x,y\in P.$$

$B:P\to P$ is an increasing sub-homogeneous operator. Assume that

(I) There is $h_0\in P_h$ such that $A(h_0,h_0)\in P_h$ and $Bh_0\in P_h$;

(II) There exists a constant $\delta_0>0$ such that $A(x,y)\geq\delta_0 Bx$ for $\forall x,y\in P$.

Then,

(1) $A:P_h\times P_h\to P_h$, $B:P_h\to P_h$;

(2) There exist $u_0,v_0\in P_h$ and $r\in(0,1)$ such that

$$rv_0\leq u_0<v_0,\ u_0\leq A(u_0,v_0)+Bu_0\leq A(v_0,u_0)+B(v_0)\leq v_0;$$

(3) The operator equation $A(x,x)+Bx=x$ has a unique solution x^* in P_h;

(4) For any initial values $x_0,y_0\in P_h$, constructing successively the sequences

$$x_n=A(x_{n-1},y_{n-1})+Bx_{n-1},\ y_n=A(y_{n-1},x_{n-1})+By_{n-1},\quad n=1,2,\ldots,$$

we have $x_n\to x^*$ and $y_n\to x^*$ as $n\to\infty$.

Lemma 2.7 ([18]) *Let $\alpha\in(0,1)$, $A:P\times P\to P$ be a mixed monotone operator satisfying*

$$A(tx,t^{-1}y)\geq tA(x,y),\quad t\in(0,1),\ x,y\in P.$$

$B:P\to P$ is an increasing α-concave operator. Assume that

(I) There is $h_0\in P_h$ such that $A(h_0,h_0)\in P_h$ and $Bh_0\in P_h$;

(II) There exists a constant $\delta_0>0$ such that $A(x,y)\leq\delta_0 Bx$ for $\forall x,y\in P$.

Then,

(1) $A:P_h\times P_h\to P_h$, $B:P_h\to P_h$;

(2) There exist $u_0,v_0\in P_h$ and $r\in(0,1)$ such that

$$rv_0\leq u_0<v_0,\ u_0\leq A(u_0,v_0)+Bu_0\leq A(v_0,u_0)+B(v_0)\leq v_0;$$

(3) The sum-type operator $T=A+B$ has a unique fixed point $x^*\in P_h$;

(4) For any initial values $x_0,y_0\in P_h$, constructing successively the sequences

$$x_n=T(x_{n-1},y_{n-1}),\ y_n=T(y_{n-1},x_{n-1}),\quad n=1,2,\ldots,$$

we have $x_n\to x^*$ and $y_n\to x^*$ as $n\to\infty$.

3. Main results

In our considerations, we work in the Banach space $C[0,1]=\{u:[0,1]\to R \text{ is continuous}\}$ with the standard norm $\|u\|=\sup\{|u(t)|:t\in[0,1]\}$. Notice that this paper can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \leq y \Leftrightarrow x(t) \leq y(t) \text{ for } t \in [0,1].$$

Let $P = \{u \in C[0,1] | u(t) \geq 0, t \in [0,1]\}$. It is clear that P is a normal cone in $C[0,1]$ and the normality constant is 1. In addition, for given $h > \theta$, set $P_h = \{x \in E | x \sim h\}$, in which $\sim$ is an equivalence relation, i.e., for all $x, y \in E$, $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \geq y \geq \mu x$.

By Lemma 2.4, we know that the boundary value problems (1.1) for conformable fractional differential equations has an integral solution:

$$u(t) = \int_0^1 G(t,s) f(s, u(s), u(s)\mathrm{d}s + \int_0^1 G(t,s) g(s, x(s))\mathrm{d}s,$$

where $G(t,s)$ is given by (2.3). Now, we define two operators,

$$A(u,v)(t) = \int_0^1 G(t,s) f(s, u(s), v(s))\mathrm{d}s \tag{3.1}$$

and

$$B(u)(t) = \int_0^1 G(t,s) g(s, u(s))\mathrm{d}s, \quad t \in [0,1]. \tag{3.2}$$

It is evident that u is a solution of boundary value problems (1.1) if and only if $u = A(u,u) + B(u)$.

Theorem 3.1 *Assume that*

(H_1) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous;

(H_2) For fixed $t \in [0,1]$, $f(t,u,v)$ is increasing in $u \in [0,+\infty)$, and decreasing in $v \in [0,+\infty)$; $g(t,u)$ is increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$;

(H_3) For $\forall t \in [0,1]$, $\lambda \in (0,1)$, there exists a constant $\gamma \in (0,1)$ such that $f(t, \lambda u, \lambda^{-1} v) \geq \lambda^\gamma f(t,u,v)$;

(H_4) For $\forall t \in [0,1]$ and $\lambda \in (0,1)$, $g(t, \lambda u) \geq \lambda g(t,u)$, and there exists a constant $\delta_0 > 0$ such that $f(t,u,v) \geq \delta_0 g(t,u)$.

Then:

(1) The boundary value problems (1.1) has a unique positive solution $x^* \in P_h$, where $h(t) = t^{n-1}$ and $n = [\alpha] + 1$;

(2) There exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 \leq v_0$ and

$$u_0 \leq \int_0^1 G(t,s)(f(s, u_0(s), v_0(s)) + g(s, u_0(s)))\mathrm{d}s,$$

$$v_0 \geq \int_0^1 G(t,s)(f(s, v_0(s), u_0(s)) + g(s, v_0(s)))\mathrm{d}s,$$

where $G(t,s)$ is defined by (2.3).

(3) For any initial value $u_0, v_0 \in P_h$, there are two iterative sequences $\{u_n\}, \{v_n\}$ for approximating u^*, that is, $u_n \to x^*, v_n \to x^*$, as $n \to \infty$, where

$$u_{n+1} = \int_0^1 G(t,s)(f(s, u_n(s), v_n(s)) + g(s, u_n(s)))\mathrm{d}s,$$

$$v_{n+1}=\int_0^1 G(t,s)(f\big(s,v_n(s),u_n(s)\big)+g(s,v_n(s)))\mathrm{d}s,$$

we have $\|u_n-u^*\|\to 0$ *and* $\|v_n-u^*\|\to 0$ *as* $n\to 0$.

Proof Step 1. We prove that A is a mixed monotone operator and B is an increasing operator. From (H_1) and Lemma 2.5, it is easy to know that $A: P\times P\to P$ and $B: P\times P\to P$. From (H_2), for $\forall\, u_i,v_i\in P$, $i=1,2$ with $u_1\ge u_2, v_1\le v_2$, we get

$$\begin{aligned}A(u_1,v_1)(t)&=\int_0^1 G(t,s)f(s,u_1(s),v_1(s))\mathrm{d}s\\&\ge\int_0^1 G(t,s)f(s,u_2(s),v_2(s))\mathrm{d}s=A(u_2,v_2)(t).\end{aligned}$$

That is, A is a mixed monotone operator. Moreover, we can conclude that the operator B is increasing from (H_2) and Lemma 2.5.

Step 2. For $\forall u,v\in P$, we show that $A(\lambda u,\lambda^{-1}v)\ge\lambda^{\gamma}A(u,v)(t)$ and B is a sub-homogeneous operator. For any $\lambda\in(0,1)$ and $u,v\in P$, from (H_3), we have

$$\begin{aligned}A(\lambda u,\lambda^{-1}v)&=\int_0^1 G(t,s)f(s,\lambda u(s),\lambda^{-1}v(s))\mathrm{d}s\\&\ge\lambda^{\gamma}\int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s=\lambda^{\gamma}A(u,v)(t),\end{aligned}$$

that is $A(\lambda u,\lambda^{-1}v)\ge\lambda^{\gamma}A(u,v)(t)$ for $\lambda\in(0,1)$ and $u,v\in P$. In addition, from (H_3), for any $\lambda\in(0,1), u\in P$, we obtain that

$$B(\lambda u)(t)=\int_0^1 G(t,s)g(s,\lambda u(s))\mathrm{d}s\ge\lambda\int_0^1 G(t,s)g(s,u(s))\mathrm{d}s=\lambda Bu(t).$$

Step 3. Let $h_0(t)=\frac{1}{2}t^{n-1}$. It is easy to know $\frac{1}{3}h(t)\le h_0(t)\le 2h(t)$, evidently, $h_0\in P_h$, we show that $A(h_0,h_0)\in P_h$ and $Bh_0\in P_h$. On the one hand, from (H_1), (H_2) and Lemma 2.5, we know that

$$\begin{aligned}A(h_0,h_0)(t)&=\int_0^1 G(t,s)f(s,h_0(s),h_0(s))\mathrm{d}s=\int_0^1 G(t,s)f(s,\frac{1}{2}s^{n-1},\frac{1}{2}s^{n-1})\mathrm{d}s\\&\le\frac{1}{\Gamma(n)}h(t)\int_0^1(1-s)^{n-m-1}s^{\alpha-n}f(s,1,0)\mathrm{d}s.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}A(h_0,h_0)(t)&=\int_0^1 G(t,s)f(s,h_0(s),h_0(s))\mathrm{d}s=\int_0^1 G(t,s)f(s,\frac{1}{2}s^{n-1},\frac{1}{2}s^{n-1})\mathrm{d}s\\&\ge\frac{1}{\Gamma(n)}h(t)\int_0^1[1-(1-s)^m](1-s)^{n-m-1}s^{\alpha-n}f(s,0,1)\mathrm{d}s.\end{aligned}$$

Furthermore, from (H_2) and (H_4), we get

$$f(s,1,0)\ge f(s,0,1)\ge\delta_0 g(s,0)\ge 0.$$

Since $g(t,0)\not\equiv 0$, it is easy to know

$$\int_0^1 f(s,1,0)\mathrm{d}s\ge\int_0^1 f(s,0,1)\mathrm{d}s\ge\delta_0\int_0^1 g(s,0)\mathrm{d}s>0.$$

Set

$$l_1 := \frac{1}{\Gamma(n)} \int_0^1 (1-s)^{n-m-1} s^{\alpha-n} f(s,1,0) \mathrm{d}s$$

and

$$l_2 := \frac{1}{\Gamma(n)} \int_0^1 [1-(1-s)^m](1-s)^{n-m-1} s^{\alpha-n} f(s,0,1) \mathrm{d}s.$$

Evidently, $l_1 \geq l_2 \geq 0$, and $l_2 h \leq A(h_0, h_0) \leq l_1 h$ for $\forall\, t \in [0,1]$, that is $A(h_0, h_0) \in P_h$. Similarly,

$$\begin{aligned}(Bh_0)(t) &= \int_0^1 G(t,s) g(s, h_0(s)) \mathrm{d}s = \int_0^1 G(t,s) g(s, \frac{1}{2} s^{n-1}) \mathrm{d}s \\ &\leq \frac{h(t)}{\Gamma(n)} \int_0^1 (1-s)^{n-m-1} s^{\alpha-n} g(s,1) \mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}(Bh_0)(t) &= \int_0^1 G(t,s) g(s, h_0(s)) \mathrm{d}s = \int_0^1 G(t,s) g(s, \frac{1}{2} s^{n-1}) \mathrm{d}s \\ &\geq \frac{h(t)}{\Gamma(n)} \int_0^1 [1-(1-s)^m](1-s)^{n-m-1} s^{\alpha-n} g(s,0) \mathrm{d}s.\end{aligned}$$

Set

$$l_3 := \frac{1}{\Gamma(n)} \int_0^1 (1-s)^{n-m-1} s^{\alpha-n} g(s,1) \mathrm{d}s$$

and

$$l_4 := \frac{1}{\Gamma(n)} \int_0^1 [1-(1-s)^m](1-s)^{n-m-1} s^{\alpha-n} g(s,0) \mathrm{d}s.$$

It is obvious that $l_3 \geq l_4 \geq 0$ and $l_4 h \leq Bh_0 \leq l_3 h$, that is $Bh_0 \in P_h$. In addition, from (H_4), for $\forall u, v \in P$ and $t \in (0,1)$, we obtain

$$\begin{aligned}A(u,v)(t) &= \int_0^1 G(t,s) f(s, u(s), v(s)) \mathrm{d}s \geq \delta_0 \int_0^1 G(t,s) g(s, u(s)) \mathrm{d}s \\ &= \delta_0 B u(t).\end{aligned}$$

Now, all conditions of Lemma 2.6 are satisfied and an application implies that there exist $u_0, v_0 \in P_h$ and $\gamma \in (0,1)$ such that $\gamma v_0 \leq u_0 < v_0$ and

$$\begin{aligned}u_0 &\leq \int_0^1 G(t,s)\Big(f\big(s, u_0(s), v_0(s)\big) + g\big(s, u_0(s)\big)\Big) \mathrm{d}s, \\ v_0 &\geq \int_0^1 G(t,s)\Big(f\big(s, v_0(s), u_0(s)\big) + g\big(s, v_0(s)\big)\Big) \mathrm{d}s.\end{aligned}$$

In addition, the boundary value problems (1.1) for conformable fractional differential equations has a unique positive solution $u^* \in P_h$. Furthermore, for any initial value $u_0, v_0 \in P_h$, constructing successively the sequences

$$\begin{aligned}u_{n+1} &= \int_0^1 G(t,s)(f(s, u_n(s), v_n(s)) + g(s, u_n(s))) \mathrm{d}s, \\ v_{n+1} &= \int_0^1 G(t,s)(f(s, v_n(s), u_n(s)) + g(s, v_n(s))) \mathrm{d}s,\end{aligned}$$

we have $\|u_n - u^*\| \to 0$ and $\|v_n - u^*\| \to 0$ as $n \to 0$. $\square$

Corollary 3.2 *Let $g(t,u)\equiv 0$. Assume that (H_1)–(H_3) hold and $f(t,0,1)\not\equiv 0$. Then, the following boundary value problem for conformable fractional differential equation*

$$\begin{cases} -T_\alpha^{0+}u(t)=f(t,u(t),u(t)), \quad 0\le t\le 1,\ n-1\le\alpha\le n;\\ u^{(i)}(0)=0, \quad i=0,1,2,3,\dots,n-2;\\ [T_\beta^{0^+}u(t)]_{t=1}=0, \quad m-1\le\beta\le m,\ m=1,2,\dots,n-1\end{cases} \tag{3.3}$$

has a unique positive solution u^* in P_h. Moreover, constructing successively the sequences

$$u_n=\int_0^1 G(t,s)f(s,u_{n-1}(s),v_{n-1}(s))\mathrm{d}s,$$
$$v_n=\int_0^1 G(t,s)f(s,v_{n-1}(s),u_{n-1}(s))\mathrm{d}s,$$

we have $u_n\to u^*$ and $v_n\to u^*$ as $n\to\infty$.

Theorem 3.3 *Assume (H_1), (H_2) hold, and suppose that*

(H_5) For $\forall\lambda\in(0,1),t\in[0,1],u,v\in[0,+\infty)$, there exists a constant $\mu\in(0,1)$ such that $g(t,\lambda u)\ge\lambda^\mu g(t,u)$, and $f(t,\lambda u,\lambda^{-1}v)\ge\lambda f(t,u,v)$.

(H_6) There exists a constant $\delta_0>0$ such that $f(t,u,v)\le\delta_0 g(t,u)$ for $\forall t\in[0,1]$ and $u,v\ge 0$. Then:

(1) The boundary value problems (1.1) has a unique positive solution $x^*\in P_h$, where $h(t)=t^{n-1}$ and $n=[\alpha]+1$;

(2) There exist $u_0,v_0\in P_h$ and $r\in(0,1)$ such that $rv_0\le u_0\le v_0$ and

$$u_0\le\int_0^1 G(t,s)(f(s,u_0(s),v_0(s))+g(s,u_0(s)))\mathrm{d}s,$$
$$v_0\ge\int_0^1 G(t,s)(f(s,v_0(s),u_0(s))+g(s,v_0(s)))\mathrm{d}s,$$

where $G(t,s)$ is defined by (2.3).

(3) For any initial value $u_0,v_0\in P_h$, constructing successively sequences

$$u_{n+1}=\int_0^1 G(t,s)(f(s,u_n(s),v_n(s))+g(s,u_n(s)))\mathrm{d}s,$$
$$v_{n+1}=\int_0^1 G(t,s)(f(s,v_n(s),u_n(s))+g(s,v_n(s)))\mathrm{d}s,$$

we have $u_n\to u^*$ and $v_n\to u^*$ as $n\to 0$.

Proof Firstly, from (H_1) and (H_2), we obtain that $A:P\times P\to P$ is a mixed monotone operator and $B:P\to P$ is increasing. For $\forall\lambda\in(0,1)$, $u,v\in P$, from (H_5), we can know that

$$\begin{aligned}A(\lambda u,\lambda^{-1}v)&=\int_0^1 G(t,s)f(s,\lambda u(s),\lambda^{-1}v(s))\mathrm{d}s\\ &\ge\lambda\int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s=\lambda A(u,v)(t)\end{aligned}$$

and

$$B(\lambda u)(t) = \int_0^1 G(t,s)g(s,\lambda u(s))\mathrm{d}s \geq \lambda^\mu \int_0^1 G(t,s)g(s,u(s))\mathrm{d}s$$
$$= \lambda^\mu Bu(t).$$

From (H_2) and (H_6), we can deduce that

$$g(s,0) \geq \frac{1}{\delta_0} f(s,0,1),\ \ f(s,1,0) \geq f(s,0,1), \quad \forall s \in (0,1).$$

Since $f(t,0,1) \not\equiv 0$, it is obvious that

$$\int_0^1 f(s,1,0)\mathrm{d}s \geq \int_0^1 f(s,0,1)\mathrm{d}s > 0 \tag{3.4}$$

and

$$\int_0^1 g(s,1)\mathrm{d}s \geq \int_0^1 g(s,0)\mathrm{d}s \geq \frac{1}{\delta_0}\int_0^1 f(s,0,1)\mathrm{d}s > 0. \tag{3.5}$$

Secondly, by employing Lemma 2.5, we get

$$\frac{1}{\Gamma(n)}\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}f(s,1,0)\mathrm{d}s$$
$$\geq \frac{1}{\Gamma(n)}\int_0^1 [1-(1-s)^m](1-s)^{n-m-1}s^{\alpha-n}f(s,0,1)\mathrm{d}s > 0$$

and

$$\frac{1}{\Gamma(n)}\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}g(s,1)\mathrm{d}s$$
$$\geq \frac{1}{\Gamma(n)}\int_0^1 [1-(1-s)^m](1-s)^{n-m-1}s^{\alpha-n}g(s,0)\mathrm{d}s > 0.$$

From the proof of Theorem 3.1, we obtain that $A(h_0,h_0) \in P_h$ and $B(h_0) \in P_h$. Furthermore, from (H_6), for $\forall u,v \in P$ and $t \in [0,1]$, we have

$$A(u,v)(t) = \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s \leq \delta_0 \int_0^1 G(t,s)g(s,u(s))\mathrm{d}s = \delta_0 Bu(t). \tag{3.6}$$

Finally, an application of Lemma 2.7 implies that there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$, and

$$u_0 \leq \int_0^1 G(t,s)(f(s,u_0(s),v_0(s)) + g(s,u_0(s)))\mathrm{d}s,$$
$$v_0 \geq \int_0^1 G(t,s)(f(s,v_0(s),u_0(s)) + g(s,v_0(s)))\mathrm{d}s.$$

Moreover, the boundary value problem (1.1) for conformable fractional differential equations has a unique solution $u^* \in P_h$. For any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$u_{n+1} = \int_0^1 G(t,s)(f(s,u_n(s),v_n(s)) + g(s,u_n(s)))\mathrm{d}s,$$
$$v_{n+1} = \int_0^1 G(t,s)(f(s,v_n(s),u_n(s)) + g(s,v_n(s)))\mathrm{d}s,$$

we have $u_n \to u^*$ and $v_n \to u^*$ as $n \to \infty$. $\square$

Corollary 3.4 *Let $f(t,u,v) \equiv 0$. Assume that g satisfied all the conditions of Theorem 3.3 and $g(t,0) \not\equiv 0$ for $t \in (0,1)$. Then, the following boundary value problem*

$$\begin{cases} -T_{\alpha}^{0+}u(t) = g(t,u(t)), \quad 0 \le t \le 1, \ n-1 \le \alpha \le n; \\ u^{(i)}(0) = 0, \quad i = 0,1,2,3,\ldots,n-2; \\ [T_{\beta}^{0^+}u(t)]_{t=1} = 0, \ m-1 \le \beta \le m, \quad m = 1,2,\ldots,n-1 \end{cases} \tag{3.7}$$

has a unique positive solution $u^* \in P_h$; Moreover, for any initial values $u_0, \in P_h$, constructing successively the sequences

$$u_{n+1} = \int_0^1 G(t,s)g(s,u_n(s))\mathrm{d}s, \quad n = 1,2,\ldots,$$

one has $u_n \to u^*$ as $n \to \infty$.

4. Applications

In this section, we conclude this article with the following two examples.

Example 4.1 Let $\alpha = \frac{9}{2}$ and $\beta = \frac{5}{2}$. We consider the following two-point boundary value problem:

$$\begin{cases} -T_{\frac{9}{2}}^{0^+}x(t) = 2(x(t))^{\frac{1}{4}} + t^2 + (x(t)+1)^{-\frac{1}{2}} + 1, \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = x'''(0) = 0, \\ T_{\frac{5}{2}}^{0^+}x(1) = \frac{1}{1+x(1)}. \end{cases} \tag{4.1}$$

Conclusion. BVP (4.1) has a unique positive solution x^* in P_{h_1}, where $h_1(t) = t^4$.

Proof Let $g(t,x) = (x(t))^{\frac{1}{4}} + t^2$ and $g(t,0) = t^2 \not\equiv 0$, $f(t,x,y) = (x(t))^{\frac{1}{4}} + (y(t)+1)^{-\frac{1}{2}} + 1$. Clearly, $g : [0,1] \times [0,+\infty) \to [0,+\infty)$, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous. And it is easy to see that $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in (0,1)$, $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in (0,1)$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in (0,1)$ and $x \in [0,+\infty)$. In addition, $\forall\, \lambda \in (0,1)$ we get

$$\begin{aligned} g(t,\lambda x) &= (\lambda x(t))^{\frac{1}{4}} + t^2 \ge \lambda^{\frac{1}{4}}(x(t))^{\frac{1}{4}} + \lambda t^2 \\ &\ge \lambda((x(t))^{\frac{1}{4}} + t^2)g(t,x) = \lambda g(t,x) \end{aligned}$$

and

$$\begin{aligned} f(t,\lambda x,\lambda^{-1}y) &\ge \lambda^{\frac{1}{4}}(x(t))^{\frac{1}{4}} + \lambda^{\frac{1}{2}}(y(t)+1)^{-\frac{1}{2}} + 1 \\ &\ge \lambda^{\frac{1}{2}}((x(t))^{\frac{1}{4}} + \lambda\frac{1}{2}(y(t)+1)^{-\frac{1}{2}} + 1) = \lambda^{\frac{1}{2}}f(t,x,y). \end{aligned}$$

Moreover, setting $\delta_0 = 1$, we have

$$f(t,x,y) = (x(t))^{\frac{1}{4}} + (y(t)+1)^{-\frac{1}{2}} + 1 \ge (x(t))^{\frac{1}{4}} + t^2 = \delta_0 g(t,x).$$

By employing Theorem 3.1, we know that BVP (4.1) has a unique positive solution x^* in P_{h_1}, where $h_1(t) = t^4$. $\square$

Example 4.2 Let $\alpha = \frac{9}{4}$ and $\beta = \frac{3}{2}$. We consider two points boundary value problem for the conformable fractional differential equation as follows:

$$\begin{cases} -T^{0^+}_{\frac{9}{4}} x(t) = 2 + 2(x(t))^{\frac{1}{4}} + \cos^2 t + \frac{1}{x(t)+1}, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \\ [T^{0^+}_{\frac{3}{2}} x(t)]_{t=1} = \frac{1}{1+x(1)^{\frac{1}{2}}}. \end{cases} \tag{4.2}$$

Conclusion. BVP (4.2) has a unique positive solution x^* in P_{h_2}, where $h_2(t) = t^2$.

Proof Let $g(t,x) = 2+(x(t))^{\frac{1}{4}}$, $f(t,x,y) = \cos^2 t+(x(t))^{\frac{1}{4}}+\frac{1}{y+1}$. Evidently, $g : [0,1]\times[0,+\infty) \to [0,+\infty)$ and $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous. And it is easy to verify that $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in (0,1)$, $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in (0,1)$ and $y \in [0,+\infty)$ and decreasing in $y \in [0,+\infty)$ for fixed $t \in (0,1)$ and $x \in [0,+\infty)$. Moreover, we get

$$g(t,\lambda x) = 2 + (\lambda x(t))^{\frac{1}{4}} \geq \lambda^{\frac{1}{4}} 2 + \lambda^{\frac{1}{4}} (x(t))^{\frac{1}{4}} = \lambda^{\frac{1}{4}} g(t,x)$$

and

$$\begin{aligned} f(t,\lambda x,\lambda^{-1} y) &= \cos^2 t + (\lambda x(t))^{\frac{1}{4}} + \frac{1}{\lambda^{-1} y + 1} \\ &\geq \lambda \cos^2 t + \lambda (x(t))^{\frac{1}{4}} + \frac{\lambda}{y+1} \geq \lambda f(t,x,y). \end{aligned}$$

Furthermore, $f(t,0,1) = \cos^2 t + \frac{1}{2} \not\equiv 0$, setting $\delta_0 = 1$, we obtain

$$f(t,x,y) = \cos^2 t + (x(t))^{\frac{1}{4}} + \frac{1}{y+1} \leq 2 + (x(t))^{\frac{1}{4}} = \delta_0 g(t,x).$$

In consequence, an application of Theorem 3.3 implies that BVP (4.2) has a unique positive solution x^* in P_{h_2}, where $h_2(t) = t^2$. $\square$

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