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Exponential Stability in Mean Square of Neutral Stochastic Functional Differential Equations

Zhi LI*, Liping XU
School of Information and Mathematics, Yangtze University, Hubei 434023*, P. R. China*

Abstract A novel approach to the exponential stability in mean square of neutral stochastic functional differential equations is presented. Consequently, some new criteria for the exponential stability in mean square of the considered equations are obtained and some known results are improved. Lastly, some examples are investigated to illustrate the theory.

Keywords exponential stability in mean square; stochastic; functional differential equations; neutral

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1. Introduction

A traditional approach to analyze the stability for stochastic functional differential equations is the Lyapunov's function method. Lyapunov functions and functionals have been successfully used to obtain the stability of stochastic differential equations [1–5]. Another widely-used approach to stability of stochastic functional differential equations is the Razumikhin-type theorems. Razumikhin-type theorems for the exponential stability of stochastic functional differential equations have been presented in [6–9]. A Razumikhin-type theorem for the asymptotic stability of stochastic functional differential equations has been given in [10–12].

In fact, it is not easy to find a Lyapunov function (functional) for stochastic differential and the stability conditions obtained by the Lyapunov's function method are often given in terms of differential inequalities, matrix inequalities and so on. The given conditions by Lyapunov function (functional) and Razumikhin-type theorems are not only a little bit strong but also general implicit and not easy to examine.

On the other hand, neutral stochastic delay differential equations are often used to describe the dynamical systems which not only involve derivatives but also depend on present and past states. Neutral stochastic delay differential equations have attracted the increasing attention due to the wide applications in the distributed networks containing lossless transmission lines [13], processes including steam or water pipes, heat exchanges, and other engineering systems [14] and population ecology [15]. For neutral stochastic functional differential equations, we refer to [16–19].

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* Corresponding author
E-mail address: lizhi_csu@126.com (Zhi LI)

By Lyapunov's function method and Razumikhin-type theorems, in [18, 20–23] some efforts have been devoted to the investigation of exponential stability in mean square of neutral stochastic functional differential equations. However, the results derived there are either difficult to demonstrate in a straightforward way for practical situations or somewhat too restricted to be applied to general neutral stochastic functional differential equations. In this paper, we will present a novel approach to the exponential stability in mean square of neutral stochastic functional differential equations. Our approach does not involve Lyapunov functions and complex calculations. Our approach is based on a comparison principle and a proof by contradiction and our conditions are also feasible. Our results improve some known results.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we present some criteria for the exponential stability in mean square of neutral stochastic functional differential equations. In Section 4, we state some comparisons with existing results and present some examples to illustrate the advantage of our results.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\tau > 0$ and $C([-\tau,0];\mathbb{R}^n)$ denote the family of all continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\|_C = \sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$, where $|\cdot|$ is Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its norm $\|A\|$ is defined by $\|A\| = \sup\{|Ax| : |x| = 1, x \in \mathbb{R}^n\}$. Moreover, let $w(t) = (w_1(t),\dots,w_m(t))^T$ be an m-dimensional Brownian motion defined over $(\Omega, \mathcal{F}, \mathbb{P})$. We also denote by $C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $C([-\tau,0];\mathbb{R}^n)$-valued random variables.

Consider the following neutral stochastic functional differential equation

$$\mathrm{d}[x(t) - G(x_t)] = f(t,x_t)\mathrm{d}t + g(t,x_t)\mathrm{d}w(t) \tag{2.1}$$

on $t \geq 0$ with initial data $x_0 = \xi$, where

$$G : C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^n,\ f : \mathbb{R}_+ \times C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^n,\ g : \mathbb{R}_+ \times C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^{n\times m}.$$

Moreover, $x_t = \{x(t+s) : -\tau \leq s \leq 0\}$ which is regarded as a $C([-\tau,0];\mathbb{R}^n)$-valued stochastic process and $\xi = \{\xi(s) : -\tau \leq s \leq 0\} \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$. An $\mathcal{F}_t$-adapted process $x(t)$, $-\tau \leq t < \infty$ is said to be the solution of the equation (2.1) if it satisfies the initial condition above and moreover for each $t \geq 0$,

$$x(t) - G(x_t) = \xi(0) - G(x_0) + \int_0^t f(s,x_s)\mathrm{d}s + \int_0^t g(s,x_s)\mathrm{d}w(s), \tag{2.2}$$

where the stochastic integral is defined in the Itô's sense. For the details on the existence and uniqueness of the solution to (2.1), we can refer to [24]. For example, when f, g, G are uniformly Lipschitz continuous, or they are locally Lipschitz continuous and satisfy the linear growth condition, Kolmanovskii and Nosov [24] proved that there is unique continuous solution to (2.1), and any moment of the solution is finite. For stability purpose, throughout the paper we always suppose that Eq. (2.1) has a unique solution for arbitrarily given initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ and the solution is denoted by $x(t,\xi)$, or simply $x(t)$, when no confusion is possible. For the purposes of stability, we shall assume that

$$G(0) \equiv 0,\ f(t,0) \equiv 0,\ g(t,0) \equiv 0 \ \text{ for any } \ t \geq 0.$$

It is well-known that for a given $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];H)$, Eq. (2.1) has a trivial solution when $\xi \equiv 0$.

Definition 2.1 *The trivial solution $x(t,\xi)$ of (2.1) is said to be exponentially stable in mean square, if for any initial value ξ, there exists a pair of positive constants $\lambda > 0$ and C such that for all $t \geq 0$*

$$\mathbb{E}|x(t,\xi)|^2 \leq C\mathbb{E}\|\xi\|_C e^{-\lambda t},$$

or, equivalently,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|x(t,\xi)|^2 \leq -\lambda.$$

Definition 2.2 *The trivial solution $x(t,\xi)$ of (2.1) is said to be almost surely exponentially stable if there exists a constant $\lambda > 0$ such that there is a finite random variable β such that for all $t \geq 0$*

$$|x(t,\xi)| \leq \beta e^{-\lambda t} \ \text{ a.s.}$$

3. Exponential stability for neutral stochastic functional equations

To state the main result of this section, let us define some functions. Let $\eta_i(t,\theta) : \mathbb{R}_+ \times [-\tau,0] \to \mathbb{R}$ $(i = 1,2)$ be non-decreasing in θ for each $t \in \mathbb{R}_+$. Furthermore, $\eta_i(t,\theta)$ is continuous in θ on $[-\tau,0]$. Assume that

$$L_i(t,\phi) := \int_{-\tau}^0 \phi(\theta)\mathrm{d}[\eta_i(t,\theta)], \quad t \in \mathbb{R}_+,\ i = 1,2, \tag{3.1}$$

is a locally bounded Borel-measurable function in t for each $\phi \in C([-\tau,0];\mathbb{R}^n)$. Here, the integral in (3.1) is the Riemann-Stieltjes integral. Furthermore, we assume that there is a constant $k \in (0,1)$ such that for all $\varphi \in L^2_{\mathcal{F}}([-\tau,0];\mathbb{R}^n)$

$$|G(\varphi)|^2 \leq k \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|^2. \tag{3.2}$$

Lemma 3.1 *Let (3.2) hold with $0 < k < 1$ and $\rho \geq 0$, $\delta > 0$, $K > 1$. If*

$$e^{\delta t}\mathbb{E}|x(t) - G(x_t)|^2 \leq K \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|x(\theta)|^2 \tag{3.3}$$

for all $0 \leq t \leq \rho$, then

$$e^{\delta t}\mathbb{E}|x(t)|^2 \leq \frac{K}{(1-\sqrt{k})^2} \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|x(\theta)|^2.$$

Proof Let $k < \varepsilon < 1$. For $0 \leq t \leq \rho$, we have

$$\mathbb{E}|x(t) - G(x_t)|^2 \geq \mathbb{E}|x(t)|^2 - 2\mathbb{E}(|x(t)||G(x_t)|) + \mathbb{E}|G(x_t)|^2$$

$$\geq (1-\varepsilon)\mathbb{E}|x(t)|^2 - (\varepsilon^{-1}-1)\mathbb{E}|G(x_t)|^2.$$

Then, by (3.2) we have

$$\mathbb{E}|x(t)|^2 \leq \frac{1}{1-\varepsilon}\mathbb{E}|x(t)-G(x_t)|^2 + \frac{k}{\varepsilon}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|x(t+\theta)|^2.$$

Using the condition (3.3), we derive that for all $0 \leq t \leq \rho$

$$\begin{aligned} e^{\delta t}\mathbb{E}|x(t)|^2 \leq & \frac{K}{1-\varepsilon}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|x(\theta)|^2 + \frac{k}{\varepsilon}\sup_{-\tau\leq\theta\leq 0}[e^{\delta t}\mathbb{E}|x(t+\theta)|^2] \\ \leq & \frac{K}{1-\varepsilon}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|x(\theta)|^2 + \frac{k}{\varepsilon}\sup_{-\tau\leq t\leq \rho}[e^{\delta t}\mathbb{E}|x(t)|^2]. \end{aligned}$$

Moreover, this holds for $-\tau \leq t \leq 0$. Thus,

$$\sup_{-\tau\leq t\leq\rho}[e^{\delta t}\mathbb{E}|x(t)|^2] \leq \frac{K}{1-\varepsilon}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|x(\theta)|^2 + \frac{k}{\varepsilon}\sup_{-\tau\leq t\leq\rho}[e^{\delta t}\mathbb{E}|x(t)|^2].$$

Since $1 > \frac{k}{\varepsilon}$, we can obtain

$$\sup_{-\tau\leq t\leq\rho}[e^{\delta t}\mathbb{E}|x(t)|^2] \leq \frac{K\varepsilon}{(1-\varepsilon)(\varepsilon-k)}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|x(\theta)|^2.$$

Lastly, letting $\varepsilon = \sqrt{k}$, we can obtain our desired result. The proof is completed. □

Theorem 3.2 *Assume that (3.2) holds with $0 < k < 1$. Let $\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ be a locally bounded Borel-measurable function such that for any $t \in \mathbb{R}_+$, $\varphi \in C([-\tau,0];\mathbb{R}^n)$,*

$$\mathbb{E}(2(\varphi(0)-G(\varphi))^T f(t,\varphi)) \leq \gamma(t)\mathbb{E}|\varphi(0)|^2 + \int_{-\tau}^{0}\mathbb{E}|\varphi(\theta)|^2 \mathrm{d}[\eta_1(t,\theta)] \tag{3.4}$$

and

$$\mathbb{E}(\mathrm{trac}[g^T(t,\varphi)g(t,\varphi)]) \leq \int_{-\tau}^{0}\mathbb{E}|\varphi(\theta)|^2\mathrm{d}[\eta_2(t,\theta)]. \tag{3.5}$$

If there exists $\beta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\gamma(t) + \int_{-\tau}^{0} e^{-\beta\theta}\mathrm{d}[\eta_1(t,\theta)] + \int_{-\tau}^{0} e^{-\beta\theta}\mathrm{d}[\eta_2(t,\theta)] \leq -(1-\sqrt{k})^2\beta, \tag{3.6}$$

then the trivial solution of (2.1) is exponentially stable in mean square. In particular, $\mathbb{E}|x(t,\xi)|^2$ exponentially decays with the rate β for any $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$.

Proof Fix $K > 1$ sufficient large and let $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ such that $\mathbb{E}\|\xi\|^2_C > 0$. For the sake of simplicity, we denote $x(t) := x(t,\xi)$, $t \geq -\tau$, where $x(t,\xi)$ is the solution to (2.1). Let $Z(t) := Ke^{-\beta t}\mathbb{E}\|\xi\|^2_C$, $t \geq 0$. Then, we deduce from $K > 1$ sufficiently large and $\mathbb{E}\|\xi\|^2_C > 0$ that $X(t) := \mathbb{E}|x(t)-G(t,x_t)|^2 \leq Z(t)$, $t \in [-\tau,0]$. We will show

$$\mathbb{E}|x(t)-G(t,x_t)|^2 \leq Z(t), \quad \forall t \geq 0. \tag{3.7}$$

Assume on the contrary that there exists $t_1 > 0$ such that $X(t_1) > Z(t_1)$. Let $t_* := \inf\{t > 0 : X(t) > Z(t)\}$. By continuity of $X(t)$ and $Z(t)$,

$$X(t) \leq Z(t), \quad t \in [0,t_*], \ X(t_*) = Z(t_*) \tag{3.8}$$

and

$$\mathbb{E}|x(t_m) - G(x_{t_m})|^2 > Ke^{-\beta t_m}\mathbb{E}\|\xi\|_C^2,$$

for some $t_m \in (t_*, t_* + \frac{1}{m})$, $m \in \mathbb{N}$.

Applying the Itô's formula to the function $V(t,x) = e^{\alpha t}|x(t) - G(x_t)|^2$, (3.4), (3.5) and the Fubini's theorem, we have

$$\begin{aligned}
&\mathbb{E}(e^{\alpha t}|x(t) - G(x_t)|^2) = \mathbb{E}|\xi(0) - G(\xi)|^2 + \mathbb{E}\int_0^t \alpha e^{\alpha s}|x(s) - G(x_s)|^2 \mathrm{d}s + \\
&\quad 2\mathbb{E}\int_0^t e^{\alpha s}(x(s) - G(x_s))^T f(s, x_s)\mathrm{d}s + \mathbb{E}\int_0^t e^{\alpha s}\mathrm{trac}[g^T(s, x_s)g(s, x_s)]\mathrm{d}s \\
&\leq \mathbb{E}|\xi(0) - G(\xi)|^2 + \mathbb{E}\int_0^t \alpha e^{\alpha s}|x(s) - G(x_s)|^2 \mathrm{d}s + \int_0^t \gamma(s)e^{\alpha s}\mathbb{E}|x(s)|^2 \mathrm{d}s + \\
&\quad \int_0^t e^{\alpha s}\Big(\int_{-\tau}^0 \mathbb{E}|x(s+\theta)|^2 d[\eta_1(s,\theta)]\Big)\mathrm{d}s + \int_0^t e^{\alpha s}\Big(\int_{-\tau}^0 \mathbb{E}|x(s+\theta)|^2 \mathrm{d}[\eta_2(s,\theta)]\Big)\mathrm{d}s.
\end{aligned}$$

Let $K_1 := K\mathbb{E}\|\xi\|_C^2$ and $K_2 = \frac{K_1}{(1-\sqrt{k})^2}$. Since $\eta_1(s,\theta)$ and $\eta_2(s,\theta)$ are increasing in θ on $[-\tau, 0]$, we derive that from (3.8) and the Lemma 3.1

$$\int_{-\tau}^0 \mathbb{E}|x(s+\theta)|^2 \mathrm{d}[\eta_1(s,\theta)] \leq K_2 e^{-\beta s}\int_{-\tau}^0 e^{-\beta\theta}\mathrm{d}[\eta_1(s,\theta)]$$

and

$$\int_{-\tau}^0 \mathbb{E}|x(s+\theta)|^2 \mathrm{d}[\eta_2(s,\theta)] \leq K_2 e^{-\beta s}\int_{-\tau}^0 e^{-\beta\theta}\mathrm{d}[\eta_2(s,\theta)],$$

for any $s \leq t_*$. Then, it follows that

$$\begin{aligned}
&e^{\alpha t_*}\mathbb{E}|x(t_*) - G(x_{t_*})|^2 \leq \mathbb{E}|\xi(0) - G(\xi)|^2 + \int_0^{t_*} e^{\alpha s}e^{-\beta s}(K_1\alpha + K_2\gamma(s))\mathrm{d}s + \\
&\quad \int_0^{t_*} e^{\alpha s}K_2 e^{-\beta s}\int_{-\tau}^0 e^{-\beta\theta}\mathrm{d}[\eta_1(s,\theta)]\mathrm{d}s + \int_0^{t_*} e^{\alpha s}K_2 e^{-\beta s}\int_{-\tau}^0 e^{-\beta\theta}\mathrm{d}[\eta_2(s,\theta)]\mathrm{d}s \\
&= \mathbb{E}|\xi(0) - G(\xi)|^2 + \int_0^{t_*} K_1 e^{\alpha s}e^{-\beta s}\cdot \\
&\quad \Big[\alpha + \frac{1}{(1-\sqrt{k})^2}\Big(\gamma(s) + \int_{-\tau}^0 e^{-\beta\theta}\mathrm{d}[\eta_1(s,\theta)] + \int_{-\tau}^0 e^{-\beta\theta}\mathrm{d}[\eta_2(s,\theta)\Big)\Big]\mathrm{d}s.
\end{aligned}$$

Taking (3.6) into account, we get for sufficient large K,

$$\begin{aligned}
e^{\alpha t_*}\mathbb{E}|x(t_*) - G(x_{t_*})|^2 &\leq \mathbb{E}|\xi(0) - G(\xi)|^2 + \int_0^{t_*} e^{\alpha s}K_1 e^{-\beta s}(\alpha - \beta)\mathrm{d}s \\
&= \mathbb{E}|\xi(0) - G(\xi)|^2 + K_1(e^{\alpha t_*}e^{-\beta t_*} - 1) \\
&= \mathbb{E}|\xi(0) - G(\xi)|^2 - K_1 + K_1 e^{\alpha t_*}e^{-\beta t_*} \\
&= \mathbb{E}|\xi(0) - G(\xi)|^2 - K\mathbb{E}\|\xi\|_C^2 + Ke^{\alpha t_*}e^{-\beta t_*}\mathbb{E}\|\xi\|_C^2 \\
&< Ke^{\alpha t_*}e^{-\beta t_*}\mathbb{E}\|\xi\|_C^2,
\end{aligned}$$

which conflicts with (3.8). Therefore,

$$\mathbb{E}|x(t) - G(x_t)|^2 \leq Ke^{-\beta t}\mathbb{E}\|\xi\|_C^2, \quad t \geq 0$$

and

$$\mathbb{E}|x(t)|^2 \le \frac{K}{(1-\sqrt{k})^2} e^{-\beta t}\mathbb{E}\|\xi\|_C^2, \quad t \ge 0.$$

Now, we are intended to show the boundness of the segment process $x_t(\xi)$. □

Remark 3.3 We remark that the conditions (3.4) and (3.5) are generalization of some existing conditions. We cannot find these conditions for ensuring the exponential stability in mean square of (2.1) in the reported literature. Our results are new and very advantageous in the applications of "mixed" delay stochastic differential equations, which include the point delay, varying delay and distributed delay.

Corollary 3.4 *Assume that (3.2) holds with* $0 < k < 1$. *Let* $\Upsilon_1(\cdot,\cdot), \Upsilon_2(\cdot,\cdot) : \mathbb{R}_+ \times [-\tau, 0] \to \mathbb{R}_+$, $\gamma_i(\cdot), \zeta_i(\cdot), h_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, $i = 0, 1, 2, \ldots, n$ *with* $0 := h_0(t) \le h_1(t) \le h_2(t) \le \cdots \le h_n(t) \le \tau$, $t \in \mathbb{R}_+$, *be locally bounded Borel measurable functions such that for any* $t \in \mathbb{R}_+$, $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$,

$$\mathbb{E}\Big(2(\varphi(0) - G(\varphi))^T f(t,\varphi)\Big) \le \sum_{i=0}^{n} \gamma_i(t)\mathbb{E}|\varphi(-h_i(t))|^2 + \int_{-\tau}^{0} \Upsilon_1(t,s)\mathbb{E}|\varphi(s)|^2 \mathrm{d}s, \tag{3.9}$$

$$\mathbb{E}\Big(\mathrm{trac}[g^T(t,\varphi)g(t,\varphi)]\Big) \le \sum_{i=0}^{n} \zeta_i(t)\mathbb{E}|\varphi(-h_i(t))|^2 + \int_{-\tau}^{0} \Upsilon_2(t,s)\mathbb{E}|\varphi(s)|^2 \mathrm{d}s. \tag{3.10}$$

If there exists $\beta > 0$ *such that for any* $t \in \mathbb{R}_+$,

$$\begin{aligned} &\sum_{i=0}^{n} e^{\beta h_i(t)}\gamma_i(t) + \int_{-\tau}^{0} e^{-\beta s}\Upsilon_1(t,s)\mathrm{d}s + \sum_{i=0}^{n} e^{\beta h_i(t)}\zeta_i(t) + \int_{-\tau}^{0} e^{-\beta s}\Upsilon_2(t,s)\mathrm{d}s \\ &\quad \le -(1-\sqrt{k})^2\beta, \end{aligned} \tag{3.11}$$

then the trivial solution of (2.1) is exponentially stable in mean square. In particular, $\mathbb{E}|x(t,\xi)|^2$ *exponentially decays with the rate* β *for any* $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Proof Define the following functions for $t \ge 0$, $s \in [-\tau, 0]$

$$u_i(t,s) := \begin{cases} 0, & \text{if } s \in [-\tau, -h_i(t)], \\ \gamma_i(t), & \text{if } s \in (-h_i(t), 0], \end{cases}$$

$$\eta_1(t,s) := \sum_{i=1}^{n} u_i(t,s) + \int_{-\tau}^{s} \Upsilon_1(t,r)\mathrm{d}r$$

and

$$v_i(t,s) := \begin{cases} 0, & \text{if } s \in [-\tau, -h_i(t)], \\ \zeta_i(t), & \text{if } s \in (-h_i(t), 0], \end{cases}$$

$$\eta_2(t,s) := \sum_{i=1}^{n} v_i(t,s) + \int_{-\tau}^{s} \Upsilon_2(t,r)\mathrm{d}r.$$

By the properties of the Riemann-Stieltjes integrals, one has for each $i = 1, 2$ that

$$\int_{-\tau}^{0} \phi(s)\mathrm{d}\Big[\int_{-\tau}^{s} \Upsilon_i(t,r)\mathrm{d}r\Big] = \int_{-\tau}^{0} \phi(s)\Upsilon_i(t,s)\mathrm{d}s, \quad t \in \mathbb{R}_+,$$

for any $\phi(\cdot)\in C([-\tau,0];\mathbb{R}^n)$. Then for any $t\in\mathbb{R}_+$, $\phi(\cdot)\in C([-\tau,0];\mathbb{R}^n)$,

$$\int_{-\tau}^{0}\phi(s)\mathrm{d}[\eta_1(t,s)]=\sum_{i=1}^{n}\gamma_i(t)\phi(-h_i(t))+\int_{-\tau}^{0}\phi(s)\Upsilon_1(t,s)\mathrm{d}s,$$

$$\int_{-\tau}^{0}\phi(s)\mathrm{d}[\eta_2(t,s)]=\sum_{i=1}^{n}\zeta_i(t)\phi(-h_i(t))+\int_{-\tau}^{0}\phi(s)\Upsilon_2(t,s)\mathrm{d}s.$$

Therefore, (3.9), (3.10) imply that (3.4), (3.5) hold and (3.11) ensures that (3.6) holds. By the Theorem 3.1 we can obtain our desired results. The proof is completed. □

Corollary 3.5 *Assume that (3.2) holds with $0<k<1$. Let γ be a constant and non-decreasing functions $\eta_i(\cdot):[-\tau,0]\to\mathbb{R}_+$, $i=1,2$ such that*

$$\mathbb{E}(2(\varphi(0)-G(\varphi))^Tf(t,\varphi))\le\gamma|\varphi(0)|^2+\int_{-\tau}^{0}\mathbb{E}|\varphi(\theta)|^2\mathrm{d}[\eta_1(\theta)] \tag{3.12}$$

and

$$\mathbb{E}(\mathrm{trac}[g^T(t,\varphi)g(t,\varphi)])\le\int_{-\tau}^{0}\mathbb{E}|\varphi(\theta)|^2\mathrm{d}[\eta_2(\theta)], \tag{3.13}$$

for any $t\in\mathbb{R}_+$, $\varphi\in C([-\tau,0];\mathbb{R}^n)$. If

$$\gamma+\eta_1(0)-\eta_1(-\tau)+\eta_2(0)-\eta_2(-\tau)<0, \tag{3.14}$$

then the trivial solution of (2.1) is exponentially stable in mean square.

Proof By continuity and (3.14) we have

$$\gamma+e^{\beta\tau}[\eta_1(0)-\eta_1(-\tau)]+e^{\beta\tau}[\eta_2(0)-\eta_2(-\tau)]<-(1-\sqrt{k})^2\beta,$$

for some $\beta>0$, sufficiently small. Since $\eta_i(\cdot)$, $i=1,2$ is non-decreasing, it follows that

$$\begin{aligned}&\gamma+\int_{-\tau}^{0}e^{-\beta\theta}\mathrm{d}[\eta_1(-\theta)]+\int_{-\tau}^{0}e^{-\beta\theta}\mathrm{d}[\eta_2(-\theta)]\\&\le\gamma+e^{\beta\tau}[\eta_1(0)-\eta_1(-\tau)]+e^{\beta\tau}[\eta_2(0)-\eta_2(-\tau)]<-(1-\sqrt{k})^2\beta,\end{aligned}$$

which means that (3.6) holds. The proof is completed. □

From the Corollaries 3.4 and 3.5, we immediate obtain the following Corollary 3.6.

Corollary 3.6 *Assume that (3.2) holds with $0<k<1$. Let $h_i(\cdot):\mathbb{R}_+\to\mathbb{R}$, $i=0,1,2,\dots,n$ with $0:=h_0(t)\le h_1(t)\le h_2(t)\le\cdots\le h_n(t)\le\tau$, $t\in\mathbb{R}_+$, be locally bounded Borel measurable functions. Suppose that there exist constants γ_i,ζ_i, $i=0,1,2,\dots,n$ and two Borel measurable functions $\theta_i:[-\tau,0]\to\mathbb{R}_+$, $i=1,2$ such that for any $t\in\mathbb{R}_+$, $\varphi\in C([-\tau,0];\mathbb{R}^n)$,*

$$\mathbb{E}(2(\varphi(0)-G(\varphi))^Tf(t,\varphi))\le\sum_{i=0}^{n}\gamma_i\mathbb{E}|\varphi(-h_i(t))|^2+\int_{-\tau}^{0}\theta_1(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s, \tag{3.15}$$

$$\mathbb{E}(\mathrm{trac}[g^T(t,\varphi)g(t,\varphi)])\le\sum_{i=0}^{n}\zeta_i\mathbb{E}|\varphi(-h_i(t))|^2+\int_{-\tau}^{0}\theta_2(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s. \tag{3.16}$$

If

$$\sum_{i=0}^{n}\gamma_i+\int_{-\tau}^{0}\theta_1(s)\mathrm{d}s+\sum_{i=0}^{n}\zeta_i+\int_{-\tau}^{0}\theta_2(s)\mathrm{d}s\leq 0, \tag{3.17}$$

then the trivial solution of (2.1) is exponentially stable in mean square.

Corollary 3.7 *Assume that (3.2) holds with $0<k<1$. Let $h_i(\cdot):\mathbb{R}_+\to\mathbb{R}$, $i=0,1,2,\ldots,n$ with $0:=h_0(t)\leq h_1(t)\leq h_2(t)\leq\cdots\leq h_n(t)\leq\tau$, $t\in\mathbb{R}_+$, be locally bounded Borel measurable functions. Suppose that there exist constants γ_i,ζ_i, $i=0,1,2,\ldots,n$ and two Borel measurable functions $\theta_i:[-\tau,0]\to\mathbb{R}_+$, $i=1,2$ such that for any $t\in\mathbb{R}_+$, $\varphi\in C([-\tau,0];\mathbb{R}^n)$,*

$$\mathbb{E}(2\varphi^T(0)f(t,\varphi))\leq\sum_{i=0}^{n}\gamma_i\mathbb{E}|\varphi(-h_i(t))|^2+\int_{-\tau}^{0}\theta_1(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s, \tag{3.18}$$

$$\mathbb{E}|f(t,\varphi)|^2\leq\sum_{i=0}^{n}\rho_i\mathbb{E}|\varphi(-h_i(t))|^2+\int_{-\tau}^{0}\theta_2(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s, \tag{3.19}$$

$$\mathbb{E}(\mathrm{trac}[g^T(t,\varphi)g(t,\varphi)])\leq\sum_{i=0}^{n}\zeta_i\mathbb{E}|\varphi(-h_i(t))|^2+\int_{-\tau}^{0}\theta_3(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s. \tag{3.20}$$

If

$$\sum_{i=0}^{n}\gamma_i+\int_{-\tau}^{0}\theta_1(s)\mathrm{d}s+\sqrt{k}\sum_{i=0}^{n}\rho_i+\sqrt{k}\int_{-\tau}^{0}\theta_2(s)\mathrm{d}s+\sum_{i=0}^{n}\zeta_i+\int_{-\tau}^{0}\theta_3(s)\mathrm{d}s+\sqrt{k}\leq 0, \tag{3.21}$$

then the trivial solution of (2.1) is exponentially stable in mean square.

Proof For any $\varphi\in C([-\tau,0];\mathbb{R}^n)$, by the (3.2), (3.18) and (3.19) we have

$$\begin{aligned}
&\mathbb{E}(2(\varphi(0)-G(\varphi))^Tf(t,\varphi))\leq 2\mathbb{E}(\varphi^T(0)f(t,\varphi)+2|G(\varphi)||f(t,\varphi)|)\\
&\leq\sum_{i=0}^{n}\gamma_i\mathbb{E}|\varphi(-h_i(t))|^2+\int_{-\tau}^{0}\theta_1(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s+\sqrt{k}\sum_{i=0}^{n}\rho_i\mathbb{E}|\varphi(-h_i(t))|^2+\\
&\sqrt{k}\int_{-\tau}^{0}\theta_2(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s+\sqrt{k}\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|\varphi(\theta)|^2.
\end{aligned}$$

So, by the Corollary 3.6 we can obtain our desired results. □

4. Comparison with existing results and some examples

Now, we state some comparisons with existing results to illustrate the advantage of our results.

To compare the results of the Corollary 3.6 with one in [25], let us introduce another new notation $\mathscr{W}([-\tau,0];\mathbb{R}_+)$, which is the family of all Borel-measurable bounded nonnegative functions $\eta(\theta)$ defined on $[-\tau,0]$ such that $\int_{-\tau}^{0}\eta(\theta)\mathrm{d}\theta=1$. In [25], conditions (3.2), (3.15) and (3.16) are strengthened as follows: There is a constant $k\in(0,1)$ and a function $\eta\in\mathscr{W}([-\tau,0];\mathbb{R}_+)$

such that

$$|G(\varphi)|^2 \le k \int_{-\tau}^{0} \eta(\theta)|\varphi(\theta)|^2 \mathrm{d}\theta \quad \text{for all } \varphi \in C([-\tau,0];\mathbb{R}^n). \tag{4.1}$$

Moreover, there exists a function $\theta_1 \in \mathscr{W}([-\tau,0];\mathbb{R}_+)$ and two positive constants λ_1 and λ_2 such that

$$2(\varphi(0) - G(\varphi))^T f(t,\varphi) + \mathrm{trac}[g^T(t,\varphi)g(t,\varphi)] \le -\lambda_1|\varphi(0)|^2 + \lambda_2 \int_{-\tau}^{0} \theta_1(s)|\varphi(s)|^2 \mathrm{d}s, \tag{4.2}$$

for all $t \ge 0$ and $\varphi \in C([-\tau,0];\mathbb{R}^n)$. These two conditions are indeed stronger than (3.2) and (3.15) and (3.16). For example, if (4.1) holds, then for any $\phi \in C([-\tau,0];\mathbb{R}^n)$,

$$\mathbb{E}|G(\phi)|^2 \le k \int_{-\tau}^{0} \eta(\theta)|\varphi(\theta)|^2 \mathrm{d}\theta \le k \sup_{-\tau\le\theta\le 0} \mathbb{E}|\varphi(\theta)|^2 \int_{-\tau}^{0} \eta(\theta)\mathrm{d}\theta = k \sup_{-\tau\le\theta\le 0} \mathbb{E}|\varphi(\theta)|^2,$$

that is, (3.2) holds. On the other hand, if (4.2) holds, we easily show that (3.15) and (3.16) hold with

$$\sum_{i=0}^{n} \gamma_i + \sum_{i=0}^{n} \zeta_i = -\lambda_1, \;\; h_i \equiv 0, \;\; \int_{-\tau}^{0} \theta_1(s)\mathrm{d}s + \int_{-\tau}^{0} \theta_2(s)\mathrm{d}s = \lambda_2.$$

In [25], Mao proved that the trivial solution to (2.1) is exponentially stable in mean square if (4.1) and (4.2) hold and $\lambda_1 > \lambda_2$. So, the Corollary 3.6 improves the Theorem 3.1 of [25].

Besides, Mao [7] considered the exponential stability in mean square of the trivial solution to (2.1) under the conditions (3.2) and the following assumption:

$$2(\varphi(0) - G(\varphi))^T f(t,\varphi) + \mathrm{trac}[g^T(t,\varphi)g(t,\varphi)] \le -\lambda_1|\varphi(0)|^2 + \lambda_2 \sup_{-\tau\le\theta\le 0} \varphi(\theta)|^2 \mathrm{d}s, \tag{4.3}$$

for all $t \ge 0$ and $\varphi \in C([-\tau,0];\mathbb{R}^n)$. The author deduced that if

$$0 < k < \frac{1}{4} \quad \text{and} \quad \lambda_1 > \frac{\lambda_2}{(1-2\sqrt{k})^2}, \tag{4.4}$$

then the trivial solution to (2.1) is exponentially stable in mean square.

Note that if (3.15) and (3.16) hold, then (4.3) holds with $\lambda_1 = -\sum_{i=0}^{n} \gamma_i - \sum_{i=0}^{n} \zeta_i$ and $\lambda_2 = \int_{-\tau}^{0} \theta_1(s)\mathrm{d}s + \int_{-\tau}^{0} \theta_2(s)\mathrm{d}s$. Although the conditions (3.15) and (3.16) are little bit stronger than (4.3), our assumption $0 < k < 1$ and $\lambda_1 > \lambda_2$ are much sharper than (4.4). On the other hand, our Corollary 3.6 can be applied to deal with the "mixed" delay case easily.

Consider the following neutral stochastic delay differential equations of the form

$$\mathrm{d}[x(t) - \bar{G}(x(t-\tau))] = \bar{f}(t,x(t),x(t-\tau))\mathrm{d}t + \bar{g}(t,x(t),x(t-\tau))\mathrm{d}w(t), \tag{4.5}$$

on $t \ge 0$ with initial data $x_0 = \xi$. Mao [7] also studied the exponential stability of the trivial solution to (4.5) under the following assumptions:

$$|\bar{G}(x)|^2 \le k|x|^2, \quad \text{for some } k \in (0,1) \text{ and all } x \in \mathbb{R}^n, \tag{4.6}$$

and there are two positive constants λ_1 and λ_2 such that

$$2(x - \bar{G}(y))^T \bar{f}(t,x,y) + \mathrm{trace}[\bar{g}^T(t,x,y)\bar{g}(t,x,y)] \le -\lambda_1|x|^2 + \lambda_2|y|^2, \tag{4.7}$$

for all $t \geq 0$ and $x, y \in \mathbb{R}^n$. Mao [7] (see the Corollary 6.1) proved that the trivial solution to (4.5) is exponentially stable in mean square if (4.4) holds.

Note that (4.7) implies that (3.15) and (3.16) hold with $\sum_{i=0}^n \gamma_i + \sum_{i=0}^n \zeta_i = -\lambda_1$ and $\int_{-\tau}^0 \theta_1(s)\mathrm{d}s + \int_{-\tau}^0 \theta_2(s)\mathrm{d}s = \lambda_2$. So we deduce that the trivial solution to (4.5) is exponentially stable in mean square if $\lambda_1 > \lambda_2$ by our Corollary 3.6. Obviously, our assumption $\lambda_1 > \lambda_2$ is much sharper than (4.4).

Now, we present some examples to illustrate the advantage of our results.

Example 4.1 Consider the neutral stochastic differential equation

$$\mathrm{d}[x(t) - G(x_t)] = (f_0(t, x(t)) + f_1(t, x_t))\mathrm{d}t + g(t, x_t)\mathrm{d}w(t), \quad t \geq 0 \tag{4.8}$$

with initial data $x_0 = \xi \in C([-\tau, 0]; \mathbb{R}^n)$, $k \in [0, 1)$, where $f_0 : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f_1 : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^{m\times n}$, $w(t)$ is an m-dimension Brownian motion. Assume that f_0, f_1, g satisfy the local Lipschitz condition and linear growth condition and (3.2) holds with $0 \leq k < 1$. We also assume that

$$\mathbb{E}(\varphi^T(0) f_0(t, \varphi(0))) \leq \alpha \mathbb{E}|\varphi(0)|^2, \quad t \in \mathbb{R}_+, \ \varphi \in C([-\tau, 0]; \mathbb{R}^n), \tag{4.9}$$

$$\mathbb{E}|f_1(t, \varphi)| \leq \int_{-\tau}^0 \eta_1(s)\mathbb{E}|\varphi(s)|\mathrm{d}s, \quad t \in \mathbb{R}_+, \ \varphi \in C([-\tau, 0]; \mathbb{R}^n) \tag{4.10}$$

and

$$\mathbb{E}(\mathrm{trac}[g^T(t, \varphi)g(t, \varphi)]) \leq \int_{-\tau}^0 \eta_2(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s, \quad t \in \mathbb{R}_+, \ \varphi \in C([-\tau, 0]; \mathbb{R}^n). \tag{4.11}$$

Let $f(t, x) = f_0(t, x(0)) + f_1(t, x)$, $t \in \mathbb{R}_+$, $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$. Note that (4.9) and (4.10) imply that

$$\mathbb{E}(2^T\varphi(0)f(t, \varphi)) \leq (2\alpha + \int_{-\tau}^0 \eta_1(s)\mathrm{d}s)\mathbb{E}|\varphi(0)|^2 + \int_{-\tau}^0 \eta_1(s)\mathbb{E}|\varphi(s)|^2\mathrm{d}s.$$

Furthermore, we assume

$$\mathbb{E}\|f_0(t, \varphi(0))\|^2 \leq \rho \mathbb{E}|\varphi(0)|^2, \quad t \in \mathbb{R}_+, \ \varphi \in C([-\tau, 0]; \mathbb{R}^n).$$

So, by the Corollary 3.7 we deduce that the trivial solution to (4.8) is exponentially stable in mean square if

$$\alpha + \sqrt{k}\rho + \sqrt{k}\tau \int_{-\tau}^0 \eta_1^2(s)\mathrm{d}s + \int_{-\tau}^0 \eta_1(s)\mathrm{d}s + \frac{1}{2}\int_{-\tau}^0 \eta_2(s)\mathrm{d}s + \frac{1}{2}\sqrt{k} < 0. \tag{4.12}$$

If $G \equiv 0$, then Eq. (4.8) reduces to the following stochastic functional differential equation:

$$\mathrm{d}x(t) = (f_0(t, x(t)) + f_1(t, x_t))\mathrm{d}t + g(t, x_t)\mathrm{d}w(t), \quad t \geq 0. \tag{4.13}$$

Using the Razumikhin-type theorem, Mao [10] has shown that the trivial solution to (4.13) is exponentially stable in mean square if

$$\alpha + \sqrt{\tau}\Big(\int_{-\tau}^0 (\eta_1(s))^2\mathrm{d}s\Big)^{1/2} + \frac{1}{2}\tau \max_{-\tau\leq s\leq 0} \eta_2(s) < 0. \tag{4.14}$$

By the Corollary 3.7 we deduce that the trivial solution to (4.13) is exponentially stable in mean square if

$$\alpha + \int_{-\tau}^{0} \eta_1(s)\mathrm{d}s + \frac{1}{2}\int_{-\tau}^{0} \eta_2(s)\mathrm{d}s < 0. \tag{4.15}$$

Using the Hölder's inequality, we have

$$\int_{-\tau}^{0} \eta_1(s)\mathrm{d}s \le \Big(\int_{-\tau}^{0} 1\mathrm{d}s\Big)^{1/2}\Big(\int_{-\tau}^{0} (\eta_1(s))^2\mathrm{d}s\Big)^{1/2} = \sqrt{\tau}\Big(\int_{-\tau}^{0} (\eta_1(s))^2\mathrm{d}s\Big)^{1/2}.$$

On the other hand, we have

$$\int_{-\tau}^{0} \eta_2(s)\mathrm{d}s \le \tau \max_{-\tau\le s\le 0} \eta_2(s).$$

So, (4.14) is more conservative than (4.15).

Example 4.2 Consider the scalar linear time-varying stochastic differential equation with delay

$$\mathrm{d}[x(t) - G(x_t)] = (-a(t)x(t) + b(t)x(t-h_1(t)))\mathrm{d}t + c(t)x(t-h_2(t))\mathrm{d}w(t), \tag{4.16}$$

where $a(t), b(t), c(t), h_1(t), h_2(t) : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and $h_1(t), h_2(t) \in [0,\tau]$ for some $\tau > 0$, and $w(t)$ is scalar Brownian motion.

We assume (3.2) holds with $k \in (0,1)$. Let

$$f(t,\varphi) := -a(t)\varphi(0) + b(t)\varphi(-h_1(t)), \quad g(t,\varphi) := c(t)\varphi(-h_2(t)),$$

for $t \in \mathbb{R}_+$, $\varphi \in C([-\tau,0];\mathbb{R})$. Then, for all $t \in \mathbb{R}_+$, $\varphi \in C([-\tau,0];\mathbb{R})$ we have

$$\begin{aligned}2\varphi(0)f(t,\varphi) =& -2a(t)|\varphi(0)|^2 + 2b(t)\varphi(0)\varphi(-h_1(t))) \\ \le& -2a(t)|\varphi(0)|^2 + |b(t)|(\varphi^2(0) + \varphi^2(-h_1(t))),\end{aligned} \tag{4.17}$$

$$\begin{aligned}2G(\varphi)f(t,\varphi) \le& \sqrt{k}[-a(t)|\varphi(0)| + b(t)\varphi(0)\varphi(-h_1(t))]^2 + \sqrt{k}\sup_{-\tau\le s\le 0}|\varphi(s)|^2 \\ \le& 2\sqrt{k}[a^2(t)|\varphi(0)|^2 + b^2(t)\varphi^2(-h_1(t))] + \sqrt{k}\sup_{-\tau\le s\le 0}|\varphi(s)|^2\end{aligned} \tag{4.18}$$

and

$$g^2(t,\varphi) = c^2(t)\varphi^2(-h_2(t)). \tag{4.19}$$

Then, by the Corollary 3.4 we deduce that if there exists $\beta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\begin{aligned}&-2a(t) + |b(t)| + e^{\beta h_1(t)}(|b(t)| + 2\sqrt{k}b^2(t)) + \sqrt{k} + 2\sqrt{k}a^2(t) + e^{\beta h_2(t)}c^2(t) \\ &\le -(1-\sqrt{k})^2\beta,\end{aligned} \tag{4.20}$$

then the trivial solution of (4.16) is exponentially stable in mean square. In particular, $\mathbb{E}|x(t,\xi)|^2$ exponentially decays with the rate β for any $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R})$.

Note that by the continuity,

$$-2a(t) + 2|b(t)| + 2\sqrt{k}b^2(t) + \sqrt{k} + 2\sqrt{k}a^2(t) + c^2(t) \le 0, \tag{4.21}$$

ensures that (4.20) holds for some sufficiently small $\beta > 0$. Therefore, we can declare that the trivial solution of (4.16) is exponentially stable in mean square if (4.21) holds.

If $G \equiv 0$, the equation (4.16) reduces to the following stochastic differential equation

$$\mathrm{d}x(t) = (-a(t)x(t) + b(t)x(t - h_1(t)))\mathrm{d}t + c(t)x(t - h_2(t))\mathrm{d}w(t). \tag{4.22}$$

By the above discussion, we know that if

$$-a(t) + |b(t)| + \frac{1}{2}c^2(t) \leq 0, \tag{4.23}$$

then the trivial solution of (4.22) is exponentially stable in mean square.

Using the spectral property of Metzler matrices, Ngoc [26] and Ngoc and Hieu [27] proved that the trivial solution of (4.22) is exponentially stable in mean square provided

$$a(t) \geq a > 0, \quad t \geq 0; \ |b(t)| + \frac{1}{2}c^2(t) \leq ka, \ t \geq 0, \tag{4.24}$$

for some $0 < k < 1$. Obviously, (4.23) has more advantages than (4.24).

Example 4.3 For simplicity, we consider the following stochastic scalar equation

$$\mathrm{d}x(t) = \Big(- a(t)x(t) + \int_{-\tau}^{0} x(t+s)\mathrm{d}[\eta(s)]\Big)\mathrm{d}t + b(t)x(t)\mathrm{d}w(t), \tag{4.25}$$

for $t \geq 0$, where $\eta(t)$ is a function of bounded variation on $[-\tau, 0]$ and $a(t)$, $b(t)$ are continuous functions and $w(t)$ the one-dimensional Brownian motion.

Define $\gamma(t) := a(t) - \mathrm{Var}_{[-\tau,0]}\eta(\cdot) - \frac{1}{2}|b(t)|$, $t \geq 0$. Using the Laypunov functional method, it has been shown in [28] that the trivial solution of (4.25) is asymptotically mean-square stable if

$$\gamma := \inf_{t \geq 0} \gamma(t) > 0. \tag{4.26}$$

Actually we can deduce (4.26) ensures that the trivial solution of (4.25) is exponentially stable in mean square. Let

$$f(t, \varphi) := -a(t)\varphi(0) + \int_{-\tau}^{0} \varphi(s)\mathrm{d}[\eta(s)], \quad g(t, \varphi) := b(t)\varphi(0),$$

for $t \geq 0$, $\varphi \in C([-\tau, 0]; \mathbb{R})$. Define $V(s) := \mathrm{Var}_{[-\tau,s]}\eta(\cdot)$, $s \in [-\tau, 0]$. Then $V(s)$ is non-decreasing on $[-\tau, 0]$. By the properties of the Riemann-Stieltjes integral, we have

$$\Big|\int_{-\tau}^{0} \varphi(0)\varphi(s)\mathrm{d}[\eta(s)]\Big| \leq \int_{-\tau}^{0} |\varphi(0)\varphi(s)|\mathrm{d}[V(s)].$$

Thus,

$$\begin{aligned}\varphi(0)f(t, \varphi) \leq & - a(t)\varphi^2(0) + \int_{-\tau}^{0} |\varphi(0)\varphi(s)|\mathrm{d}[V(s)] \\ \leq & \Big(- a(t) + \frac{1}{2}\int_{-\tau}^{0} \mathrm{d}[V(s)]\Big)\varphi^2(0) + \frac{1}{2}\int_{-\tau}^{0} \varphi^2(s)\mathrm{d}[V(s)].\end{aligned}$$

By the Theorem 3.2, the trivial solution of (4.25) is exponentially mean-square stable if there exists $\beta > 0$ such that

$$-a(t) + \frac{1}{2}\int_{-\tau}^{0} \mathrm{d}[V(s)] + \frac{1}{2}\int_{-\tau}^{0} e^{-\beta s}\mathrm{d}[V(s)] + \frac{1}{2}b^2(t) \leq -\beta, \tag{4.27}$$

for all $t \geq 0$. It follows from (4.27)

$$-a(t) + V(0) + \frac{1}{2}b^2(t) \leq -\gamma,$$

for all $t \geq 0$. Setting $\beta \in (0, \frac{\gamma}{2})$ sufficiently small, we know that $\frac{1}{2}(e^{\beta\tau} - 1)V(0) < \frac{\gamma}{2}$, which implies for any $t \geq 0$

$$-a(t) + \frac{1}{2}V(0) + \frac{1}{2}e^{\beta\tau}V(0) + \frac{1}{2}b^2(t) \leq -\frac{\gamma}{2} \leq -\beta.$$

Since $V(\cdot)$ is non-decreasing, it follows that $\int_{-\tau}^{0} e^{-\beta s}\mathrm{d}[V(s)] \leq e^{\beta\tau}V(0)$. Therefore, we obtain for any $t \geq 0$

$$\begin{aligned} &-a(t) + \frac{1}{2}\int_{-\tau}^{0}\mathrm{d}[V(s)] + \frac{1}{2}\int_{-\tau}^{0} e^{-\beta s}\mathrm{d}[V(s)] + \frac{1}{2}b^2(t) \\ &\leq -a(t) + \frac{1}{2}V(0) + \frac{1}{2}e^{\beta\tau}V(0) + \frac{1}{2}b^2(t) \leq -\beta. \end{aligned}$$

5. Conclusion

By a novel approach, we presented some new criteria for the mean square exponential stability of neutral stochastic functional differential equations. Some known results are improved and generalized. It is important to note that the approach utilized in the present paper can be applied to study exponential stability of various stochastic dynamical systems. Some of which will be studied in the near future.

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References

[1] L. ARNOLDA, B. SCHMALFUSS. *Lyapunov's second method for random dynamical systems*. J. Differential Equations, 2001, **177**(1): 235–265.

[2] Xiaodi LI, Shiji SONG, Jianhong WU. *Exponential stability of nonlinear systems with delayed impulses and applications*. IEEE Trans. Automat. Control, 2019, **64**(10): 4024–4034.

[3] Ruifeng LIU, V. MANDREKAR. *Stochastic semilinear evolution equations: Lyapunov function, stability and ultimate boundedness*. J. Math. Anal. Appl., 1997, **212**(2): 537–553.

[4] Xuerong MAO. *Stability of stochastic differential equations with Markovian switching*. Stochastic Process. Appl., 1999, **79**(1): 45–67.

[5] J. C. TRIGEASSOUA, N. MAAMRIB, J. SABATIERA, et al. *A Lyapunov approach to the stability of fractional differential equations*. Signal Processing, 2011, **91**: 437–445.

[6] Bin LIU, H. J. MARQUEZ. *Razumikhin-type stability theorems for discrete delay systems*. Automatica J. IFAC, 2007, **43**(7): 1219–1225.

[7] Xuerong MAO. *Razumikhin-type theorems on exponential stability of neutral stochastic differential equations*. SIAM J. Math. Anal., 1997, **28**(2): 389–401.

[8] Shiguo PENG, Yun ZHANG. *Razumikhin-type theorems on p-th moment exponential stability of impulsive stochastic delay differential equations*. IEEE Trans. Automat. Control, 2010, **55**(8): 1917–1922.

[9] Quanqin ZHU. *Razumikhin-type theorem for stochastic functional differential equations with Lévy noise and Markov switching*. Internat. J. Control, 2017, **90**(8): 1703–1712.

[10] Xuerong MAO. *Razumikhin-type theorems on exponential stability of stochastic functional differential equations*. Stochastic Process. Appl., 1996, **65**(2): 233–250.

[11] Bugong XU, Yongqing LIU. *An improved Razumikhin-type theorem and its applications*. IEEE Trans. Automat. Control, 1994, **39**(4): 839–841.

[12] Xueyan ZHAO, Feiqi DENG. *New type of stability criteria for stochastic functional differential equations via Lyapunov functions*. SIAM J. Control Optim., 2014, **52**(4): 2319–2347.

[13] R. K. BRAYTON. *Bifurcation of periodic solutions in a nonlinear difference-differential equation of neutral type*. Quart. Appl. Math., 1966, **24**: 215–224.

[14] V. B. KOLMANOVSKII, A. D. MYSHKIS. *Applied Theory of Functional Differential Equations*. Kluwer Academic, Dordrecht, 1992.

[15] Y. KUANG. *Delay Differential Equations with Applications in Population Dynamics*. Academic Press, San Diego, 1993.

[16] Jing CUI, Litan YAN, Xichao SUN. *Exponential stability for neutral stochastic partial differential equations with delays and Poisson jumps*. Statist. Probab. Lett., 2011, **81**(12): 1970–1977.

[17] Guixin HU, Ke WANG. *Stability in distribution of neutral stochastic functional differential equations with Markovian switching*. J. Math. Anal. Appl., 2012, **385**(2): 757–769.

[18] Kai LIU, Xuewen XIA. *On the exponential stability in mean square of neutral stochastic functional differential equations*. Systems Control Lett., 1999, **37**(4): 207–215.

[19] Fuke WU, Xuerong MAO. *Numerical solutions of neutral stochastic functional differential equations*. SIAM J. Numer. Anal., 2008, **46**(4): 1821–1841.

[20] Lirong HUANG, Feiqi DENG. *Razumikhin-type theorems on stability of neutral stochastic functional differential equations*. IEEE Trans. Automat. Control, 2008, **53**(7): 1718–1723.

[21] S. JANKOVIĆ, J. RANDJELOVIĆ, M. JOVANOVIĆ. *Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations*. J. Math. Anal. Appl., 2009, **355**(2): 811–820.

[22] J. RANDJELOVIĆ, S. JANKOVIĆ. *On the pth moment exponential stability criteria of neutral stochastic functional differential equations*. J. Math. Anal. Appl., 2007, **326**(1): 266–280.

[23] Yinfang SONG, Yi SHEN. *New criteria on asymptotic behavior of neutral stochastic functional differential equations*. Automatica J. IFAC, 2013, **49**(2): 626-632.

[24] V. B. KOLMANOVSKII, V. R. NOSOV. *Stability of Functional Differential Equations*. Academic Press, New York, 1986.

[25] Xuerong MAO. *Exponential stability in mean square of neutral stochastic differential-functional equations*. Systems Control Lett., 1995, **26**(4): 245–251.

[26] P. H. A. NGOC. *Novel criteria for exponential stability in mean square of stochastic functional differential equations*. Proc. Amer. Math. Soc., 2020, **148**(8): 3427–3436.

[27] P. H. A. NGOC, L. T. HIEU. *A novel approach to mean square exponential stability of stochastic delay differential equations*. IEEE Trans. Automat. Control, 2021, **66**(5): 2351–2356.

[28] V. KOLMANOVSKII, A. MYSHKIS. *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic Publishers, Dordrecht, 1999.