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On Starlike Meromorphic Functions of Order α

Lei LIU, Jinhua FAN*
Department of Mathematics, Nanjing University of Science and Technology, Jiangsu 210094*, P. R. China*

Abstract Let $S(p)$ be the class of all univalent meromorphic functions f on the unit disk $\mathbb{D}$ with a simple pole at $p \in (0,1)$. For $\alpha \in [0,1)$, we denote by $\Sigma^*(p,\omega_0,\alpha)$ the class of $f \in S(p)$ such that $\hat{\mathbb{C}} \setminus f(\mathbb{D})$ is a starlike domain of order α with respect to fixed point $\omega_0 \neq 0, \infty$. In this paper, some analytic characterizations and coefficient estimates of $f \in \Sigma^*(p,\omega_0,\alpha)$ are considered.

Keywords meromorphic function; starlike function; Taylor coefficient; Laurent coefficient

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1. Introduction

Let S be the class of analytic univalent functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. For $f \in S$, it has the following Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad z \in \mathbb{D}.$$

The famous Bieberbach Conjecture, which was proposed by Bieberbach [1] in 1916, claimed that $|a_n(f)| \leq n$ for $n \in \mathbb{N}$, strict inequality holds for all n unless f is the Koebe function or one of its rotation. Since then, many mathematicians have devoted to this conjecture [2–4]. As we know, in 1984, the conjecture was finally proved by de Branges [5].

During the study of Bieberbach Conjecture, many important subclasses of S have been considered, such as convex functions, starlike functions, close-to-convex functions and so on. For the definitions, basic properties and more details about these subclasses, we refer to the monograph of Duren [6] and Pommerenke [7]. Other properties of these subclasses can be seen in [8–10] and so on. By [6] or [7], a function $f \in S$ is called starlike if the image $f(\mathbb{D})$ is starlike domain with respect to the origin. The class of starlike function is denoted by S^*. It is well-known that $f \in S^*$ if and only if

$$\mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\alpha \in [0,1)$. A function $f \in S$ is called starlike of order α if it satisfies

$$\mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha, \quad z \in \mathbb{D}. \tag{1.2}$$

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* Corresponding author
E-mail address: liulei970205@njust.edu.cn (Lei LIU); jinhuafan@hotmail.com (Jinhua FAN)

The class of starlike function of order α is denoted by $S^*(\alpha)$. Let $f \in S^*(\alpha)$. Robertson [11] studied the Taylor coefficient $a_n(f)$ and proved that

$$|a_n(f)| \le \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}, \quad n \ge 2.$$

We call Ω starlike domain of order α with respect to ω_0, if there exist $f \in S^*(\alpha)$ and a suitable constant a such that $\Omega = \tilde{f}(\mathbb{D})$, where $\tilde{f} = af + \omega_0$. Since

$$\mathrm{Re}(\frac{z\tilde{f}'(z)}{\tilde{f}(z)-\omega_0}) = \mathrm{Re}(\frac{zf'(z)}{f(z)}),$$

then Ω is a starlike domain of order α with respect to ω_0 if and only if there exists an analytic univalent function $\tilde{f}: \mathbb{D} \to \Omega$ with $\tilde{f}(0) = \omega_0$ and $\mathrm{Re}(\frac{z\tilde{f}'(z)}{\tilde{f}(z)-\omega_0}) > \alpha$.

The class Σ is the counterpart to the class S, which maps the outside of the unit circle conformally onto a simply connected domain in $\hat{\mathbb{C}}$. The subclasses of Σ with especial geometry were considered, such as starlike meromorphic functions and concave functions. Originally starlike meromorphic functions map the the outside of the unit circle conformally to the outside of a starlike domain and fix the point at infinity. Later, it turned out to be more convenient to analyze univalent meromorphic functions defined in $\mathbb{D}$ with a simple at some point in $\mathbb{D}$. In the early time, Miller [12,13] and other scholors considered the geometry of a function being starlike meromorphic and deduced several analytic characterizations.

When $0 < p < 1$, let $S(p)$ be the class of univalent meromorphic function in $\mathbb{D}$ with a simple pole at p and the standard normalization $f(0) = f'(0) - 1 = 0$. The class $S(p)$ and its subclasses have been investigated by many scholars [12–14]. When $\omega_0 \ne 0, \infty$, a function $f \in S(p)$ is called starlike meromorphic function with respect to ω_0, if $\hat{\mathbb{C}} \backslash f(\mathbb{D})$ is starlike domain with respect to ω_0. Following [15–17], we let $\Sigma^*(p, \omega_0)$ be the class of starlike meromorphic function with respect to ω_0.

In 1994, Livingston gave analytic characterization for functions in $\Sigma^*(p, \omega_0)$.

Theorem 1.1 ([18]) *Let $f \in S(p)$. Then $f \in \Sigma^*(p, \omega_0)$ if and only if*

$$\mathrm{Re}(\frac{(z-p)(1-zp)f'(z)}{f(z)-\omega_0}) < 0, \quad z \in \mathbb{D}.$$

Theorem 1.2 ([18]) *Let $f \in S(p)$. Then $f \in \Sigma^*(p, \omega_0)$ if and only if*

$$\mathrm{Re}(\frac{zf'(z)}{f(z)-\omega_0} + \frac{p}{z-p} - \frac{pz}{1-pz}) < 0, \quad z \in \mathbb{D}.$$

In 1988, Zhang gave an equivalent integral representation to characterize $f \in \Sigma^*(p, \omega_0)$.

Theorem 1.3 ([19]) *Let $f \in S(p)$. Then $f \in \Sigma^*(p, \omega_0)$ if and only if there exists a probability measure $\mu(x)$ on $\partial\mathbb{D}$ such that*

$$f(z) = \omega_0 + \frac{p\omega_0}{(z-p)(1-zp)} \exp \int_{\partial\mathbb{D}} 2\log(1-xz)\mathrm{d}\mu(x),$$

where ω_0 and μ satisfy the equation $\omega_0 = -\frac{1}{p+\frac{1}{p}-2\int_{\partial\mathbb{D}} x\mathrm{d}\mu(x)}$.

When $f \in \Sigma^*(p,\omega_0)$, it has the Laurent expansion at p

$$f(z) = \sum_{n=-1}^{\infty} b_n (z-p)^n, \quad |z-p| < 1-p \tag{1.3}$$

and the Taylor expansion at the origin

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < p. \tag{1.4}$$

Some estimation results of the Laurent coefficient in (1.3) and the Taylor coefficient in (1.4) were obtained.

Theorem 1.4 ([14]) *Let $f \in \Sigma^*(p,\omega_0)$ with the expansion (1.3). Then*

$$|b_0 - \omega_0| \le \frac{2+p}{1-p^2}|b_{-1}| \tag{1.5}$$

and

$$|b_1| \le \frac{|b_{-1}|}{(1-p^2)^2}. \tag{1.6}$$

Theorem 1.5 ([20]) *Let $f \in \Sigma^*(p,\omega_0)$ with the expansion (1.4). Then the second Taylor coefficient a_2 is determined by*

$$\left|a_2 - (p + \frac{1}{p} + \omega_0) + \frac{1}{4}\omega_0(p + \frac{1}{p} + \omega_0)^2\right| \le |\omega_0|(1 - \frac{1}{4}|p + \frac{1}{p} + \omega_0|^2). \tag{1.7}$$

Other coefficient estimates of $f \in \Sigma^*(p,\omega_0)$ can be found in [14, 16, 21] and so on.

In the whole paper, we restrict $\alpha \in [0,1)$ and $p \in (0,1)$, parallel to the consideration of $S^*(\alpha)$, we call $f \in S(p)$ starlike meromorphic function of order α with respect to ω_0, if $\hat{\mathbb{C}}\backslash f(\mathbb{D})$ is starlike domain of order α with respect to ω_0 ($\neq 0, \infty$). The class of starlike meromorphic function of order α respect to ω_0 is denoted by $\Sigma^*(p,\omega_0,\alpha)$. In this paper, we will give analytic characterizations and the coefficient estimates of $f \in \Sigma^*(p,\omega_0,\alpha)$.

2. Characterizations for starlike meromorphic functions of order α

In this section, similar to Theorems 1.1–1.3, we will give characterizations for $\Sigma^*(p,\omega_0,\alpha)$ as following Theorems 2.1–2.3.

Theorem 2.1 *Let $f \in S(p)$. Then $f \in \Sigma^*(p,\omega_0,\alpha)$ if and only if*

$$\mathrm{Re}(\frac{(z-p)(1-zp)f'(z)}{f(z)-\omega_0}) < -\alpha(1-p^2), \quad z \in \mathbb{D}. \tag{2.1}$$

Theorem 2.2 *Let $f \in \Sigma^*(p,\omega_0,\alpha)$. Then*

$$\mathrm{Re}(\frac{zf'(z)}{f(z)-\omega_0} + \frac{p}{z-p} - \frac{pz}{1-pz}) < -\frac{\alpha(1-p)}{1+p}, \quad z \in \mathbb{D}. \tag{2.2}$$

Theorem 2.3 *Let $f \in \Sigma^*(p,\omega_0,\alpha)$. Then there exists an analytic function $\varphi(z)$ in $\mathbb{D}$ such that*

$$f(z) = \omega_0 + \frac{p\omega_0}{(z-p)(1-zp)} \exp \int_0^z -\frac{2[1-\frac{\alpha(1-p)}{1+p}]\varphi(\zeta)}{1-\zeta\varphi(\zeta)} \mathrm{d}\zeta. \tag{2.3}$$

In order to prove Theorem 2.1, we introduce the following lemmas.

Lemma 2.4 ([22]) *Let* $\mathbb{D}^* = \{z \in \hat{\mathbb{C}} : |z| > 1\}$ *and* $f\colon \mathbb{D}^* \to \hat{\mathbb{C}}$ *be a univalent meromorphic function which maps* $\mathbb{D}^*$ *onto the outside of a bounded Jordan curve* Γ *and* $f(\infty) = \infty$. *Then the curve* Γ *is analytic if and only if* f *is analytic univalent in* $\{z \in \hat{\mathbb{C}} : |z| > r\}$ *for some* $r < 1$.

Lemma 2.5 ([23]) *Let* h *map* $\mathbb{D}$ *conformally onto the inner domain of the Jordan curve* $\Gamma \cap \mathbb{C}$. *Then* Γ *is an analytic curve if and only if* h *is analytic and univalent in* $\{z \in \mathbb{C} : |z| < r\}$ *for some* $r > 1$.

Proof of Theorem 2.1 We denote by $\Omega^* = f(\mathbb{D})$, $\Omega = \hat{\mathbb{C}}\backslash\bar{\Omega}^*$, $\Gamma = \partial\Omega = \partial\Omega^*$. For $r \in (0, \infty)$, we let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D}_r^* = \{z \in \mathbb{C} : |z| > r\}$. We divide the proof of Theorem 2.1 into two parts.

Sufficient part. Let $f \in \Sigma^*(p, \omega_0, \alpha)$. Then (2.1) is satisfied.

Let $u(z) = \frac{1+zp}{z+p}$ map $\mathbb{D}^*$ onto $\mathbb{D}$, $g = f \circ u$ map $\mathbb{D}^*$ onto Ω^* with $g(\infty) = \infty$. By the Riemann mapping theorem, we let $h(z)$ map $\mathbb{D}$ onto Ω and denote by $\Gamma_{1-\frac{1}{k}} = \{h(z) : |z| = 1-\frac{1}{k}\}$, $k = 2, 3, 4, \dots$, $\Omega_{1-\frac{1}{k}}$ and $\Omega^*_{1-\frac{1}{k}}$ are the interior domain and exterior domain of $\Gamma_{1-\frac{1}{k}}$, respectively, and we know $\Gamma_{1-\frac{1}{k}}$ are analytic curves. Let $g_{1-\frac{1}{k}}$ map $\mathbb{D}^*$ onto $\Omega^*_{1-\frac{1}{k}}$ with $g'_{1-\frac{1}{k}}(\infty) > 0$, $g_{1-\frac{1}{k}}(\infty) = \infty$.

Due to the definition of $\Gamma_{1-\frac{1}{k}}$ and Lemma 2.4, each curve can be expressed as $g_{1-\frac{1}{k}}(e^{i\theta})$, $\theta \in [0, 2\pi)$. Since the interior of the curve $\Gamma_{1-\frac{1}{k}}$ is starlike domain of order α with respect to ω_0, then by the geometric property of $\Gamma_{1-\frac{1}{k}}$, we have $\frac{\partial}{\partial\theta}\arg(g_{1-\frac{1}{k}}(e^{i\theta}) - \omega_0) > \alpha$. Therefore,

$$\mathrm{Re}\Big(\frac{zg'_{1-\frac{1}{k}}(z)}{g_{1-\frac{1}{k}(z)} - \omega_0}\Big) = \frac{\partial}{\partial\theta}\arg(g_{1-\frac{1}{k}}(e^{i\theta}) - \omega_0) > \alpha, \quad |z| = 1. \tag{2.4}$$

Since $g_{1-\frac{1}{k}}(\mathbb{D}^*) = \Omega^*_{1-\frac{1}{k}}$, $g_{1-\frac{1}{k}}(\infty) = \infty$ and $g_{1-\frac{1}{k}}(z) = r_{-1}z + r_0 + \sum_{n=1}^{\infty} r_n z^{-n}$, straightforward computation gives

$$\lim_{z\to\infty}\mathrm{Re}\Big(\frac{zg'_{1-\frac{1}{k}}(z)}{g_{1-\frac{1}{k}}(z) - \omega_0}\Big) = 1 > \alpha. \tag{2.5}$$

By (2.4), (2.5) and the maximum principle of harmonic function $\mathrm{Re}(\frac{zg'_{1-\frac{1}{k}}(z)}{g_{1-\frac{1}{k}}(z)-\omega_0})$, we have

$$\mathrm{Re}\Big(\frac{zg'_{1-\frac{1}{k}}(z)}{g_{1-\frac{1}{k}}(z) - \omega_0}\Big) > \alpha, \quad |z| > 1.$$

Since $\Gamma_{1-\frac{1}{k}}$ converges to Γ in the sense of kernel convergence for $k \to \infty$, $g_{1-\frac{1}{k}}$ converges locally uniformly to g due to the Carathéodory kernel theorem [7]. Therefore,

$$\mathrm{Re}\Big(\frac{zg'(z)}{g(z) - \omega_0}\Big) > \alpha, \quad |z| > 1. \tag{2.6}$$

Considering $u(z) = \frac{1+zp}{z+p}$, $g = f \circ u$, simple calculations give

$$g'(z) = -\frac{(u-p)^2}{1-p^2}f'(u), \quad u \in \mathbb{D}, \tag{2.7}$$

and

$$\alpha < \mathrm{Re}(\frac{zg'(z)}{g'(z)-\omega_0}) = -\mathrm{Re}(\frac{(1-up)(u-p)f'(u)}{(1-p^2)(f(u)-\omega_0)}), \quad u \in \mathbb{D}. \tag{2.8}$$

Then (2.1) is satisfied.

Necessary part. Let $f \in S(p)$. If (2.1) is satisfied, then $f \in \Sigma^*(p, \omega_0, \alpha)$.

Let $u(z) = \frac{1+zp}{z+p}$ map $\mathbb{D}^*$ onto $\mathbb{D}$ and $g = f \circ u$ map $\mathbb{D}^*$ onto Ω^* with $g(\infty) = \infty$. We denote by $\Gamma_{1+\frac{1}{k}} = \{g(z) : |z| = 1 + \frac{1}{k}\}$, $k = 2, 3, 4 \ldots$ and we know $\Gamma_{1+\frac{1}{k}}$ are analytic curves. Let $h(z)$ map $\mathbb{D}$ onto Ω, $h_{1+\frac{1}{k}}$ map $\mathbb{D}$ onto $\Omega_{1+\frac{1}{k}}$, where $\Omega_{1+\frac{1}{k}}$ is the interior domain of $\Gamma_{1+\frac{1}{k}}$. If (2.1) is satisfied, by the same computation as (2.7) and (2.8), we have

$$\mathrm{Re}(\frac{zg'(z)}{g(z)-\omega_0}) > \alpha, \quad |z| > 1.$$

By the definition of $\Gamma_{1+\frac{1}{k}}$ and Lemma 2.5, we know each curve can be described by $h_{1+\frac{1}{k}}(e^{i\theta}), \theta \in [0, 2\pi)$. Since the interior of $\Gamma_{1+\frac{1}{k}}$ is starlike domain of order α with respect to ω_0, then by the geometric property of $\Gamma_{1+\frac{1}{k}}$, we have $\frac{\partial}{\partial\theta} arg(h_{1+\frac{1}{k}}(e^{i\theta}) - \omega_0) > \alpha$. Therefore,

$$\mathrm{Re}(\frac{zh'_{1+\frac{1}{k}}(z)}{h_{1+\frac{1}{k}}(z)-\omega_0}) = \frac{\partial}{\partial\theta} arg(h_{1+\frac{1}{k}}(e^{i\theta}) - \omega_0) > \alpha, |z| = 1.$$

By the maximum principle of harmonic function $\mathrm{Re}(\frac{zh'_{1+\frac{1}{k}}(z)}{h_{1+\frac{1}{k}}(z)-\omega_0})$, we have

$$\mathrm{Re}(\frac{zh'_{1+\frac{1}{k}}(z)}{h_{1+\frac{1}{k}}(z)-\omega_0}) > \alpha, \quad |z| < 1.$$

Since $\Gamma_{1+\frac{1}{k}}$ converges to Γ in the sense of kernel convergence for $k \to \infty$, $h_{1+\frac{1}{k}}$ converges locally uniformly to h due to the Carathéodory kernel theorem. Therefore,

$$\mathrm{Re}(\frac{zh'(z)}{h(z)-\omega_0}) > \alpha, \quad |z| < 1.$$

Hence, we have Ω is starlike domain of order α with respect to ω_0 and $f \in \Sigma^*(p, \omega_0, \alpha)$, which completes the proof of Theorem 2.1. □

Using the methods in [18], we give the proof of Theorem 2.2.

Proof of Theorem 2.2 When $p < r < 1$, we let $\sigma = (r-1)p/(r-p^2)$ and $L_r(z) = r(z - \sigma)/(1 - z\bar{\sigma})$. Direct computations give $L_r(p) = p$ and $L_r(\mathbb{D}) = \{z : |z| < r\}$.

For $f \in \Sigma^*(p, \omega_0, \alpha)$, we let

$$P(z) = -\frac{(z-p)(1-pz)f'(z)}{(1-p^2)(f(z)-\omega_0)} \tag{2.9}$$

and

$$Q_r(z) = \frac{z(1-p^2)P(L_r(z)) - p(1-z^2)}{(z-p)(1-zp)}. \tag{2.10}$$

When $|z| = 1$, we have

$$\mathrm{Re}(\frac{-p(1-z^2)}{(z-p)(1-pz)}) = \mathrm{Re}(\frac{-p(\bar{z}-z)}{|1-pz|^2}) = 0 \tag{2.11}$$

and

$$\frac{z}{z-p}=\frac{1}{\overline{1-zp}}. \tag{2.12}$$

Since $L_r(z)\in\mathbb{D}$, by (2.11), (2.12) and Theorem 2.1, when $|z|=1$, we have

$$\begin{aligned}\mathrm{Re}(Q_r(z)) =&\mathrm{Re}(\frac{z(1-p^2)P(L_r(z))}{(z-p)(1-pz)})+\mathrm{Re}(\frac{-p(1-z^2)}{(z-p)(1-pz)})\\ =&\mathrm{Re}(\frac{(1-p^2)P(L_r(z))}{|1-pz|^2})>\frac{\alpha(1-p)}{1+p}.\end{aligned}$$

Since $Q_r(z)$ is analytic for $|z|\le 1$, $L_r(z)\to z$ as $r\to 1$, letting $r\to 1$, we have

$$\mathrm{Re}(\frac{z(1-p^2)P(z)-p(1-z^2)}{(z-p)(1-zp)})>\frac{\alpha(1-p)}{(1+p)},\quad |z|=1. \tag{2.13}$$

By the maximum principle of harmonic function $\mathrm{Re}(\frac{z(1-p^2)P(z)-p(1-z^2)}{(z-p)(1-zp)})$, we have

$$\mathrm{Re}(\frac{z(1-p^2)P(z)-p(1-z^2)}{(z-p)(1-zp)})>\frac{\alpha(1-p)}{(1+p)},\quad |z|<1. \tag{2.14}$$

A straightforward computation gives

$$-\frac{zf'(z)}{f(z)-\omega_0}-\frac{p}{z-p}+\frac{zp}{1-zp}=\frac{z(1-p^2)P(z)-p(1-z^2)}{(z-p)(1-zp)}. \tag{2.15}$$

Then (2.2) follows by (2.14) and (2.15), which completes the proof of Theorem 2.2. □

Proof of Theorem 2.3 It is well-known fact that for an analytic function $p(z)$ in $\mathbb{D}$ with $\mathrm{Re}(p(z))>0$ and $p(0)=1$, then there exists an analytic function $\varphi:\mathbb{D}\to\mathbb{D}$ such that $p(z)=\frac{1+z\varphi(z)}{1-z\varphi(z)}$. We combine this fact with Theorem 2.2, for $f\in\Sigma^*(p,\omega_0,\alpha)$ and

$$p(z)=-\frac{1}{1-\frac{\alpha(1-p)}{1+p}}\{\frac{zf'(z)}{f(z)-\omega_0}+\frac{p}{z-p}-\frac{pz}{1-pz}+\frac{\alpha(1-p)}{1+p}\},$$

then there exists

$$-\frac{1}{1-\frac{\alpha(1-p)}{1+p}}\{\frac{zf'(z)}{f(z)-\omega_0}+\frac{p}{z-p}-\frac{pz}{1-pz}+\frac{\alpha(1-p)}{1+p}\}=\frac{1+z\varphi(z)}{1-z\varphi(z)}. \tag{2.16}$$

Simplifying (2.16), we have

$$\begin{aligned}&\frac{zf'(z)}{f(z)-\omega_0}+\frac{p}{z-p}-\frac{pz}{1-pz}+\frac{\alpha(1-p)}{1+p}\\ &=[-1+\frac{\alpha(1-p)}{1+p}]\frac{1-z\varphi(z)+2z\varphi(z)}{1-z\varphi(z)}.\end{aligned} \tag{2.17}$$

It is easy to check (2.17) is equivalent to

$$\frac{zf'(z)}{f(z)-\omega_0}+\frac{z}{z-p}-\frac{pz}{1-pz}=-\frac{2z[1-\frac{\alpha(1-p)}{1+p}]\varphi(z)}{1-z\varphi(z)}. \tag{2.18}$$

Dividing by z and then integrating from 0 to z on both sides of (2.18), we obtain

$$\log(f(z)-\omega_0)(z-p)(1-zp)-\log p\omega_0=\int_0^z-\frac{2[1-\frac{\alpha(1-p)}{1+p}]\varphi(\zeta)}{1-\zeta\varphi(\zeta)}\mathrm{d}\zeta. \tag{2.19}$$

It is easy to check (2.19) is equivalent to

$$f(z)=\omega_0+\frac{p\omega_0}{(z-p)(1-zp)}\exp\int_0^z-\frac{2[1-\frac{\alpha(1-p)}{1+p}]\varphi(\zeta)}{1-\zeta\varphi(\zeta)}\mathrm{d}\zeta,$$

which completes the proof of Theorem 2.3. □

3. The Laurent coefficient and Taylor coefficient estimates of $f\in\Sigma^*(p,\omega_0,\alpha)$

In this section, let $f\in\Sigma^*(p,\omega_0,\alpha)$. We will estimate the Laurent coefficient in (1.3) and the Taylor coefficient in (1.4). Our main results are Theorems 3.1 and 3.2.

Theorem 3.1 *Let $f\in\Sigma^*(p,\omega_0,\alpha)$ have the Laurent expansion (1.3). Then*

$$|b_1|\le\frac{1-\alpha}{(1-p^2)^2}|b_{-1}| \tag{3.1}$$

and

$$|b_0-\omega_0|\le\frac{p+2(1-\alpha)}{1-p^2}|b_{-1}|. \tag{3.2}$$

Theorem 3.2 *Let $f\in\Sigma^*(p,\omega_0,\alpha)$ have the Taylor expansion (1.4). Then the second coefficient a_2 is determined by*

$$\begin{aligned}&\big|a_2+(\frac{1}{2}-\frac{1}{4\lambda})\omega_0(\frac{1}{\omega_0}+\frac{1}{p}+p)^2-(\omega_0+p+\frac{1}{p})\big|\\&\le|\omega_0|\lambda|1-\frac{1}{4\lambda^2}(\frac{1}{\omega_0}+\frac{1}{p}+p)^2|,\end{aligned} \tag{3.3}$$

where $\lambda=1-\frac{\alpha(1-p)}{1+p}$.

Remark 3.3 When $\alpha=0$, Theorems 3.1 and 3.2 correspond to Theorems 1.4 and 1.5, respectively.

In order to prove Theorems 3.1 and 3.2, we need the following lemmas.

Lemma 3.4 ([6]) *Let $q(z)=1+\sum_{n=1}^{\infty}q_nz^n$, $|z|<1$ be analytic and satisfy the condition* $\mathrm{Re}(q(z))>0$. *Then $|q_n|\le2$, $n\ge1$.*

Lemma 3.5 ([24]) *Let $q(z)=1+\sum_{n=1}^{\infty}q_nz^n,|z|<1$ be analytic and satisfy the condition* $\mathrm{Re}(q(z))>0$. *Then*

$$|q_2-\nu q_1^2|\le2,\quad0\le\nu\le1.$$

Lemma 3.6 ([25]) *Let $\omega(z)=s_1z+s_2z^2+\cdots$, $|z|<1$ be analytic with $|\omega(z)|\le1$. Then*

$$|s_1|\le1,\quad|s_2|\le1-|s_1^2|.$$

Proof of Theorem 3.1 For $f\in\Sigma^*(p,\omega_0,\alpha)$, we let

$$P(z)=-\frac{(z-p)(1-zp)f'(z)}{(1-p^2)(f(z)-\omega_0)}-\alpha. \tag{3.4}$$

We know $P(z)$ is analytic and $\mathrm{Re}(P(z))>0$ by Theorem 2.1.

When f has the expansion (1.3), by (3.4), we have

$$\begin{aligned}P(z)=&\frac{-(1-zp)\sum_{n=-1}^{\infty}nb_n(z-p)^n}{(1-p^2)(\sum_{n=-1}^{\infty}b_n(z-p)^n-\omega_0)}-\alpha\\=&\frac{-(1-zp)[-b_{-1}(z-p)^{-1}+b_1(z-p)+2b_2(z-p)^2+\cdots]}{(1-p^2)[b_{-1}(z-p)^{-1}+b_0+b_1(z-p)+\cdots-\omega_0]}-\alpha\\=&\frac{-(1-zp)[-b_{-1}+b_1(z-p)^2+2b_2(z-p)^3+\cdots]}{(1-p^2)[b_{-1}+b_0(z-p)+b_1(z-p)^2+\cdots-\omega_0(z-p)]}-\alpha.\end{aligned} \tag{3.5}$$

Then, we obtain $P(p)=1-\alpha$. Furthermore, we have the following expansion of $P(z)$

$$P(z)=1-\alpha+\sum_{n=1}^{\infty}c_n(z-p)^n,\quad |z-p|<1-p. \tag{3.6}$$

By (1.3), (3.4) and (3.6), we obtain

$$\begin{aligned}&(1-p^2)(1+\sum_{n=1}^{\infty}c_n(z-p)^n)(\sum_{n=-1}^{\infty}b_n(z-p)^n-\omega_0)\\&=-(1-p^2)\sum_{n=-1}^{\infty}nb_n(z-p)^n+p\sum_{n=-1}^{\infty}nb_n(z-p)^{n+1}.\end{aligned} \tag{3.7}$$

Comparing the constant term and the coefficient of $(z-p)$ in (3.7), we obtain the relations

$$(1-p^2)b_0-\omega_0(1-p^2)+(1-p^2)c_1b_{-1}=-pb_{-1}, \tag{3.8}$$

$$(1-p^2)b_1+(1-p^2)c_1b_0+(1-p^2)c_2b_{-1}-\omega_0(1-p^2)c_1=-(1-p^2)b_1. \tag{3.9}$$

Further, by (3.8) and (3.9), we have

$$\frac{b_1}{b_{-1}}=\frac{pc_1+(1-p^2)(c_1^2-c_2)}{2(1-p^2)}. \tag{3.10}$$

Let

$$Q(z)=\frac{1}{1-\alpha}P(\frac{z+p}{1+zp}).$$

Then $\mathrm{Re}(Q(z))>0$ and $Q(0)=1$. By the Herglotz representation [7] of $Q(z)$, we have

$$\frac{1}{1-\alpha}P(\frac{z+p}{1+zp})=Q(z)=\int_0^{2\pi}\frac{1+e^{it}z}{1-e^{it}z}\mathrm{d}m(t), \tag{3.11}$$

where $m(t)$ is an increasing function with $\int_0^{2\pi}\mathrm{d}m(t)=1$. By (3.11), it is easy to check

$$P(z)=(1-\alpha)\int_0^{2\pi}\frac{1-pz+e^{it}(z-p)}{1-pz-e^{it}(z-p)}\mathrm{d}m(t). \tag{3.12}$$

By (3.6) and (3.12), we have

$$c_1=(1-\alpha)\int_0^{2\pi}\frac{2e^{it}}{1-p^2}\mathrm{d}m(t) \tag{3.13}$$

and

$$c_2=(1-\alpha)\int_0^{2\pi}\frac{2e^{it}(p+e^{it})}{(1-p^2)^2}\mathrm{d}m(t). \tag{3.14}$$

Let

$$T(z) = \int_0^{2\pi} \frac{1+e^{it}z}{1-e^{it}z}\mathrm{d}m(t).$$

Then $\mathrm{Re}(T(z)) > 0$, $T(0) = 1$. Hence, $T(z)$ has the expansion $T(z) = 1 + p_1 z + p_2 z^2 + \cdots$, $|z| < 1$. Direct computation gives

$$p_1 = 2\int_0^{2\pi} e^{it}\mathrm{d}m(t), \tag{3.15}$$

$$p_2 = 2\int_0^{2\pi} e^{2it}\mathrm{d}m(t). \tag{3.16}$$

From (3.13) to (3.16), we have

$$pc_1 + (1-p^2)(c_1^2 - c_2) = \frac{1-\alpha}{1-p^2}(p_1^2(1-\alpha) - p_2).$$

By Lemma 3.5, we have $|p_1^2(1-\alpha) - p_2| \le 2$. Hence

$$|pc_1 + (1-p^2)(c_1^2 - c_2)| \le \frac{2(1-\alpha)}{1-p^2}.$$

Following this fact with (3.10), we obtain (3.1).

By (3.8), (3.13) and (3.15), we have

$$|b_0 - \omega_0| = \frac{|p + (1-p^2)c_1|}{1-p^2}|b_{-1}| = \frac{|p + (1-\alpha)p_1|}{1-p^2}|b_{-1}|.$$

Then by Lemma 3.4, we have

$$\frac{|p + (1-\alpha)p_1|}{1-p^2}|b_{-1}| \le \frac{p + 2(1-\alpha)}{1-p^2}|b_{-1}|.$$

Hence, (3.2) is obtained. □

Proof of Theorem 3.2 Let

$$P(z) = -\frac{1}{1-\frac{\alpha(1-p)}{1+p}}\{\frac{zf'(z)}{f(z)-\omega_0} + \frac{p}{z-p} - \frac{pz}{1-pz} + \frac{\alpha(1-p)}{1+p}\}. \tag{3.17}$$

Then $P(z)$ is analytic and $P(0) = 1$. By Theorem 2.2, we know $\mathrm{Re}(P(z)) > 0$.

In order to compute conveniently, we write

$$P(z) = \frac{1+\omega(z)}{1-\omega(z)}, \quad z \in \mathbb{D}, \tag{3.18}$$

where $\omega(z) : \mathbb{D} \to \mathbb{D}$ is analytic function with $\omega(0) = 0$.

We write

$$P(z) = 1 + d_1 z + d_2 z^2 + \cdots \tag{3.19}$$

and

$$\omega(z) = s_1 z + s_2 z^2 + \cdots . \tag{3.20}$$

Noting (1.4) and (3.19), comparing the coefficients of z^n of (3.17) for $n = 1, 2$, we obtain

$$\begin{cases} d_1 = \frac{1}{1-\frac{\alpha(1-p)}{1+p}}(\frac{1}{\omega_0} + \frac{1}{p} + p), \\ d_2 = \frac{1}{1-\frac{\alpha(1-p)}{1+p}}(\frac{2a_2}{\omega_0} + \frac{1}{\omega_0^2} + \frac{1}{p^2} + p^2). \end{cases} \tag{3.21}$$

Eliminating ω_0 from (3.21), we get

$$d_2 = \frac{1}{1-\frac{\alpha(1-p)}{1+p}}\{\frac{1}{p^2}+p^2+[(1-\frac{\alpha(1-p)}{1+p})d_1-p-\frac{1}{p}]^2\}+ \frac{1}{1-\frac{\alpha(1-p)}{1+p}}\{2a_2[(1-\frac{\alpha(1-p)}{1+p})d_1-p-\frac{1}{p}]\}. \tag{3.22}$$

From (3.18) to (3.20), we have

$$d_1 = 2s_1 \tag{3.23}$$

and

$$d_2 = 2(s_1^2+s_2). \tag{3.24}$$

Let $\lambda = 1-\frac{\alpha(1-p)}{1+p}$, by (3.22) to (3.24). We obtain

$$\begin{aligned} a_2 =& \frac{2(s_2+s_1^2)-\frac{1}{\lambda}(p^2+\frac{1}{p^2})-\frac{1}{\lambda}(2s_1\lambda-\frac{1}{p}-p)^2}{\frac{2}{\lambda}(2s_1\lambda-\frac{1}{p}-p)} \\ =& \frac{1}{p}+p\frac{2s_1^2\lambda^2-s_1^2\lambda-s_2\lambda+p^2-2s_1p\lambda}{1+p^2-2ps_1\lambda}. \end{aligned} \tag{3.25}$$

By Lemma 3.6, we obtain

$$\begin{aligned} &|a_2-\frac{1}{p}-p\frac{2s_1^2\lambda^2-s_1^2\lambda+p^2-2s_1p\lambda}{1+p^2-2ps_1\lambda}| = |\frac{s_2p\lambda}{1+p^2-2ps_1\lambda}| \\ &\le \frac{p\lambda(1-|s_1|^2)}{|1+p^2-2ps_1\lambda|}. \end{aligned} \tag{3.26}$$

By (3.21), (3.23) and (3.26), we have

$$\begin{aligned} &|a_2+(\frac{1}{2}-\frac{1}{4\lambda})\omega_0(\frac{1}{\omega_0}+\frac{1}{p}+p)^2-(\omega_0+p+\frac{1}{p})| \\ &\le |\omega_0|\lambda|1-\frac{1}{4\lambda^2}(\frac{1}{\omega_0}+\frac{1}{p}+p)^2|, \end{aligned}$$

which completes the proof of Theorem 3.2. □

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