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Algebraic Properties of Little-Hankel Operators on Cutoff Harmonic Bergman Space

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Abstract In this paper, we study the finite rank product, commutators and semi-commutators of little-Hankel operators with quasihomogeneous symbols on the cutoff harmonic Bergman space b_n^2. We obtain the conclusions that the commutator and semi-commutator of little-Hankel operators with qusihomogeneous symbols are finite rank operators.

Keywords little-Hankel operator; cutoff Harmonic Bergman space; commutator; finite rank; semi-commutator

MR(2020) Subject Classification 47B35; 47B47

1. Introduction

Let D be the open unit disk in the complex plane C and $\mathrm{d}A$ be the normalized area measure on D. $L^2(D,\mathrm{d}A)$ is the Hilbert space of Lebesgue square integrable functions with respect to $\mathrm{d}A$ on D with the inner product

$$\langle f,g\rangle=\int_D f(z)\overline{g(z)}\mathrm{d}A(z).$$

The classical Bergman space L_a^2 consists of all L^2-holomorphic functions on D. It is a reproducing function space with reproducing kernel

$$K_z(w)=\frac{1}{(1-\bar{z}w)^2},\quad z,w\in D.$$

Harmonic Bergman space $L_h^2(D)$ is the closed subspace of $L^2(D,\mathrm{d}A)$ consisting of the harmonic functions on D. It is clear that

$$L_h^2(D)=L_a^2(D)+\overline{zL_a^2(D)}.$$

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Moreover, L_h^2 is a reproducing space, and its reproducing kernel is denoted by $R_z(w)$. From the above relationship, we have

$$R_z(w) = K_z(w) + \overline{K_z(w)} - 1.$$

For a fixed positive integer n, it is clear that $\{\overline{z}, \overline{z}^2, \ldots, \overline{z}^n\} \subset \overline{zL_a^2}$. We define

$$W_n = \{\overline{z}, \overline{z}^2, \ldots, \overline{z}^n\}^{\vee},$$

in which $\{\cdot\}^{\vee}$ denotes the linear closed space spanned by $\{\cdot\}$. The cutoff harmonic Bergman space b_n^2 is defined by

$$b_n^2 = L_a^2 \bigoplus W_n. \tag{1.1}$$

From the relationship of (1.1), we know that b_n^2 is a reproducing Hilbert space, and its kernel is denoted by $R_z^{(n)}$, and given by

$$R_z^{(n)} = K_z(w) + \sum_{i=1}^{n} (i+1)(z\overline{w})^i.$$

Let P denote the orthogonal projection from $L^2(D)$ onto $L_a^2(D)$. Then for any $f \in L^2(D)$

$$Pf(z) = \langle f, K_z \rangle.$$

Let Q be the orthogonal projection from $L^2(D)$ onto $L_h^2(D)$. Then

$$Qf(z) = \langle f, R_z \rangle, \quad \forall\, f \in L^2.$$

Let P_n denote the orthogonal projection from $L^2(D)$ onto W_n, and $Q_n = P \bigoplus P_n$ be the projection from $L^2(D)$ onto b_n^2. It is clear that

$$Q_n f(z) = \langle f, R_z^{(n)} \rangle, \quad \forall\, f \in L^2(D).$$

Let $U : L^2(D, \mathrm{d}A) \to L^2(D, \mathrm{d}A)$ be the unitary operator defined by $Uf(z) = f(\overline{z})$, where f belongs to $L^2(D, \mathrm{d}A)$. For $\varphi \in L^\infty(D)$, the little-Hankel operator $h_\varphi : b_n^2 \to b_n^2$ with symbol φ is defined by

$$h_\varphi f(z) = Q_n U(\varphi f) = \int_D f(\overline{w})\varphi(\overline{w})\overline{R_z^{(n)}(w)}\mathrm{d}A(w).$$

In the classical Hardy space $H^2(T)$, Brown and Halmos [1] proved that if $T_f T_g = 0$, then either f or g must be identically zero. In the Bergman space, Ahern and Čučković [2] obtained the result is analogous to Brown and Halmos's result about two Toeplitz operators with harmonic symbols. Furthermore, it was shown in [3] that if $T_f T_g = 0$, where f is arbitrary bounded and g is radial, then either $f \equiv 0$ or $g \equiv 0$. Those zero product results have been generalized to finite rank product result in [4]. Čučković and Louhichi characterized finite rank product of several quasihomogeneous Toeplitz operators on the Bergman space of the unit disk in [5]. Moreover, Yang and Lu [6, 7] studied finite rank product of quasihomogeneous Toeplitz operators on the harmonic Bergman space and cutoff harmonic Bergman space, respectively.

For the finite-rank commutator or semi-commutator problems, Axler [8] and Ding [9] have solved it on the Hardy space. On the Bergman space the problem seems to be far from solution. On the Bergman space of the unit disk, Guo, Sun and Zheng [10] completely characterized the finite rank commutator and semi-commutator of two Toeplitz operators with bounded harmonic symbols. Ding [11] characterized the finite rank commutators of Toeplitz operator with bounded harmonic symbols on the cutoff harmonic Bergman space.

For the Hankel operator and little-Hankel operator, many researchers have studied them and have obtained many results for them [12–16].

For two little-Hankel operators h_φ and h_ψ the commutator and semi-commutator are defined by

$$[h_\varphi, h_\psi] = h_\varphi h_\psi - h_\psi h_\varphi, \quad (h_\varphi, h_\psi] = h_{\varphi\psi} - h_\varphi h_\psi.$$

In this paper, we mainly study the commutators and semi-commutators of little-Hankel operators with quasihomogeneous symbols. Meanwhile, the zero-product problem for little-Hankel operators is also discussed.

2. Preliminaries

It is necessary to introduce the Mellin transform of the tradial function before we state our works.

Definition 2.1 *Let $\varphi \in L^1([0,1], r\mathrm{d}r)$. The Mellin transform $\hat{\varphi}$ of a function φ is defined by*

$$\widehat{\varphi}(z) = \int_0^1 \varphi(r) r^{z-1} \mathrm{d}r.$$

It is clear that $\hat{\varphi}$ is well defined on the right half-plane $\{z : \mathrm{Re}\, z \geq 2\}$ and analytic on $\{z : \mathrm{Re}\, z > 2\}$.

Let $p \in \mathbb{Z}$, $\varphi \in L^1(D, \mathrm{d}A)$ is called a quasihomogeneous function of degree p if φ has the expression as below

$$\varphi(re^{i\theta}) = e^{ip\theta} f(r),$$

where f is a radial function. The main reason for studying Toeplitz operators with quasihomogeneous symbols is that any function f in $L^2(D, \mathrm{d}A)$ has the polar decomposition

$$f(re^{i\theta}) = \sum_{k \in \mathbb{Z}} e^{ik\theta} f_k(r),$$

where f_k are radial functions in $L^2([0,1], r\mathrm{d}r)$.

Lemma 2.2 *Let $p \geq 0$ and φ be a bounded radial function. Then we have*

(1) For $h_{e^{ip\theta}\varphi}(z^k)$

If $p > n$,

$$h_{e^{ip\theta}\varphi}(z^k) = 0, \quad \forall\, k \geq 0.$$

If $p \leq n$,

$$h_{e^{ip\theta}\varphi}(z^k) = \begin{cases} 2(k+p+1)\widehat{\varphi}(2k+p+2) z^{k+p}, & 0 \leq k \leq n-p, \\ 0, & k > n-p. \end{cases}$$

(2) For $h_{e^{-ip\theta}\varphi}(z^k)$

$$h_{e^{-ip\theta}\varphi}(z^k)=\begin{cases}2(p-k+1)\widehat{\varphi}(p+2)z^{p-k}, & 0\le k\le p,\\ 2(k-p+1)\widehat{\varphi}(2k-p+2)\overline{z}^{k-p}, & p<k\le p+n,\\ 0, & k>p+n.\end{cases}$$

(3) For $h_{e^{ip\theta}\varphi}(\overline{z}^k)$

If $p>n$,

$$h_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}0, & 1\le k<p-n,\\ 2(p-k+1)\widehat{\varphi}(p+2)\overline{z}^{p-k}, & p-n\le k\le n.\end{cases}$$

If $p\le n$,

$$h_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)\widehat{\varphi}(p+2)\overline{z}^{p-k}, & 1\le k<p,\\ 2(k-p+1)\widehat{\varphi}(2k-p+2)z^{k-p}, & p\le k\le n.\end{cases}$$

(4) For $h_{e^{-ip\theta}\varphi}(\overline{z}^k)$

$$h_{e^{-ip\theta}\varphi}(\overline{z}^k)=2(k+p+1)\widehat{\varphi}(2k+p+2)z^{k+p},\quad \forall 1\le k\le n.$$

3. Finite rank product of little-Hankle operators

We will discuss the finite rank product of little-Hankel operators with quasihomogeneous symbols in this section.

Theorem 3.1 *Let $p_1,\dots,p_m\in\mathbb{Z}^+\bigcup\{0\}$ and $\varphi_1,\dots,\varphi_m$ be bounded radial functions. Then $h_{e^{ip_m\theta}\varphi_m}\cdots h_{e^{ip_1\theta}\varphi_1}$ is the finite rank operator.*

Proof We denote by S the product of little-Hankel operators $h_{e^{ip_m\theta}\varphi_m}\cdots h_{e^{ip_1\theta}\varphi_1}$.

For any $\overline{z}^k\in b_n^2$ $(1\le k\le n)$, it is clear that $\{S(\overline{z}^k)|1\le k\le n\}$ has finite rank, and its rank is less than n.

For $\{S(z^k):k\ge 0\}$, by Lemma 2.4 we know that

(1) When $p_1+p_2+\cdots+p_m\le n$ and $0\le k\le n-(p_1+p_2+\cdots+p_m)$, we have

$$\begin{aligned}S(z^k)=&h_{e^{ip_m\theta}\varphi_m}\cdots h_{e^{ip_1\theta}\varphi_1}(z^k)\\ =&2(k+p_1+\cdots p_m+1)\widehat{\varphi_m}(2k+2p_1+\cdots 2p_{m-1}+p_m+2)\cdots\\ &2(k+p_1+1)\widehat{\varphi_1}(2k+p_1+2)\overline{z}^{k+p_1+\cdots p_m}.\end{aligned}$$

(2) For the other cases $S(z^k)=0$.

From above tow cases, we obtain that

$$\begin{aligned}\{S(z^k):k\ge 0\}\subseteq&\{S(z^k):0\le k\le n-(p_1+p_2+\cdots+p_m),p_1+p_2+\cdots+p_m\le n\}\\ =&\operatorname{span}\{\overline{z}^{k+p_1+p_2+\cdots+p_m}:0\le k\le n-(p_1+p_2+\cdots+p_m)\},\end{aligned}$$

this means the rank of $\{S(z^k):k\ge 0\}$ is at most equal to n. So we know that the rank of S must be finite. □

Corollary 3.2 *Let $p_1, \ldots, p_m \in \mathbb{Z}$ and $\varphi_1, \ldots, \varphi_m$ be bounded radial functions. Then $h_{e^{-ip_m\theta}\varphi_m} \cdots h_{e^{-ip_1\theta}\varphi_1}$ must have the finite rank on b_n^2.*

4. Finite rank Commutator

In this section, we investigate the commutator $[h_{e^{ip\theta}\varphi}, h_{e^{is\theta}\psi}]$ and $[h_{e^{ip\theta}\varphi}, h_{e^{-is\theta}\psi}]$.

Theorem 4.1 *Let p, s be non-negative integers and at least one of them is nonzero. For any integrable radial functions φ and ψ such that $h_{e^{ip\theta}\varphi}$ and $h_{e^{is\theta}\psi}$ are bounded operators, the commutators $[h_{e^{ip\theta}\varphi}, h_{e^{is\theta}\psi}]$ must be finite rank on b_n^2. Especially, if p and s satisfy one of the next conditions:*

(1) $s \geq 2n$, $p \geq 2n$;

(2) $s > 2n$, $n < p < 2n$, $s - p > n - 1$;

(3) $s > 2n$, $p \leq n$;

(4) $p > 2n$, $s < p < 2n$, $p - s > n - 1$;

(5) $p > 2n$, $s \leq n$,

the operators $h_{e^{ip\theta}\varphi}$ and $h_{e^{is\theta}\psi}$ are commutable.

Proof Let S denote the commutator $[h_{e^{ip\theta}\varphi}, h_{e^{is\theta}\psi}]$, by direct calculation we have

(1) If $s > n,\ p > n$,

$$S(z^k) = 0, \quad \forall\, k \geq 0.$$

(2) If $s > n$, $p \leq n$, $s - p > n$,

$$S(z^k) = \begin{cases} 0, & 0 \leq k < s - n - p, \\ -2(k+p+1)2(s-k-p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & s - n - p \leq k \leq n - p, \\ 0, & k > n - p. \end{cases}$$

(3) If $s > n$, $p \leq n$, $s - p \leq n$,

$$S(z^k) = \begin{cases} -2(k+p+1)2(s-k-p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & 0 \leq k \leq n - p, \\ 0, & k > n - p. \end{cases}$$

(4) If $s \leq n$, $p > n$, $p - s > n$,

$$S(z^k) = \begin{cases} 0, & 0 \leq k < p - n - s, \\ 2(k+s+1)2(p-k-s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)\overline{z}^{p-k-s}, & p - n - s \leq k \leq n - s, \\ 0, & k > n - s. \end{cases}$$

(5) If $s \leq n$, $p > n$, $p - s \leq n$,

$$S(z^k) = \begin{cases} 2(k+s+1)2(p-k-s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)\overline{z}^{s-k-p}, & 0 \leq k \leq n - s, \\ 0, & k > n - s. \end{cases}$$

(6) If $s \le n$, $p \le n$, $p > s$, $p - s > n - p$,

$$S(z^k) = \begin{cases} 2(k+s+1)2(p-k-s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)\overline{z}^{p-k-s} & \\ -2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & 0 \le k \le n-p, \\ 2(k+s+1)2(p-k-s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)\overline{z}^{p-k-s}, & n-p < k \le p-s, \\ 2(k+s+1)2(k+s-p+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)z^{k+s-p}, & p-s < k \le n-s, \\ 0, & k > n-s. \end{cases}$$

(7) If $s \le n,\ p \le n,\ p > s,\ p - s \le n - p$,

$$S(z^k) = \begin{cases} 2(k+s+1)2(p-k-s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)\overline{z}^{p-k-s} & \\ -2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & 0 \le k \le p-s, \\ 2(k+s+1)2(k+s-p+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)z^{k+s-p} & \\ -2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & p-s < k \le n-p, \\ 2(k+s+1)2(k+s-p+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)z^{k+s-p}, & n-p < k \le n-s, \\ 0, & k > n-s. \end{cases}$$

(8) If $s \le n,\ p \le n,\ p \le s,\ s - p \le n - s$,

$$S(z^k) = \begin{cases} 2(k+s+1)2(k+s-p+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)z^{k+s-p} & \\ -2(k+p+1)2(s-k-p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & 0 \le k \le n-s, \\ -2(k+p+1)2(s-k-p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & n-s < k \le s-p, \\ -2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & s-p < k \le n-p, \\ 0, & k > n-p. \end{cases}$$

(9) If $s \le n,\ p \le n,\ p \le s,\ s - p < n - s$,

$$S(z^k) = \begin{cases} 2(k+s+1)2(k+s-p+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)z^{k+s-p} & \\ -2(k+p+1)2(s-k-p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & 0 \le k \le s-p, \\ 2(k+s+1)2(k+s-p+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)z^{k+s-p} & \\ -2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & s-p < k \le n-s, \\ -2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & n-s < k \le n-p, \\ 0, & k > n-p. \end{cases}$$

By the above expressions, we have

(1) If $s > n,\ p > n$,

$$S(z^k) = 0, \quad \forall\, k \ge 0.$$

(2) If $s > n,\ p \le n,\ s - p > n$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : s - n \le l \le n\}.$$

(3) If $s > n,\ p \le n,\ s - p \le n$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : s - n \le l \le s - p\}.$$

(4) If $s \le n,\ p > n,\ p - s > n$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : p - n \le l \le n\}.$$

(5) If $s \le n,\ p > n,\ p - s \le n$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : p - n \le l \le p - s\}.$$

(6) If $s \le n,\ p \le n,\ p > s,\ p - s > n - p$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : 0 \le l \le p - s\} \cup \mathrm{span}\{z^l : 0 < l \le n - s\}.$$

(7) If $s \le n,\ p \le n,\ p > s,\ p - s \le n - p$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : 0 \le l \le p - s\} \cup \mathrm{span}\{z^l : 0 < l \le n - s\}.$$

(8) If $s \le n,\ p \le n,\ p \le s,\ s-p \le n-s$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : 0 \le l \le s-p\} \cup \mathrm{span}\{z^l : 0 < l \le n-p\}.$$

(9) If $s \le n,\ p \le n,\ p \le s,\ s-p < n-s$,

$$S(z^k) \subseteq \mathrm{span}\{\overline{z}^l : 0 \le l \le s-p\} \cup \mathrm{span}\{z^l : 0 < l \le n-p\}.$$

From these we know that $\{S(z^k) : k \ge 0\}$ must have finite rank, especially, when $s > n, p > n$ the rank of $\{S(z^k) : k \ge 0\}$ is equal to zero.

Next, we give the expression of $S(\overline{z}^k)$. Firstly, we give the expressions of $h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k)$ and $h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k)$ as below.

Case 1. $s > n,\ p > n$

(1) If $s > 2n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2) If $p > s,\ p-s+k > n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(3) If $p > s,\ p-s+k \le n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < s-n, \\ 2(s-k+1)2(p-s+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{p-s+k}, & s-n \le k \le n. \end{cases}$$

(4) If $n < p \le s$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < s-n, \\ 2(s-k+1)2(p-s+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{p-s+k}, & s-n \le k \le n. \end{cases}$$

(5) If $p = 2n,\ s = 2n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < n, \\ 4(n+1)^2\widehat{\psi}(2n+2)\widehat{\varphi}(2n+2)\overline{z}^{n}, & k = n. \end{cases}$$

(1') If $p > 2n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2') If $s > p,\ s-p+k > n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(3') If $s > p,\ s-p+k \le n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < s-n, \\ 2(p-k+1)2(s-p+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{s-p+k}, & p-n \le k \le n. \end{cases}$$

(4') If $n < s < p$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < s-n, \\ 2(p-k+1)2(s-p+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{s-p+k}, & p-n \le k \le n. \end{cases}$$

(5') If $p = 2n,\ s = 2n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < n, \\ 4(n+1)^2\widehat{\psi}(2n+2)\widehat{\varphi}(2n+2)\overline{z}^n, & k = n. \end{cases}$$

Case 2. $s > n,\ p \le n$

(1) If $s - k \ge p$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < s-n, \\ 2(s-k+1)2(p-s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k-p+2)z^{s-k-p}, & s-n \le k \le n. \end{cases}$$

(2) If $s - k < p$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < s-n, \\ 2(s-k+1)2(p-s+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{p-s+k}, & s-n \le k \le n. \end{cases}$$

(1') If $p \le n,\ s > p + n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2') If $p \le n,\ n < s \le p + n$, but not satisfy $s - p + k < n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(3') If $p \le n,\ n < s \le p + n$ and $0 < s - p + k < n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(s-p+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{s-p+k}, & 1 \le k \le p, \\ 0, & p < k \le n. \end{cases}$$

Case 3. $s \le n,\ p > n$

(1) If $s \le n,\ n < p \le s + n$, but not satisfy $p - s + k \le n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2) If $s \le n,\ n < p \le s + n$, $0 < p - s + k \le n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(s-k+1)2(p-s+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{p-s+k}, & 1 \le k \le s, \\ 0, & s < k \le n. \end{cases}$$

(3) If $s \le n,\ p > s + n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(1') If $s \le n,\ p > n,\ p - k \le s$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < p-n, \\ 2(p-k+1)2(s-p+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{s-p+k}, & p-n \le k \le n. \end{cases}$$

(2') If $s \le n,\ p > n,\ p - k > s$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k < p-n, \\ 2(p-k+1)2(s-p-k+1)\widehat{\psi}(2p-2k-s+2)\widehat{\varphi}(p+2)z^{p-k-s}, & p-n \le k \le n. \end{cases}$$

Case 4. $s \le n,\ p \le n$

(1) If $s \le n,\ 1 \le p \le s,\ p > s-k$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(s-k+1)2(s-k-p+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k-p+2)z^{s-k+p} \\ +2(s-k+1)2(p-s+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{p-s+k}, & 1 \le k \le s, \\ 2(k-s+1)2(p-s+k+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2)\overline{z}^{p-s+k}, & s < k \le n. \end{cases}$$

(2) If $s \le n,\ 1 \le p \le s,\ p \le s-k$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(s-k+1)2(s-k-p+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k-p+2)z^{s-k+p}, & 1 \le k \le s, \\ 2(k-s+1)2(p-s+k+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2)\overline{z}^{p-s+k}, & s < k \le n. \end{cases}$$

(3) If $s \le n,\ s < p \le n$ and $1 \le p+k-s \le n$,

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(s-k+1)2(p-s+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{p-s+k}, & 1 \le k \le s, \\ 2(k-s+1)2(p-s+k+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2)\overline{z}^{p-s+k}, & s < k \le n. \end{cases}$$

(4) If $s \le n,\ s < p \le n$, but $1 \le p+k-s \le n$ is not satisfied

$$h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(s-k+1)2(p-s+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{p-s+k}, & 1 \le k \le s, \\ 0, & s < k \le n. \end{cases}$$

(1') If $p \le n,\ s \le p,\ s < p-k$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(p-s-k+1)\widehat{\psi}(2p-2k-s+1)\widehat{\varphi}(p+2)z^{p-s-k}, & 1 \le k < p, \\ 2(k-p+1)2(k-p+s+1)\widehat{\psi}(2k-2p+s+1)\widehat{\varphi}(2k-p+2)\overline{z}^{k-p+s+1}, & p \le k \le n. \end{cases}$$

(2') If $p \le n,\ s \le p$, but $s < p-k$ is not satisfied

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(s-p+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{s-p+k}, & 1 \le k < p, \\ 2(k-p+1)2(k-p+s+1+1)\widehat{\psi}(2k-2p+s+1)\widehat{\varphi}(2k-p+2)\overline{z}^{k-p+s+1}, & p \le k \le n. \end{cases}$$

(3') If $s \le n,\ p < s \le n$ and $1 \le k-p+s \le n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(s-p+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{s-p+k}, & 1 \le k \le p, \\ 2(k-p+1)2(k-p+s+1)\widehat{\psi}(2k-2p+s+1)\widehat{\varphi}(2k-p+2)\overline{z}^{k-p+s}, & p < k \le n. \end{cases}$$

(4') If $s \le n,\ p < s \le n$, but $1 \le k-p+s \le n$ is not satisfied

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(s-p+k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\overline{z}^{s-p+k}, & 1 \le k \le p, \\ 0, & p < k \le n. \end{cases}$$

From the above calculations, we have expression of $S(\overline{z}^k)$ as below

(1) If $s \ge 2n,\ p \ge 2n$,

$$S(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2) If $s \ge 2n,\ n < p < 2n,\ s-p > n-1$ or $p \ge 2n,\ n < s < 2n,\ p-s > n-1$,

$$S(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(3) If $s \ge 2n,\ p \le n$ or $p \ge 2n,\ s \le n$,

$$S(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(4) If $s \ge 2n,\ n < p < 2n,\ s-p \le n-1$,

$$S(\overline{z}^k) = -h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : s-n \le l < n+s-p\}.$$

(5) If $n < s < 2n,\ p > 2n,\ p-s < n-1$,

$$S(\overline{z}^k) = h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : p-n \le l \le n+p-s\}.$$

(6) If $n < s < 2n,\ n < p \le 2n,\ s > p,\ s-p+k > n$,

$$S(\overline{z}^k) = h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : p-n \le l \le n+p-s\}.$$

(7) If $n < s < 2n,\ n < p \le 2n,\ s > p,\ s-p+k \le n$,

$$S(\overline{z}^k) = h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) - h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : p-n \le l \le n+s-p\}.$$

(8) If $n < s < 2n,\ p \le n$,

$$S(\overline{z}^k) = h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) - h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le s\}.$$

(9) If $s \le n,\ n < p \le 2n$,

$$S(\overline{z}^k) = h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) - h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le p\}.$$

(10) If $s \le n,\ p \le n,\ p \le s$,

$$\begin{aligned} S(\overline{z}^k) =& h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) - h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \\ &\subseteq \operatorname{span}\{\overline{z}^l : 0 < l < n+s-p\} \cup \operatorname{span}\{z^l : 0 < l < s+p\}. \end{aligned}$$

(11) If $s \le n,\ p \le n,\ s < p$,

$$\begin{aligned} S(\overline{z}^k) &= h_{e^{ip\theta}\varphi}h_{e^{is\theta}\psi}(\overline{z}^k) - h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \\ &\subseteq \operatorname{span}\{\overline{z}^l : 0 < l < n+p-s\} \cup \operatorname{span}\{z^l : 0 < l < p-s\}. \end{aligned}$$

From the above analysis, we can see that S has the finite rank on b_n^2. Furtheremore, $h_{e^{ip\theta}\varphi}$ and $h_{e^{is\theta}\psi}$ commutate when $s \ge 2n$, $p \ge 2n$ or $s > 2n$, $n < p < 2n$, $s-p > n-1$ or $s \ge 2n$, $p \le n$ or $p > 2n$, $n < s < 2n$, $p-s > n-1$ or $p \ge 2n$, $s \le n$. This completes the proof. □

Corollary 4.2 *Let s be non-negative integers and at least one of them is nonzero. For any integrable radial functions φ and ψ such that h_φ and $h_{e^{is\theta}\psi}$ are bounded operators, the commutators $[h_\varphi, h_{e^{is\theta}\psi}]$ must be of finite rank on b_n^2. Especially, if $s > 2n$, the operators h_φ and $h_{e^{is\theta}\psi}$ are commutable.*

Theorem 4.3 *Let $p, s \ge 0$ and at least one of them is nonzero. Let φ and ψ be two integrable radial functions on D such that $h_{e^{ip\theta}\varphi}$ and $h_{e^{-is\theta}\psi}$ are bounded operators. Then commutator $[h_{e^{ip\theta}\varphi}, h_{e^{-is\theta}\psi}]$ must have finite rank. Furthermore, $h_{e^{ip\theta}\varphi}$ and $h_{e^{-is\theta}\psi}$ are commutable when $p > 2n$.*

Proof Let S denote the commutator $[h_{e^{ip\theta}\varphi}, h_{e^{-is\theta}\psi}]$. We first give the expression of

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k),\ h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k),\ h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k),\ h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k).$$

For the expression of $h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k)$, we have

(1) If $p > n$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k) = \begin{cases} 0, & 0 \le k \le s, \\ 0, & s < k < p-n+s, \\ 2(k-s+1)2(p-k+s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(p+2)\overline{z}^{p-k+s}, & p-n+s \le k \le n+s, \\ 0, & k > n+s. \end{cases}$$

Especially, if $p > 2n$, $h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k) = 0$.

(2) If $p \le n$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k) = \begin{cases} 2(s-k+1)2(s-k+p+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k+p+2)\overline{z}^{s-k+p}, & 0 \le s-k \le n-p, \\ 0, & n-p < s-k, \\ 2(k-s+1)2(p-k+s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(p+2)\overline{z}^{p-k+s}, & 0 < k-s < p, \\ 2(k-s+1)2(k-s-p+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2)z^{k-s-p}, & p \le k-s \le n, \\ 0, & k-s > n. \end{cases}$$

For $h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k)$, we have

(1) If $p > n$,

$$h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) = 0, \quad \forall\, k \ge 0.$$

(2) If $p \le n$,

$$h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) = \begin{cases} 2(k+p+1)2(s+k+p)\widehat{\psi}(2k+2p+s+2)\widehat{\varphi}(2k+p+2)z^{k+p+s}, & 0 \le k \le n-p, \\ 0, & k > n-p. \end{cases}$$

For the expression of $h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k)$, we have

(1) If $p > n$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2) If $p \le n$, $p+s+1 > n$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(3) If $p \le n$, $p+s+1 \le n$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(k+s+p+1)2(s+k+p)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2)\overline{z}^{k+p+s}, & 1 \le k \le n-p-s, \\ 0, & n-p-s < k \le n. \end{cases}$$

For $h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k)$, we have

(1) If $p > n$,

$$h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k \le p-n, \\ 2(p-k+1)2(s+p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k+s+2)z^{s+p-k}, & p-n \le k \le n. \end{cases}$$

Especially, if $p > 2n$, $h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0$.

(2) If $p \le n$, $s > n-p$,

$$h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(s+p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k+s+2)z^{p-k+s}, & 1 \le k < p, \\ 2(k-p+1)2(s-k+p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(s+2)z^{s+p-k}, & p \le k \le n. \end{cases}$$

(3) If $p \le n$, $s \le n-p$,

$$h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(s+p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k+s+2)z^{p-k+s}, & 1 \le k < p, \\ 2(k-p+1)2(s-k+p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(s+2)z^{s+p-k}, & p \le k \le p+s, \\ 2(k-p+1)2(k-p-s+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p-s+2)\overline{z}^{k-p-s}, & p+s \le k \le n. \end{cases}$$

From the above calculations, we know that the expression of $S(z^k)$ is as below

(1) If $p > n$,

$$s(z^k) = h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k) \subseteq \operatorname{span}\{\overline{z}^l : p-n \le k \le n\}.$$

Especially, if $p > 2n$, then $S(z^k) = 0$, $\forall\, k \ge 0$.

(2) If $p \le n$,

$$s(z^k) = h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k) - h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 < k \le n\} \cup \operatorname{span}\{z^l : 0 \le l \le n+s\}.$$

For the expression of $S(\overline{z}^k)$ we have

(1) If $p > n$,

$$s(\overline{z}^k) = h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{z^l : s+p-n \le k \le s+n\}.$$

Especially, if $p > 2n$, then $S(\overline{z}^k) = 0,\ \forall\, 1 \le k \le n$.

(2) If $p \le n,\ s > n-p$,

$$s(\overline{z}^k) = -h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{z^l : s+p-n \le k \le s+p\}.$$

(3) If $p \le n,\ s \le n-p$,

$$s(\overline{z}^k) = h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k)h_{e^{-is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{z^l : 0 \le l \le s+p\} \cup \operatorname{span}\{\overline{z}^l : 0 < l \le n\}.$$

Then we can imply that S has the finite rank on b_n^2. Especially, when $p > 2n$, operators $h_{e^{ip\theta}\varphi}$ and $h_{e^{-is\theta}\psi}$ are commutable. This completes the proof. □

5. Finite rank semi-commutators

In this section, we will study the semi-commutators of two little-Hankle operators with quasi-homogeneous symbols.

Theorem 5.1 *Let $p, s \ge 0$ and at least one of them be nonzero. Let φ and ψ be integrable radial functions on D such that $h_{e^{ip\theta}\varphi}$ and $h_{e^{is\theta}\psi}$ are bounded operators, then $(h_{e^{is\theta}\psi}, h_{e^{ip\theta}\varphi}]$ must have finite rank on b_n^2. Especially, $(h_{e^{is\theta}\psi}, h_{e^{ip\theta}\varphi}] = 0$ when $p > n,\ s > p+n$ or $p \le n$, $s > 2n$.*

Proof Let S denote $(h_{e^{is\theta}\psi}, h_{e^{ip\theta}\varphi}]$. By Lemma 2.2, we can obtain that

For $h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k)$

(1) When $p > n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) = 0, \quad \forall\, k \ge 0.$$

(2) When $p \le n,\ s > n,\ s-p > n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) = \begin{cases} 0, & 0 \le k \le s-n-p, \\ 2(k+p+1)2(s-p-k+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & s-n-p \le k \le n-p, \\ 0, & k > n-p. \end{cases}$$

(3) When $p \le n,\ s > n,\ s-p \le n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) = \begin{cases} 2(k+p+1)2(s-p-k+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & 0 \le k \le n-p, \\ 0, & k > n-p. \end{cases}$$

(4) When $p \le n,\ s \le n,\ s \ge p$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) = \begin{cases} 2(k+p+1)2(s-p-k+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & 0 \le k \le s-p, \\ 2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & s-p \le k \le n-p, \\ 0, & k > n-p. \end{cases}$$

(5) When $p \le n,\ s \le n,\ s < p$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k) = \begin{cases} 2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k+p-s}, & 0 \le k \le s-p, \\ 0, & k > n-p. \end{cases}$$

For $h_{e^{i(p+s)\theta}\varphi\psi}(z^k)$

(1) When $p+s > n$,

$$h_{e^{i(p+s)\theta}\varphi\psi}(z^k) = 0, \quad \forall\, k \ge 0.$$

(2) When $p+s \le n$,

$$h_{e^{i(p+s)\theta}\varphi\psi}(z^k) = 2(k+s+p+1)\widehat{\varphi\psi}(2k+s+p+2)z^{k+p+s}, \quad \forall\, k \ge 0.$$

For $h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k)$

(1) When $p > n,\ s > p,\ s-p+k > n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0.$$

Especially, when $p > n,\ s > n+p$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2) When $p > n,\ s > p,\ s-p+k \le n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k \le p-n, \\ 2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s-p+k}, & p-n < k \le n. \end{cases}$$

(3) When $p > n,\ n < s \le p$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k \le p-n, \\ 2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s-p+k}, & p-n < k \le n. \end{cases}$$

(4) When $p \le n,\ s > p+n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(5) When $p \le n,\ n < s \le p+n$, but $0 < s-p+k < n$ is not satisfied

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(6) When $p \le n,\ n < s \le p+n,\ s-p+k < n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s-p+k}, & 1 \le k < p, \\ 0, & p \le k \le n. \end{cases}$$

(7) When $p > n,\ s \le n,\ k > p-s$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k \le p-n, \\ 2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s-p+k}, & p-n < k \le n. \end{cases}$$

(8) When $p > n,\ s \le n,\ k \le p-s$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 0, & 1 \le k \le p-n, \\ 2(p-k+1)2(p-k-s+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-k-s}, & p-n < k \le n. \end{cases}$$

(9) When $p\le n,\ p<s\le n$, but $s-p+k<n$ is not satisfied

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s-p+k}, & 1\le k<p\\ 0, & p\le k\le n.\end{cases}$$

(10) When $p\le n,\ p<s\le n$, and $s-p+k<n$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s-p+k}, & 1\le k\le p,\\ 2(k-p+1)2(k-p+s+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+1)\overline{z}^{k-p+s}, & p<k\le n.\end{cases}$$

(11) When $p\le n,\ s\le p,\ k>p-s$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s-p+k}, & 1\le k\le p,\\ 2(k-p+1)2(k-p+s+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+1)\overline{z}^{k-p+s}, & p<k\le n.\end{cases}$$

(12) When $p\le n,\ s\le p,\ k\le p-s$,

$$h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(p-k-s+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+1)z^{p-k-s}, & 1\le k\le p,\\ 2(k-p+1)2(k-p+s+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+1)\overline{z}^{k-p+s}, & p<k\le n.\end{cases}$$

For $h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k)$

(1) When $p+s>n$,

$$h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k)=\begin{cases}0, & 1\le k\le p+s-n,\\ 2(p+s-k+1)\widehat{\varphi\psi}(p+s+2)\overline{z}^{p+s-k}, & p+s-n\le k\le n.\end{cases}$$

Especially, when $p+s>2n$,

$$h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k)=0,\quad \forall\, 1\le k\le n.$$

(2) When $p+s\le n$,

$$h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k)=\begin{cases}2(p+s-k+1)\widehat{\varphi\psi}(p+s+2)\overline{z}^{p+s-k}, & 1\le k\le p+s,\\ 2(k-p-s+1)\widehat{\varphi\psi}(2k-p-s+2)z^{k-p-s}, & p+s\le k\le n.\end{cases}$$

From the above results, we have the expression of $S(z^k)$ as below

(1) When $p>n$,

$$S(z^k)=h_{e^{i(p+s)\theta}\varphi\psi}(z^k)-h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k)=0,\quad \forall\, k\ge 0.$$

(2) When $p\le n,\ s>n,\ s>p+n$,

$$S(z^k)=h_{e^{i(p+s)\theta}\varphi\psi}(z^k)-h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k)\subseteq \operatorname{span}\{\overline{z}^l: s-n\le l\le n\}.$$

Especially, when $p\le n,\ s>2n$,

$$S(z^k)=h_{e^{i(p+s)\theta}\varphi\psi}(z^k)-h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k)=0,\quad \forall\, k\ge 0.$$

(3) When $p\le n,\ s>n,\ s\le p+n$,

$$S(z^k)=h_{e^{i(p+s)\theta}\varphi\psi}(z^k)-h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k)\subseteq \operatorname{span}\{\overline{z}^l: s-n\le l\le s-p\}.$$

(4) When $p\le n,\ s\le n,\ s\ge p,\ s+p>n$,

$$S(z^k)=h_{e^{i(p+s)\theta}\varphi\psi}(z^k)-h_{e^{is\theta}\psi}h_{e^{ip\theta}\varphi}(z^k)\subseteq \operatorname{span}\{\overline{z}^l: 0\le l\le s-p\}\cup\operatorname{span}\{z^l: 0\le l\le n-s\}.$$

(5) When $p \le n,\ s \le n,\ s < p,\ s + p > n$,

$$S(z^k) = h_{e^{i(p+s)\theta}\varphi\psi}(z^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(z^k) \subseteq \operatorname{span}\{z^l : p - s \le l \le n - s\}.$$

(6) When $p \le n,\ s \le n,\ s \ge p,\ s + p \le n$,

$$S(z^k) = h_{e^{i(p+s)\theta}\varphi\psi}(z^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(z^k) \subseteq \operatorname{span}\{z^l : 0 \le l \le n\} \cup \operatorname{span}\{\overline{z}^l : 0 < l < s - p\}.$$

For the expression of $S(\overline{z}^k)$, we have

(1) When $p > n,\ s > p,\ k > n - s + p$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0.$$

Especially, when $p > n,\ s > p + n$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(2) When $p > n,\ s > p,\ s - p + k \le n$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : s - n \le l \le s\}.$$

(3) When $p > n,\ n < s \le p$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : s - n \le l \le s\}.$$

(4) When $\frac{n}{2} < p \le n,\ s > p + n$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

(5) When $p \le \frac{n}{2},\ s > p + n$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : p + s - n \le l \le n\}.$$

(6) When $p \le n,\ n < s \le p + n$ but $s - p + k < n$ is not satisfied,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : p + s - n \le l \le n\}.$$

(7) When $\frac{n}{2} \le p \le n,\ n < s \le p + n$ and $s - p + k < n$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : p + s - n \le l \le n\}.$$

(8) When $p < \frac{n}{2},\ n < s \le p + n$ and $s - p + k < n$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : s - p \le l \le n\}.$$

(9) When $p > n,\ s \le n,\ k > p - s$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le n\}.$$

(10) When $p > n,\ s \le n,\ k \le p - s$,

$$\begin{aligned} S(\overline{z}^k) &= h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \\ &\subseteq \operatorname{span}\{\overline{z}^l : p + s - n \le l \le n\} \cup \operatorname{span}\{z^l : p - s - k < l \le n - s\}. \end{aligned}$$

(11) When $p \le n, p < s < n, p + s > n$ but $1 \le k - p + s \le n$ is not satisfied,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \le n\}.$$

(12) When $p \le n,\ p < s < n,\ p + s \le n$ but $1 \le k - p + s \le n$ is not satisfied,

$$\begin{aligned} S(\overline{z}^k) &= h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \\ &\subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le p + s\} \cup \operatorname{span}\{z^l : 0 \le l \le n - p - s\}. \end{aligned}$$

(13) When $p \le n,\ s \le p,\ p + s > n,\ k > p - s$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \le n + s + 1\}.$$

(14) When $p \le n,\ s \le p,\ p + s \le n,\ k > p - s$,

$$\begin{aligned} S(\overline{z}^k) &= h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \\ &\subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le n + s + 1\} \cup \operatorname{span}\{z^l : 0 \le l \le n - p - s\}. \end{aligned}$$

(15) When $\frac{n}{2} < p \le n,\ s \le p,\ p + s \le n,\ k \le p - s$,

$$\begin{aligned} S(\overline{z}^k) &= h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \\ &\subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le n + s - p + 1\} \cup \operatorname{span}\{z^l : 0 \le l \le n - p - s\}. \end{aligned}$$

(16) When $p \le \frac{n}{2},\ s \le p,\ p + s \le n,\ k \le p - s$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le p + s\} \cup \operatorname{span}\{z^l : 0 \le l \le p - s\}.$$

(17) When $p \le n,\ s \le p,\ p + s > n,\ k \le p - s$,

$$S(\overline{z}^k) = h_{e^{i(p+s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi}(\overline{z}^k) \subseteq \operatorname{span}\{\overline{z}^l : 0 \le l \le p - s\} \cup \operatorname{span}\{z^l : p + s - n \le l \le n\}.$$

From these expressions, we have the conclusion that the semi-commutator $(h_{e^{ip\theta}\varphi}, h_{e^{is\theta}\psi}]$ must have finite rank on b_n^2, especially when $p > n,\ s > p + n$ or $p \le n,\ s > 2n,\ h_{e^{is\theta}\psi} h_{e^{ip\theta}\varphi} = h_{e^{i(p+s)\theta}\varphi\psi}$. $\square$

Theorem 5.2 *Let $p, s \ge 0$, $s \ge p$ and at least one of them be nonzero. Let φ and ψ be integrable radial functions on D such that $h_{e^{ip\theta}\varphi}$ and $h_{e^{-is\theta}\psi}$ are bounded operators, then $(h_{e^{ip\theta}\varphi}, h_{e^{-is\theta}\psi}]$ must have finite rank on b_n^2. Especially, $(h_{e^{ip\theta}\varphi}, h_{e^{-is\theta}\psi}] = 0$ when $p - s > 2n$.*

Proof Let S denote $(h_{e^{ip\theta}\varphi}, h_{e^{-is\theta}\psi}]$. We will discuss the rank of $\{S(z^k) : k \ge 0\}$ and $\{S(\overline{z}^k) : 1 \le k \le n\}$.

Firstly, we characterize $\{S(z^k) : k \ge 0\}$. By Lemma 2.2, we obtain the following results directly. For the expression of $h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k)$, we have

(1) When $p > n$,

$$h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k) = \begin{cases} 0, & 0 \le k \le s, \\ 0, & s < k < p - n + s, \\ 2(k - s + 1) 2(p - k + s + 1) \widehat{\varphi}(p + 2) \widehat{\psi}(2k - s + 2) \overline{z}^{p-k+s}, & p + s - n \le k \le n + s, \\ 0, & k > n + s. \end{cases}$$

Especially, when $p > 2n$,

$$h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k) = 0, \quad \forall\, k \ge 0.$$

(2) When $p \leq n$,

$$h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k) = \begin{cases} 0, & 0 \leq k < p-n+s, \\ 2(s-k+1)2(s-k+p+1)\widehat{\varphi}(2s-2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k+p}, & p-n+s \leq k \leq s, \\ 2(k-s+1)2(p-k+s+1)\widehat{\varphi}(p+2)\widehat{\psi}(2k-s+2)\overline{z}^{p-k+s}, & s < k < s+p, \\ 2(k-s+1)2(k-s-p+1)\widehat{\varphi}(2k-2s-p+2)\widehat{\psi}(2k-s+2)z^{k-s-p}, & p+s \leq k \leq n+s, \\ 0, & k > n+s. \end{cases}$$

For $h_{e^{i(p-s)\theta}\varphi\psi}(z^k)$, we have

(1) When $p > s+n$,

$$h_{e^{i(p-s)\theta}\varphi\psi}(z^k) = 0, \quad \forall\, k \geq 0.$$

(2) When $s \leq p \leq s+n$,

$$h_{e^{i(p-s)\theta}\varphi\psi}(z^k) = \begin{cases} 2(k+p-s+1)\widehat{\varphi\psi}(2k+p-s+2)\overline{z}^{k+p-s}, & 0 \leq k \leq n-p+s, \\ 0, & k > n-p+s. \end{cases}$$

(3) When $p < s$,

$$h_{e^{i(p-s)\theta}\varphi\psi}(z^k) = \begin{cases} 2(s-p-k+1)\widehat{\varphi\psi}(s-p+2)z^{s-p-k}, & k \leq s-p, \\ 2(k+p-s+1)\widehat{\varphi\psi}(2k+p-s+2)\overline{z}^{k+p-s}, & s-p < k \leq n-p+s, \\ 0, & k > n-p+s. \end{cases}$$

For $h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(\overline{z}^k)$, we have

(1) When $p > n$,

$$h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \leq k \leq n.$$

(2) When $p \leq n,\ p+s+1 > n$,

$$h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \leq k \leq n.$$

(3) When $p \leq n,\ p+s+1 \leq n$,

$$h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(s+k+1)2(s+k+p+1)\widehat{\varphi}(2k+2s+p+2)\widehat{\psi}(2k+s+2)\overline{z}^{k+s+p}, & 1 \leq k < -p-s, \\ 0, & n-p-s \leq k \leq n. \end{cases}$$

For the expression of $h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k)$, we obtain that

(1) When $p-s > n$,

$$h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) = \begin{cases} 0, & 1 \leq k < p-s-n, \\ 2(p-s-k+1)\widehat{\varphi\psi}(p-s+2)\overline{z}^{p-s-k}, & p-s-n \leq k \leq n. \end{cases}$$

Especially, when $p-s > 2n$

$$h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \leq k \leq n.$$

(2) When $0 \leq p-s \leq n$,

$$h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) = \begin{cases} 2(p-s-k+1)\widehat{\varphi\psi}(p-s+2)\overline{z}^{p-s-k}, & 1 \leq k < p-s, \\ 2(k-p+s+1)\widehat{\varphi\psi}(2k-p+s+2)z^{k-p+s}, & p-s \leq k \leq n. \end{cases}$$

(3) When $p < s$,

$$h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) = 2(s-p+k+1)\widehat{\varphi\psi}(2k+s-p+2)z^{s-p+k}, \quad \forall\, 1 \leq k \leq n.$$

Form the above results, we give the expression of set of $\{S(z^k) : k \geq 0\}$ and $\{S(\overline{z}^k) : k \geq 0\}$ as below. For $\{S(z^k) : k \geq 0\}$, we have

Case 1. $s \geq 2n$

(1) When $p > s + 2n$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) = 0, \quad \forall\, k \geq 0.$$

So we obtain that $\{S(z^k) = 0 : k \geq 0\}$.

(2) When $s < p \leq s + n$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k),$$

then we have

$$\{S(z^k) : k \geq 0\} \subseteq \text{span}\{\overline{z}^l : p - s \leq l \leq n\}.$$

(3) When $2n < p \leq s$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k),$$

then we have

$$\{S(z^k) : k \geq 0\} \subseteq \text{span}\{\overline{z}^l : 0 < l \leq n\} \cup \text{span}\{z^l : 0 \leq l \leq s - p\}.$$

(4) When $n < p \leq 2n$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k),$$

then we have

$$\{S(z^k) : k \geq 0\} \subseteq \text{span}\{\overline{z}^l : 0 < l \leq n\} \cup \text{span}\{z^l : 0 \leq l \leq s - p\}.$$

(5) When $p \leq n$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k),$$

then we have

$$\{S(z^k) : k \geq 0\} \subseteq \text{span}\{\overline{z}^l : 0 < l \leq n\} \cup \text{span}\{z^l : 0 \leq l \leq s - p\}.$$

Case 2. $n \leq s < 2n$

(1) When $p > s + n$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) = 0, \quad \forall\, k \geq 0,$$

then we obtain that $\{S(z^k) = 0 : k \geq 0\}$.

(2) When $2n < p \leq s + n$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k),$$

then we have

$$\{S(z^k) : k \geq 0\} \subseteq \text{span}\{\overline{z}^l : p - s \leq l \leq n\}.$$

(3) When $s < p \leq 2n$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(z^k),$$

then we have
$$\{S(z^k) : k \ge 0\} \subseteq \operatorname{span}\{\overline{z}^l : p - s \le l \le n\}.$$

(4) When $n < p \le s$,
$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k),$$

then we have
$$\{S(z^k) : k \ge 0\} \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \le n\} \cup \operatorname{span}\{z^l : 0 \le l \le s - p\}.$$

(5) When $p \le n$,
$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k),$$

then we have
$$\{S(z^k) : k \ge 0\} \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \le n\} \cup \operatorname{span}\{z^l : 0 \le l \le s - p\}.$$

Case 3. $s < n$

(1) When $p > 2n$,
$$h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) = 0, \quad \forall\, k \ge 0,$$
then we obtain that $\{S(z^k) = 0 : k \ge 0\}$.

(2) When $s + n < p \le 2n$,
$$S(z^k) = -h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k),$$

then we have
$$\{S(z^k) : k \ge 0\} \subseteq \operatorname{span}\{\overline{z}^l : p - n \le l \le n\}.$$

(3) When $n < p \le s + n$,
$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k),$$

then we have
$$\{S(z^k) : k \ge 0\} \subseteq \operatorname{span}\{\overline{z}^l : p - n \le l \le n\}.$$

(4) When $s < p \le n$,
$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k),$$

then we have
$$\{S(z^k) : k \ge 0\} \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \le n\} \cup \operatorname{span}\{z^l : 0 \le l \le n - p\}.$$

(5) When $p \le s$,
$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(z^k) - h_{e^{ip\theta}\varphi} h_{e^{-is\theta}\psi}(z^k),$$

then we have
$$\{S(z^k) : k \ge 0\} \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \le n\} \cup \operatorname{span}\{z^l : 0 \le l \le n - p\}.$$

For $\{S(\overline{z}^k) : k \ge 0\}$, we have

Case 1. For any $s \geq 0$

(1) When $p > 2n + s$,

$$h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) = 0, \quad \forall\, 1 \leq k \leq n,$$

then we obtain that $\{S(\overline{z}^k) = 0 : 1 \leq k \leq n\}$.

(2) When $s + n < p \leq 2n + s$,

$$S(\overline{z}^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k),$$

then we have

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{\overline{z}^l : p - s - n \leq l \leq n\}.$$

Case 2. $s \leq n$

(1) When $n < p \leq s + n$,

$$S(\overline{z}^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k),$$

then we have

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \leq p - s\} \cup \operatorname{span}\{z^l : 0 \leq l \leq n - p + s\}.$$

(2) When $s < p \leq n$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k),$$

then we have

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{\overline{z}^l : 0 \leq l \leq n\} \cup \operatorname{span}\{z^l : 0 < l \leq n - p + s\}.$$

(3) When $p \leq s$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k),$$

then we have

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{\overline{z}^l : p + s + 1 \leq l \leq n\} \cup \operatorname{span}\{z^l : s - p + 1 \leq l \leq n - p + s\}.$$

Case 3. $s > n$

(1) When $s < p \leq s + n$,

$$S(\overline{z}^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k),$$

then we have

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{\overline{z}^l : 0 < l \leq p - s\} \cup \operatorname{span}\{z^l : 0 \leq l \leq n - p + s\}.$$

(2) When $n < p \leq s$,

$$S(\overline{z}^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k),$$

then we have

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{z^l : s - p + 1 \leq l \leq n - p + s\}.$$

(3) When $p \leq n$,

$$S(z^k) = h_{e^{i(p-s)\theta}\varphi\psi}(\overline{z}^k) - h_{e^{ip\theta}\varphi}h_{e^{-is\theta}\psi}(\overline{z}^k),$$

then we have

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \text{span}\{\overline{z}^l : p+s+1 \leq l \leq n\} \cup \text{span}\{z^l : s-p+1 \leq l \leq n-p+s\}.$$

From the characterization of the expression of $\{S(z^k) : k \geq 0\}$ and $\{S(\overline{z}^k) : 1 \leq k \leq n\}$, we can obtain that the operator S has finite rank on b_n^2. Especially, S is equal to zero when $p > 2n+s$, this completes the proof. $\square$

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