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A New Modified Two-Subgradient Extragradient Algorithm for Solving Variational Inequality Problems

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Abstract In this paper, we propose a modified two-subgradient extragradient algorithm (MTSEGA) for solving monotone and Lipschitz continuous variational inequalities with the feasible set being a level set of a smooth convex function in Hilbert space. The advantage of MTSEGA is that all the projections are computed onto a half-space per iteration. Moreover, MTSEGA only needs one computation of the underlying mapping per iteration. Under the same assumptions with the known algorithm, we show that the sequence generated by this algorithm is weakly convergent to a solution of the concerned problem.

Keywords two-subgradient extragradient algorithm; monotone; Lipschitz continuous; variational inequality; Hilbert space

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1. Introduction

Let H be a real Hilbert space and $C \subseteq H$ be a nonempty closed and convex set. We consider the following classical variational inequality problem: find vector $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $F : H \to H$ is a continuous mapping and $\langle \cdot, \cdot \rangle$ is the usual inner product in H. We let S denote the solution set of Problem (1.1).

Denote by $P_C(x)$ the metric projection of vector x onto nonempty closed and convex set C.

$$P_C(x) := \arg\min\{\|x - y\| : y \in C\}.$$

In this paper, we focus on the projection based algorithm for solving variational inequality problems. The simplest projection based algorithm is Goldstein-Levitin-Polyak algorithm, in which new iterate point x_{n+1} is updated by the following formula:

$$x_{n+1} = P_C(x_n - \lambda F(x_n)). \tag{1.2}$$

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However, the global convergence of this algorithm is established on the conditions that F is L-Lipschitz continuous on C and strongly monotone with modulus γ, when H reduces to a Euclidean space [1, 2].

To weaken the strong monotonicity of F, Korpelevich [3] proposed an extragradient algorithm (EGA for short) as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = P_C(x_n - \lambda F(y_n)). \end{cases} \tag{1.3}$$

Under the assumption that F is pseudomonotone on C, the global convergence is obtained by taking the step-size $\lambda \in (0, \frac{1}{L})$ with L being the Lipschitz modulus of F. In recent years, EGA was generalized in various ways; see, for example, [4–7] and the references therein. From (1.3), we see that EGA needs to compute two projections onto C.

Note that the projection onto a half-space is easy to implement (see Lemma 2.4 below). In 2011, Censor et al. [8] proposed the subgradient extragradient algorithm (SEGA for short) for variational inequality problems, where the new iterate point x_{n+1} is computed by projecting a vector onto a specific half-space. SEGA can be considered as an improvement of EGA when the projection onto C is difficult to compute. The iterative scheme of SEGA is as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda F(y_n)), \end{cases} \tag{1.4}$$

where $T_n = \{\omega \in H : \langle x_n - \lambda F(x_n) - y_n, \omega - y_n\rangle \leq 0\}$ is a half-space and $\lambda \in (0, \frac{1}{L})$.

In 2017, He and Wu in [9] proposed a modified subgradient extragradient algorithm (MSEGA for short) for solving Lipschitz continuous and monotone variational inequalities with

$$C = \{x \in H : c(x) \leq 0\}, \tag{1.5}$$

where $c : H \to R$ is a smooth and convex function and the Slater's condition holds, i.e.,

$$\{x \in C : c(x) < 0\} \neq \emptyset.$$

In MSEGA, the first projection in (1.4) is replaced by computing the projection onto a specially constructed half-space C_n (The idea of replacing the projection onto C with a projection onto a half-space was first suggested by Fukushima [10]). Hence, all the projections of MSEGA are implemented onto the half-space, respectively. Recently, He et al. [11] generalized MSEGA to solve variational inequalities when C is an intersection of finitely many level sets. Cao and Guo [12] proposed an inertial MSEGA for solving variational inequalities. The iterative scheme of MSEGA is as follows:

$$\begin{cases} y_n = P_{C_n}(x_n - \lambda_n F(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda_n F(y_n)), \end{cases} \tag{1.6}$$

where

$$C_n := \{\omega \in H : c(x_n) + \langle c'(x_n), \omega - x_n\rangle \leq 0\}$$

and

$$T_n = \{\omega \in H : \langle x_n - \lambda_n F(x_n) - y_n, \omega - y_n\rangle \leq 0\}.$$

The step-size λ_n is chosen by using the following line-search; i.e., $\lambda_n = \sigma\rho^{m_n}$, $\sigma > 0$, $\rho \in (0,1)$ and m_n is the smallest nonnegative integer, such that

$$\lambda_n^2\|F(x_n) - F(y_n)\|^2 + 2M\lambda_n\|x_n - y_n\|^2 \le \nu^2\|x_n - y_n\|^2,$$

where $\nu \in (0,1)$ and $M = M_1M_2 > 0$ (M_1 is the Lipschitz modulus of $c'(\cdot)$ and M_2 is a positive constant such that $\|F(x)\| \le M_2\|c'(x)\|$, $\forall x \in \partial C$). In this paper, under the same assumption that there exists $M_2 > 0$ such that $\|F(x)\| \le M_2\|c'(x)\|$ for any x belonging to the boundary of C, we explore the new algorithm for Problem (1.1), see Assumption 2.10 (H4) below.

However, all the algorithms in [3, 8, 9] require two evaluations of mapping F per iteration. This may be a disadvantage when the value of F is complicated to compute.

In this paper, we modify MSEGA to solve variational inequalities by using Popov method. This method was suggested by Popov [13], and further studied by Malitsky and Semenov [14].

We present a new modified two-subgradient extragradient algorithm (MTSEGA for short) for solving the monotone and Lipschitz continuous variational inequalities with C defined in (1.5). In MTSEGA, all the projections are implemented onto a half-space, respectively. Moreover, MTSEGA needs only one evaluation of F per iteration. The weak convergence of MTSEGA is established under the suitable assumptions.

The remainder of this paper is organized as follows. Some basic definitions and preliminary materials of projection operator are introduced in Section 2. MTSEGA and its convergence analysis is introduced in Section 3. Some concluding remarks are presented in Section 4.

2. Preliminaries

Let H be a real Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. We use the notations $x_n \to x$ and $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges strongly and weakly to x, respectively.

In this section, we first recall some basic definitions and well-known lemmas.

Definition 2.1 *Let $C \subseteq H$ be a nonempty closed and convex set and $F : H \to H$ be a mapping. Then*

(i) F is monotone on C, if $\langle F(x) - F(y), x - y\rangle \ge 0$, $\forall x, y \in C$;

(ii) F is L-Lipschitz continuous on C, if $L > 0$ and $\|F(x) - F(y)\| \le L\|x - y\|$, $\forall x, y \in C$.

Lemma 2.2 ([15, Theorem 3.16 and Proposition 4.16]) *Let $C \subseteq H$ be a nonempty closed and convex set. Then, for each $x \in H$, the following inequalities hold:*

(i) $\langle P_C(x) - x, y - P_C(x)\rangle \ge 0$, $\forall y \in C$;

(ii) $\|y - P_C(x)\|^2 \le \|y - x\|^2 - \|P_C(x) - x\|^2$, $\forall y \in C$.

Lemma 2.3 ([16, Lemma 1]) *For any sequence $\{x_n\}$ in H such that $x_n \rightharpoonup x$, then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \ne x.$$

Lemma 2.4 ([17, Lemma 2.7]) *Let $u \in H$, $a \in R$ and $T = \{v \in H : \langle u, v\rangle - a \le 0\}$. If $x \notin T$*

and $u \neq 0$, *then*

$$P_T(x) = x - \frac{\langle u, x\rangle - a}{\|u\|^2} u.$$

Lemma 2.5 ([18, Theorem 4.1]) *Consider the variational inequality problem (1.1), assume its solution set* S *is nonempty and* C *is defined by (1.5), where* $c : H \to R$ *is a continuously differentiable convex function and* $\{x \in C : c(x) < 0\} \neq \emptyset$. *If* $u \in C$, *then* $u \in S$ *if and only if either*

(i) $F(u) = 0$, *or*

(ii) $u \in \partial C$ *and there exists a positive constant* η *such that* $F(u) = -\eta c'(u)$.

Lemma 2.6 ([15, Theorem 9.1]) *Let* $f : H \to (-\infty, +\infty]$ *be convex. Then the following statements are equivalent:*

(i) f *is weakly sequential lower semicontinuous;*

(ii) f *is lower semicontinuous.*

Definition 2.7 ([15, P35]) *Let* $f : H \to (-\infty, +\infty]$ *and* $x \in H$. *Then* f *is weakly sequential lower semicontinuous at* x, *if for every sequence* $\{x_n\}$ *in* H,

$$x_n \rightharpoonup x \Rightarrow f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Definition 2.8 ([15, Definition 20.1]) *Let* $A : H \to 2^H$ *be a set-valued mapping. Then* A *is monotone if for all* $(x, u) \in \operatorname{gra} A$ *and* $(y, v) \in \operatorname{gra} A$, *it holds that* $\langle x - y, u - v\rangle \geq 0$, *where*

$$\operatorname{gra} A = \{(x, u) \in H \times H : u \in A(x)\}.$$

Definition 2.9 ([15, Definition 20.20]) *Let* $A : H \to 2^H$ *be monotone. Then* A *is maximally monotone if there exists no monotone operator* $B : H \to 2^H$ *such that* $\operatorname{gra} B$ *properly contains* $\operatorname{gra} A$, *i.e., for every* $(x, u) \in H \times H$,

$$(x, u) \in \operatorname{gra} A \iff \langle x - y, u - v\rangle \geq 0, \quad \forall\, (y, v) \in \operatorname{gra} A.$$

Throughout this paper, we need the following assumptions.

Assumption 2.10 *(H1) The solution set* S *of Problem (1.1) is nonempty.*

(H2) The mapping F *is monotone and Lipschitz continuous on* H *with modulus* $L > 0$.

(H3) The feasible set $C = \{x \in H : c(x) \leq 0\}$ *is nonempty with* $c : H \to R$ *being* K_1 *smooth convex on* H *(i.e., c is convex on* H *and its derivative function* $c'(\cdot)$ *is* K_1*-Lipschitz continuous on* H*). Moreover,* $\{x \in C : c(x) < 0\} \neq \emptyset$.

(H4) There exists a positive constant K_2 *such that* $\|F(x)\| \leq K_2\|c'(x)\|$ *for any* $x \in \partial C$, *where* ∂C *denotes the boundary of* C.

3. Algorithm and its convergence

In this section, we present MTSEGA for solving the monotone and Lipschitz continuous variational inequalities with C defined in (1.5). We will introduce the well-definedness of MTSEGA and analyze its convergence.

Algorithm 3.1 Initialization: choose $x_0, y_0 \in H$ and $\lambda > 0$. Compute

$$x_1 = P_C(x_0 - \lambda F(y_0)), \quad y_1 = P_C(x_1 - \lambda F(y_0)).$$

Iterative Steps: Starting from $x_n, y_n, y_{n-1} \in H$, calculate x_{n+1}, y_{n+1} for each $n \geq 1$ as follows:

Step 1. Compute $x_{n+1} = P_{T_n}(x_n - \lambda F(y_n))$, where

$$T_n = \{x \in H : \langle x_n - \lambda F(y_{n-1}) - y_n, x - y_n \rangle \leq 0\}.$$

Step 2. Compute $y_{n+1} = P_{C_{n+1}}(x_{n+1} - \lambda F(y_n))$, where

$$C_{n+1} = \{\omega \in H : c(x_{n+1}) + \langle c'(x_{n+1}), \omega - x_{n+1} \rangle \leq 0\}.$$

Step 3. If $x_{n+1} = y_{n+1} = y_n$, then stop. Otherwise, let $n \leftarrow n+1$ and go to Step 1.

Lemma 3.2 *Let $\{C_n\}$ and $\{T_n\}$ be two sequences generated by Algorithm 3.1 and C be the set defined by (1.5). Then, $C \subseteq C_n$ and $C \subseteq T_n$ for each n.*

Proof From the fact that c is convex and smooth, we conclude that, for any $x \in C$ and for each n,

$$c(x_n) + \langle c'(x_n), x - x_n \rangle \leq c(x) \leq 0,$$

where the second inequality follows from the definition of set C. Using this together with the definition of C_n, we obtain that $x \in C_n$. So we assert that $C \subseteq C_n$ for each n.

Next, we show that $C \subseteq T_n$ for each n. To this end, we take an arbitrary $x \in C$. By using the facts $y_n = P_{C_n}(x_n - \lambda F(y_{n-1}))$ and $C \subseteq C_n$, together with Lemma 2.2 (i), we obtain that $\langle x_n - \lambda F(y_{n-1}) - y_n, x - y_n \rangle \leq 0$. □

Lemma 3.3 *If $x_{n+1} = y_{n+1} = y_n$ in Algorithm 3.1, then $y_n \in S$.*

Proof Assume that $x_{n+1} = y_{n+1} = y_n$, we have

$$y_{n+1} = P_{C_{n+1}}(y_{n+1} - \lambda F(y_{n+1})).$$

We firstly show that $y_n \in C$. By using the definition of C_{n+1}, we have

$$c(y_{n+1}) + \langle c'(y_{n+1}), y_{n+1} - y_{n+1} \rangle \leq 0,$$

which implies that $c(y_{n+1}) \leq 0$. Hence, $y_n = y_{n+1} \in C$.

Next, we show that $y_n \in S$. By using Lemma 2.2 (i), we have

$$\langle y_{n+1} - \lambda F(y_{n+1}) - y_{n+1}, y - y_{n+1} \rangle \leq 0, \quad \forall y \in C.$$

This implies that

$$\lambda \langle F(y_{n+1}), y - y_{n+1} \rangle \geq 0, \quad \forall y \in C.$$

Using this together with the fact that $\lambda > 0$, we also assert that $y_n = y_{n+1} \in S$. □

The following conclusions are crucial in the subsequent convergence analysis.

Lemma 3.4 *Assume that Assumption 2.10 holds. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated* *by Algorithm 3.1. For any fixed $u \in S$, we have*

$$\Phi_{n+1}(u) \leq \Phi_n(u) - [1 - \lambda(L + 2K)]\|x_n - y_n\|^2 - (1 - 3\lambda L)\|x_{n+1} - y_n\|^2, \tag{3.1}$$

where

$$\Phi_n(u) = \|x_n - u\|^2 + \lambda L\|x_n - y_{n-1}\|^2 \text{ and } K = K_1 K_2 > 0.$$

Proof For any fixed $u \in S$, by the definition of x_{n+1}, the fact that $u \in C \subseteq T_n$ and Lemma 2.2 (ii), it follows that

$$\begin{aligned}
\|x_{n+1} - u\|^2 \leq & \|x_n - \lambda F(y_n) - u\|^2 - \|x_n - \lambda F(y_n) - x_{n+1}\|^2 \\
= & \|x_n - u\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda\langle F(y_n), u - x_{n+1}\rangle \\
= & \|x_n - u\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda[\langle F(y_n) - F(u), u - y_n\rangle + \langle F(u), u - y_n\rangle + \\
& \langle F(y_n), y_n - x_{n+1}\rangle] \\
\leq & \|x_n - u\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda[\langle F(u), u - y_n\rangle + \langle F(y_n), y_n - x_{n+1}\rangle] \\
= & \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 - 2\langle x_n - y_n, y_n - x_{n+1}\rangle + \\
& 2\lambda[\langle F(u), u - y_n\rangle + \langle F(y_n), y_n - x_{n+1}\rangle] \\
= & \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + 2\lambda\langle F(u), u - y_n\rangle + \\
& 2\langle x_n - \lambda F(y_n) - y_n, x_{n+1} - y_n\rangle \\
= & \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + 2\lambda\langle F(u), u - y_n\rangle + \\
& 2\langle x_n - \lambda F(y_{n-1}) - y_n, x_{n+1} - y_n\rangle + 2\lambda\langle F(y_{n-1}) - F(y_n), x_{n+1} - y_n\rangle,
\end{aligned} \tag{3.2}$$

where the second inequality holds from the monotonicity of F.

From the definition of T_n and the fact $x_{n+1} \in T_n$, we get that

$$\langle x_n - \lambda F(y_{n-1}) - y_n, x_{n+1} - y_n\rangle \leq 0. \tag{3.3}$$

Next, we estimate the value of $\langle F(y_{n-1}) - F(y_n), x_{n+1} - y_n\rangle$. By using Cauchy-Schwartz inequality and the fact $\lambda > 0$, we have

$$\begin{aligned}
2\lambda\langle F(y_{n-1}) - F(y_n), x_{n+1} - y_n\rangle & \leq 2\lambda\|F(y_{n-1}) - F(y_n)\|\|x_{n+1} - y_n\| \\
& \leq 2\lambda L\|y_{n-1} - y_n\|\|x_{n+1} - y_n\| \\
& \leq 2\lambda L(\|y_{n-1} - x_n\| + \|x_n - y_n\|)\|x_{n+1} - y_n\| \\
& \leq \lambda L(\|y_{n-1} - x_n\|^2 + 2\|x_{n+1} - y_n\|^2 + \|x_n - y_n\|^2),
\end{aligned} \tag{3.4}$$

where the second inequality holds from the fact that the mapping F is Lipschitz continuous with modulus $L > 0$, and the last inequality holds from the fact that $a^2 + b^2 \geq 2ab$ for all $a, b \in R$.

Substituting (3.3) and (3.4) into the inequality (3.2), we get that

$$\begin{aligned}
\|x_{n+1} - u\|^2 \leq & \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + 2\lambda\langle F(u), u - y_n\rangle + \\
& \lambda L\|y_{n-1} - x_n\|^2 + 2\lambda L\|x_{n+1} - y_n\|^2 + \lambda L\|x_n - y_n\|^2 \\
= & \|x_n - u\|^2 + \lambda L\|x_n - y_{n-1}\|^2 - (1 - \lambda L)\|x_n - y_n\|^2 - \\
& (1 - 2\lambda L)\|x_{n+1} - y_n\|^2 + 2\lambda\langle F(u), u - y_n\rangle.
\end{aligned} \tag{3.5}$$

Adding $\lambda L\|x_{n+1}-y_n\|^2$ to the both sides of (3.5), we have

$$\|x_{n+1}-u\|^2+\lambda L\|x_{n+1}-y_n\|^2 \le \|x_n-u\|^2+\lambda L\|x_n-y_{n-1}\|^2-(1-\lambda L)\|x_n-y_n\|^2- (1-3\lambda L)\|x_{n+1}-y_n\|^2+2\lambda\langle F(u),u-y_n\rangle.$$

Using this together with the definition of $\Phi_n(u)$ in Lemma 3.4 gives

$$\Phi_{n+1}(u)\le\Phi_n(u)-(1-\lambda L)\|x_n-y_n\|^2-(1-3\lambda L)\|x_{n+1}-y_n\|^2+2\lambda\langle F(u),u-y_n\rangle. \tag{3.6}$$

If $F(u)=0$, then (3.1) holds immediately. So, we assume that $F(u)\neq 0$. In this case, from Lemma 2.5, we know that $u\in\partial C$ and there exists a constant $\eta_u>0$ such that $F(u)=-\eta_u c'(u)$. Since $u\in\partial C$ and $c(\cdot)$ is continuous, we have $c(u)=0$. Using this together with the convexity of $c(\cdot)$, we have

$$c(y_n)\ge c(u)+\langle c'(u),y_n-u\rangle=-\frac{1}{\eta_u}\langle F(u),y_n-u\rangle.$$

By rearranging the terms of the above inequality, we obtain that

$$\langle F(u),u-y_n\rangle\le\eta_u c(y_n). \tag{3.7}$$

From the definition of C_n and the fact that $y_n\in C_n$, we have

$$c(x_n)+\langle c'(x_n),y_n-x_n\rangle\le 0.$$

It follows from the fact $c(\cdot)$ is convex that

$$\langle c'(y_n),x_n-y_n\rangle+c(y_n)\le c(x_n).$$

Hence, by adding the above two inequalities, we obtain that

$$c(y_n)\le\langle c'(x_n)-c'(y_n),x_n-y_n\rangle. \tag{3.8}$$

Combining (3.7) and (3.8), by using Cauchy-Schwartz inequality and Assumption 2.10 (H3), we get that

$$\begin{aligned}\langle F(u),u-y_n\rangle &\le \eta_u c(y_n)\le\eta_u\langle c'(x_n)-c'(y_n),x_n-y_n\rangle\\ &\le\eta_u\|c'(x_n)-c'(y_n)\|\|x_n-y_n\|\le\eta_u K_1\|x_n-y_n\|^2.\end{aligned}$$

On the other hand, from Lemma 2.5 (ii) and Assumption 2.10 (H4), we see that K_2 is an upper bound of η_u. Hence, it follows that

$$\langle F(u),u-y_n\rangle\le\eta_u K_1\|x_n-y_n\|^2\le K_1K_2\|x_n-y_n\|^2. \tag{3.9}$$

Let $K=K_1K_2>0$. Then, from (3.9) and (3.6), we get (3.1). □

Lemma 3.5 *Assume that Assumption 2.10 holds. Let $S\neq\emptyset$ and $\lambda\in(0,\min\{\frac{1}{L+2K},\frac{1}{3L}\})$, where $K=K_1K_2>0$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 3.1. Then, we have*

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|x_{n+1}-y_n\|=\lim_{n\to\infty}\|y_n-y_{n-1}\|=0.$$

Moreover, the sequences $\{x_n\}$,$\{y_n\}$ and $\{c'(x_n)\}$ are all bounded.

Proof From Lemma 3.4, we know that

$$[1-\lambda(L+2K)]\|x_n-y_n\|^2+(1-3\lambda L)\|x_{n+1}-y_n\|^2 \le \Phi_n(u)-\Phi_{n+1}(u), \quad \forall u\in S, \tag{3.10}$$

where

$$\Phi_n(u)=\|x_n-u\|^2+\lambda L\|x_n-y_{n-1}\|^2 \text{ and } K=K_1K_2>0.$$

Using this together with the fact that $\lambda\in(0,\min\{\frac{1}{L+2K},\frac{1}{3L}\})$, we get that the sequence $\{\Phi_n(u)\}$ is nonincreasing and has a lower bound zero. Hence, the limit of $\{\Phi_n(u)\}$ exists. Passing to the limit in (3.10), together with the fact $\lambda\in(0,\min\{\frac{1}{L+2K},\frac{1}{3L}\})$, we have

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{3.11}$$

This together with the fact $\|y_n-y_{n-1}\|\le\|y_n-x_n\|+\|x_n-y_{n-1}\|$ gives

$$\lim_{n\to\infty}\|y_n-y_{n-1}\|=0.$$

Moreover, together with the fact the limit of $\{\Phi_n(u)\}$ exists, we get that $\lim_{n\to\infty}\|x_n-u\|$ exists and further $\{x_n\}$ is bounded. Using this together with the fact that $\lim_{n\to\infty}\|x_n-y_n\|=0$, we see that $\{y_n\}$ is bounded. This together with the Lipschitz continuity of $c'(\cdot)$ and the fact that $\{x_n\}$ is bounded deduces that $\{c'(x_n)\}$ is also bounded. □

Now, we are in the position to establish the convergence of the Algorithm 3.1.

Theorem 3.6 *Assume that Assumption 2.10 holds. Let $S\neq\emptyset$, $\lambda\in(0,\min\{\frac{1}{L+2K},\frac{1}{3L}\})$ with $K=K_1K_2>0$ and $\{x_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 3.1. Then $\{x_n\}$ weakly converges to a vector in S.*

Proof From Lemma 3.5, we see that the sequence $\{x_n\}$ is bounded. Hence, there exists a subsequence $\{x_{n_k}\}$ that weakly converges to $x^*\in H$. Then, from the fact that $\lim_{n\to\infty}\|x_n-y_n\|=0$, we know that $y_{n_k}\rightharpoonup x^*$, $k\to\infty$. By using the definition of C_{n_k} and the fact that $y_{n_k}\in C_{n_k}$, we obtain that

$$c(x_{n_k})+\langle c'(x_{n_k}),y_{n_k}-x_{n_k}\rangle\le 0.$$

From the fact $\{c'(x_{n_k})\}$ is bounded, there exists $M>0$ such that $\|c'(x_{n_k})\|\le M$. By using Cauchy-Schwartz inequality, we have

$$c(x_{n_k})\le\|c'(x_{n_k})\|\|y_{n_k}-x_{n_k}\|\le M\cdot\|y_{n_k}-x_{n_k}\|. \tag{3.12}$$

Recall that $c(\cdot)$ is smooth. By using this together with the fact that $c(\cdot)$ is convex and Lemma 2.6, we get that $c(\cdot)$ is weakly sequential lower semicontinuous. It follows from (3.12) and the facts that $x_{n_k}\rightharpoonup x^*$ and $\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0$, combining Definition 2.7, we have

$$c(x^*)\le\liminf_{k\to\infty}c(x_{n_k})\le\liminf_{k\to\infty}M\cdot\|y_{n_k}-x_{n_k}\|=0.$$

Hence, one has $x^*\in C$.

Next, we show that $x^*\in S$. Let

$$T(x):=\begin{cases}F(x)+N_C(x) & x\in C;\\ \emptyset & x\notin C,\end{cases}$$

with $N_C(x)$ being the normal cone of C at x; i.e.,

$$N_C(x) := \{\varepsilon \in H : \langle \varepsilon, z - x\rangle \le 0,\ \forall z \in C\}. \tag{3.13}$$

It is well-known that N_C is maximal monotone [15, Example 20.26]. Since F is monotone and continuous, we know that F is also maximal monotone [15, Corollary 20.28]. Using this together with the fact that the domain of F is H, from [15, Corollary 25.5], we obtain that T is maximal monotone. For arbitrary $(x, y) \in \mathrm{gra}T$, by the definition of $\mathrm{gra}T$ in Definition 2.8, we have $y \in T(x) = F(x) + N_C(x)$, or equivalently, $y - F(x) \in N_C(x)$.

It follows from (3.13) and the fact $x^* \in C$, we have

$$\langle y - F(x), x^* - x\rangle \le 0, \quad \forall (x, y) \in \mathrm{gra}T.$$

Hence, we see that

$$\langle y, x - x^*\rangle \ge \langle F(x), x - x^*\rangle, \quad \forall (x, y) \in \mathrm{gra}T. \tag{3.14}$$

By using the definition of T_n and the fact $x \in C \subseteq T_n$, we obtain that

$$\langle x_n - \lambda F(y_{n-1}) - y_n, x - y_n\rangle \le 0.$$

From the fact $\lambda > 0$, by rearranging the inequality above, we have

$$\langle F(y_{n-1}), x - y_n\rangle \ge \frac{1}{\lambda}\langle x_n - y_n, x - y_n\rangle. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\langle F(x), x - x^*\rangle =&\langle F(x), x - y_n\rangle + \langle F(x), y_n - x^*\rangle \\ =&\langle F(x) - F(y_n), x - y_n\rangle + \langle F(y_n) - F(y_{n-1}), x - y_n\rangle + \\ &\langle F(y_{n-1}), x - y_n\rangle + \langle F(x), y_n - x^*\rangle \\ \ge&\langle F(y_n) - F(y_{n-1}), x - y_n\rangle + \frac{1}{\lambda}\langle x_n - y_n, x - y_n\rangle + \langle F(x), y_n - x^*\rangle,\end{aligned} \tag{3.16}$$

where the last inequality follows from the monotonicity of mapping F and the inequality (3.15). Combining (3.14) and (3.16), for any $(x, y) \in \mathrm{gra}T$, we get that

$$\langle y, x - x^*\rangle \ge \langle F(y_n) - F(y_{n-1}), x - y_n\rangle + \frac{1}{\lambda}\langle x_n - y_n, x - y_n\rangle + \langle F(x), y_n - x^*\rangle. \tag{3.17}$$

From the facts that the mapping F is Lipschitz continuous, $\lim_{n\to\infty} \|y_n - y_{n-1}\| = 0$ and $\{y_n\}$ is bounded, we obtain that

$$\langle F(y_n) - F(y_{n-1}), x - y_n\rangle \to 0, \quad n \to \infty. \tag{3.18}$$

Similarly, from the facts that $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\{y_n\}$ is bounded, we obtain that

$$\langle x_n - y_n, x - y_n\rangle \to 0, \quad n \to \infty. \tag{3.19}$$

Moreover, from the fact that $y_{n_k} \rightharpoonup x^*$, we have

$$\langle F(x), y_{n_k} - x^*\rangle \to 0, \quad k \to \infty. \tag{3.20}$$

Hence, from (3.18), (3.19) and (3.20), by passing limits in (3.17) along the subsequences, for any fixed $(x, y) \in \text{gra}T$, we see that

$$\langle y, x - x^* \rangle \geq 0. \tag{3.21}$$

Since T is a maximal monotone operator, (3.21) implies that $0 \in T(x^*)$. Consequently, one has $x^* \in T^{-1}(0) = S$.

Finally, we show that $\{x_n\}$ has only one weak cluster point. To this end, supposing to the contrary that $x^* \in S$ and $\bar{x} \in S$ are two different weak cluster points of $\{x_n\}$. Hence, there exist two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ weakly converging to x^* and $\bar{x}$, respectively. It follows from the fact that the limit of $\{\Phi_n(x^*)\}$ exists for any fixed $x^* \in S$. Using this together with the fact $\lim_{n\to\infty} \|x_n - y_{n-1}\| = 0$ and the definition of $\Phi_n(\cdot)$, we get that for any fixed $x^* \in S$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Moreover, together with Lemma 2.3, it follows that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{k\to\infty} \|x_{n_k} - x^*\| = \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}\| = \lim_{k\to\infty} \|x_{m_k} - \bar{x}\| \\
&= \liminf_{k\to\infty} \|x_{m_k} - \bar{x}\| < \liminf_{k\to\infty} \|x_{m_k} - x^*\| \\
&= \lim_{k\to\infty} \|x_{m_k} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|.
\end{aligned}$$

We obtain an inconsistent inequality. Therefore, we have $x^* = \bar{x}$. □

4. Concluding remarks

In this paper, we present a modified two-subgradient extragradient algorithm for solving monotone and Lipschitz continuous variational inequalities with the feasible set being a level set of a convex and smooth function. Under Assumption 2.10 (H1)–(H4), we show that the sequence generated by MTSEGA is weakly convergent to the solution of Problem (1.1). The proposed algorithm requires only one evaluation of F and two projections onto two different half-spaces per iteration, respectively. Hence, we generalize some recent results in the literature. However, how to ensure Assumption 2.10 (H4) is unknown. This is an interesting future research direction.

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