Journal of Mathematical Research with Applications
Jul., 2022, Vol. 42, No. 4, pp. 427–440
DOI:10.3770/j.issn:2095-2651.2022.04.008
Http://jmre.dlut.edu.cn

Deviation Inequalities for a Supercritical Branching Process in a Random Environment

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Abstract Let $\{Z_n, n \geq 0\}$ be a supercritical branching process in an independent and identically distributed random environment $\xi = (\xi_n)_{n\geq 0}$. In this paper, we get some deviation inequalities for $\ln(Z_{n+n_0}/Z_{n_0})$. And some applications are given for constructing confidence intervals.

Keywords deviation inequalities; branching processes; random environment

MR(2020) Subject Classification 60J80; 60K37; 60F10

1. Introduction

The branching process in a random environment (BPRE) is a generalization of the Galton-Watson process by adding environment random variables. It was first introduced by Smith and Wilkinson [1]. The BPRE can be described in the following form. Assume that $\xi = (\xi_0, \xi_1, \ldots)$ is a sequence of independently identically distributed (i.i.d.) random variables and ξ_n stands for the random environment at time n. Each random variable ξ_n corresponds to a probability law $p(\xi_n) = \{p_n(i) : i \in \mathbb{N}\}$ on $\mathbb{N} = \{0, 1, \ldots\}$, that is $\mathbb{P}(\xi_n = i) = p_n(i)$, $i \geq 0$. Hence, $p_n(i)$ is non-negative and satisfies $\sum_{i=0}^{\infty} p_n(i) = 1$. In the random environment ξ, a branching process $\{Z_n, n \geq 0\}$ can be defined by the following equations:

$$Z_0 = 1,\ Z_{n+1} = \sum_{i=1}^{Z_n} N_{n,i} \ \text{ for all } \ n \geq 0,$$

where $N_{n,i}$ represents the number of children of the i-th individual in generation n. Conditioned on the environment ξ, the random variables $\{N_{n,i}, n \geq 0, i \geq 1\}$ are independent of each other and the random variables $\{N_{n,i}, i \geq 1\}$ have a common law $p(\xi_n)$. In the sequel, denote by $\mathbb{P}_\xi$ the conditional probability when the environment ξ is given, called the quenched law as usual. And $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$ stands for the total law of the process, called annealed law, where τ is the law of the environment ξ. The corresponding quenched and annealed expectations are represented by $\mathbb{E}_\xi$ and $\mathbb{E}$, respectively. For $n \geq 0$, denote

$$m_n := m_n(\xi) = \sum_{i=0}^{\infty} i p_n(i) \ \text{ and } \ \Pi_n = \mathbb{E}_\xi Z_n = \prod_{i=0}^{n-1} m_i.$$

By the definition of expectation, it is easy to see that $m_n = \mathbb{E}_\xi N_{n,i}$ for each $i \geq 1$. The

Received June 17, 2021; Accepted October 17, 2021
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asymptotic behavior of $\log Z_n$ is crucially affected by the associated random walk

$$S_n = \ln \Pi_n = \sum_{i=1}^{n} X_i, \quad n \geq 1.$$

For simplicity, let

$$X = X_1 = \ln m_0, \ \mu = \mathbb{E}X \ \text{ and } \ \sigma^2 = \mathbb{E}(X-\mu)^2.$$

We call μ the criticality parameter. According to the value $\mu > 0$, $\mu = 0$, or $\mu < 0$, the BPRE is respectively called supercritical, critical, or subcritical.

Because critical and subcritical BPRE's will inevitably go extinct, the study of these two cases mainly focuses on the survival probability and conditional limit theorems for the branching processes, see, for instance, Afanasyev et al. [2, 3] and Vatutin [4]. For the supercritical BPRE, a number of researches have been focused on moderate and large deviations, see Böinghoff and Kersting [5], Bansaye and Berestycki [6], Huang and Liu [7], Kozlo [8], Nakashima [9], Bansaye and Böinghoff [10], Böinghoff [11] and Wang and Liu [12].

In this paper, we assume that

$$p_0(\xi_0) = 0 \ \ \mathbb{P}\text{-a.s.} \ \text{ and } \ 0 < \sigma^2 < \infty,$$

which implies that the BPRE is supercritical, $Z_n \to \infty$ and the random walk $\{S_n, n \geq 0\}$ is non-degenerate. Under the conditions: $\mathbb{E}\frac{Z_1^p}{m_0} < \infty$ for a constant $p > 1$ and $\mathbb{E}\exp\{t(X-\mu)\} < \infty$ for some t in a neighborhood of 0, Grama et al. [13] have established the Cramér moderate deviation expansion for the BPRE, which implies in particular that for $0 \leq x = o(\sqrt{n})$ as $n \to \infty$,

$$\Big|\ln \frac{\mathbb{P}(\frac{\ln Z_n - n\mu}{\sigma\sqrt{n}} \geq x)}{1-\Phi(x)}\Big| \leq C\frac{1+x^3}{\sqrt{n}}, \tag{1.1}$$

where C is a positive constant. See also Fan et al. [14] with more general conditions. Asymptotic expansions, no matter how precise, do not diminish the need for probability inequalities valid for all n, x. For the critical Galton-Watson process, such type inequalities have been well studied by Nagaev [15]. However, there are few papers on probability inequality for the BPRE. In order to fill this gap, we try to establish some deviation inequalities for the supercritical BPRE under various moment conditions on X.

2. Main results

To shorten notations, denote

$$Z_{n_0,n} = \frac{\ln \frac{Z_{n+n_0}}{Z_{n_0}} - n\mu}{\sigma\sqrt{n}}, \quad n_0, n \in \mathbb{N}.$$

In this section, we present some deviation inequalities for $Z_{n_0,n}$, under various moment conditions on X.

When X satisfies Bernstein's condition, we have the following Bernstein type inequality for $\ln \frac{Z_{n+n_0}}{Z_{n_0}}$. We refer to De la Peña [16] for similar results, where Bernstein type inequality for martingales is established.

Theorem 2.1 *Assume that there exists a positive constant H such that*

$$\mathbb{E}(X-\mu)^k \leq \frac{1}{2}k!H^{k-2}\mathbb{E}(X-\mu)^2 \quad \text{for all} \quad k \geq 2. \tag{2.1}$$

Then for all $x > 0$,

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq 2\exp\{-\frac{x^2}{2(1+6(1+H)\frac{x}{\sigma\sqrt{n}})}\}. \tag{2.2}$$

Condition (2.1) is known as Bernstein's condition. It is known that Bernstein's condition is equivalent to Cramér's condition: that is $\mathbb{E}\exp\{t(X-\mu)\} < \infty$ for some t in a neighborhood of 0, see Fan, Grama and Liu [17].

From (2.2), it is easy to see that for $0 \leq x = o(\sqrt{n})$, the bound (2.2) behaves as $2\exp\{-x^2/2\}$; while for $x \geq \sqrt{n}$, it behaves as $2\exp\{-c\,x\sqrt{n}\}$ for a constant $c > 0$.

When X has a semi-exponential moment, the following theorem holds. This theorem can be compared to the corresponding results in Borovkov [18] for partial sums of independent random variables, Dedecker et al. [19] for Lipschitz functionals of composition of random functions, and Fan et al. [20] for martingales.

Theorem 2.2 *Assume $\mathbb{E}[(X-\mu)^2\exp\{((X-\mu)^+)^\alpha\}] < \infty$ for some $\alpha \in (0,1)$. Then for all $x > 0$,*

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq 3\exp\{-\frac{x^2}{8(u+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha})}\}, \tag{2.3}$$

where

$$u = \frac{1}{\sigma^2}\mathbb{E}[(X-\mu)^2\exp\{((X-\mu)^+)^\alpha\}].$$

For moderate $0 \leq x = o(n^{\alpha/(4-2\alpha)})$, the bound (2.3) is a sub-Gaussian bound and is of the order

$$3\exp\{-\frac{x^2}{8u}\}. \tag{2.4}$$

For large $x \geq n^{\alpha/(4-2\alpha)}$, the bound (2.3) is a semi-exponential bound and is of the order

$$3\exp\{-c\,x^\alpha n^{\alpha/2}\}, \tag{2.5}$$

where c does not depend on x and n. In particular, inequality (2.3) implies the following large deviation result: there exists a positive constant c such that for all $x > 0$,

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq nx) \leq 3\exp\{-c\,x^\alpha n^\alpha\}, \tag{2.6}$$

where c does not depend on x and n.

When X has an absolute moment of order $p \geq 2$, we have the following Fuk-Nagaev type inequality for $Z_{n_0,n}$.

Theorem 2.3 *Let $p \geq 2$. Assume that $\mathbb{E}|X-\mu|^p < \infty$. Then for all $x > 0$,*

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq \exp\{-\frac{x^2}{2V^2}\} + \frac{C_p}{n^{(p-2)/2}x^p}, \tag{2.7}$$

where

$$V^2 = (p+2)^2e^p \quad \text{and} \quad C_p = 2^{p+1}(1+\frac{2}{p})^p\mathbb{E}|\frac{X-\mu}{\sigma}|^p.$$

The last inequality implies the following large deviation result: there exists a positive constant c such that for all $x > 0$,

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq nx) \leq \exp\{-\frac{x^2}{c}n\} + \frac{c}{n^{p-1}x^p}. \tag{2.8}$$

Thus for any $x > 0$,

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq nx) = O(\frac{1}{n^{p-1}})$$

as $n \to \infty$. Note that the last equality is optimal under the stated condition.

When the random variable X has an absolute moment of order $p \in (1, 2]$, we have the following von Bahr-Esseen inequality. Notice that in the next theorem, the variance of X may not exist. Thus we consider the large deviation inequality for $\ln\frac{Z_{n+n_0}}{Z_{n_0}}$ instead of $Z_{n_0,n}$.

Theorem 2.4 *Let $p \in (1, 2]$. Assume that $\mathbb{E}|X - \mu|^p < \infty$. Then for all $x > 0$,*

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq x) \leq \frac{C_p}{x^p n^{p-1}}, \tag{2.9}$$

where

$$C_p = 2^{p+1}\mathbb{E}|X - \mu|^p + (2p)^p e^{-p}.$$

From the inequality (2.9), it is easy to see that for any $x > 0$,

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq x) = O(\frac{1}{n^{p-1}}), \quad n \to \infty. \tag{2.10}$$

The last convergence rate is the best possible under the stated condition [21].

When X is bounded from above, we obtain the following Hoeffding type inequality for $Z_{n_0,n}$. We refer to Fan, Grama and Liu [22] for similar results, where Bernstein type inequality for martingales is established.

Theorem 2.5 *Assume that there exists a positive constant H such that*

$$X \leq \mu + H.$$

Then for all $0 < x \leq \frac{\sigma\sqrt{n}}{2}$,

$$\begin{aligned}\mathbb{P}(Z_{n_0,n} \geq x) &\leq 2\exp\{-\frac{x}{2H}[(1 + \frac{2\sigma\sqrt{n}}{Hx})\ln(1 + \frac{Hx}{2\sigma\sqrt{n}}) - 1]\} \\ &\leq 2\exp\{-\frac{x^2}{8(1 + \frac{Hx}{6\sigma\sqrt{n}})}\};\end{aligned} \tag{2.11}$$

and for $x > \frac{\sigma\sqrt{n}}{2}$,

$$\begin{aligned}\mathbb{P}(Z_{n_0,n} \geq x) &\leq \exp\{-\frac{x}{2H}[(1 + \frac{2\sigma\sqrt{n}}{Hx})\ln(1 + \frac{Hx}{2\sigma\sqrt{n}}) - 1]\} + \exp\{-\frac{x\sigma\sqrt{n}}{2}\} \\ &\leq \exp\{-\frac{x^2}{8(1 + \frac{Hx}{6\sigma\sqrt{n}})}\} + \exp\{-\frac{x\sigma\sqrt{n}}{2}\}.\end{aligned} \tag{2.12}$$

When X is bounded, we obtain the following Rio type inequality for $\ln\frac{Z_{n+n_0}}{Z_{n_0}}$.

Theorem 2.6 *Assume that there exist two positive constants H_1 and H_2 such that*

$$H_1 \leq X - \mu \leq H_2.$$

Then for all $x \in [0, 2(H_2 - H_1))$,

$$\begin{aligned}\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq x) &\leq \exp\{-n\max(\psi_1(x), \psi_2(x))\} + \exp\{-\frac{1}{2}nx\} \\ &\leq (1 - \frac{x}{2(H_2 - H_1)})^{\frac{nx}{H_2 - H_1}(1 - \frac{x}{4(H_2 - H_1)})} + \exp\{-\frac{1}{2}nx\},\end{aligned}$$

where

$$\psi_1(x) = \frac{x^2}{2(H_2 - H_1)^2} + \frac{x^4}{36(H_2 - H_1)^4}, \quad x > 0,$$

and

$$\psi_2(x) = (\frac{x^2}{4(H_2 - H_1)^2} - \frac{x}{H_2 - H_1})\ln(1 - \frac{x}{2(H_2 - H_1)}), \quad x \in [0, 2(H_2 - H_1)). \tag{2.13}$$

By Rio's remark [23], for all x in $[0, 1]$, we have

$$(1 - x)^{nx(2-x)} \leq \exp\{-2nx^2\},$$

which leads to, for all $x \in [0, 2(H_2 - H_1))$,

$$(1 - \frac{x}{2(H_2 - H_1)})^{\frac{nx}{H_2 - H_1}(1 - \frac{x}{4(H_2 - H_1)})} \leq \exp\{-\frac{nx^2}{2(H_2 - H_1)^2}\}.$$

So we get the following corollary, a simple consequence of the Rio type inequality.

Corollary 2.7 *Assume the condition of Theorem 2.6. Then for all $x \geq 0$,*

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq x) \leq \exp\{-\frac{nx^2}{2(H_2 - H_1)^2}\} + \exp\{-\frac{1}{2}nx\}. \tag{2.14}$$

When $0 \leq x \leq (H_2 - H_1)^2$, the second term in the right hand side of (2.14) is less than the first one. Thus we have the following Azuma-Hoeffding inequality: for all $0 \leq x \leq (H_2 - H_1)^2$,

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq x) \leq 2\exp\{-\frac{nx^2}{2(H_2 - H_1)^2}\}. \tag{2.15}$$

3. Application to construction of confidence intervals

Deviation inequalities can be applied to establishing confidence intervals for the criticality parameter μ in terms of Z_{n_0}, Z_{n+n_0} and n, or to preview Z_{n+n_0} in terms of Z_{n_0}, μ and n.

3.1. Construction of confidence intervals for μ

When Z_{n_0}, Z_{n+n_0} and σ^2 are known, we can use Theorem 2.1 to estimate μ.

Proposition 3.1 *Assume that there exists a positive constant H such that*

$$\mathbb{E}(X - \mu)^k \leq \frac{1}{2}k!H^{k-2}\mathbb{E}(X - \mu)^2 \quad \textit{for all} \quad k \geq 3.$$

Let $\delta_n \in (0,1)$ *and*

$$\Delta_n = \frac{6(1+H)}{n}\ln(2/\delta_n) + \sqrt{\frac{36(1+H)^2}{n^2}\ln^2(2/\delta_n) + \frac{2}{n}\sigma^2\ln(2/\delta_n)}.$$

Then $[A_n, +\infty)$, *with*

$$A_n = \frac{1}{n}\ln(\frac{Z_{n_0+n}}{Z_{n_0}}) - \Delta_n,$$

is a $1-\delta_n$ *confidence interval for* μ.

Proof By Theorem 2.1, we have

$$\mathbb{P}(Z_{n_0,n} > \frac{x\sqrt{n}}{\sigma}) \le 2\exp\{-\frac{nx^2}{2(\sigma^2 + 6(1+H)x)}\}. \tag{3.1}$$

Let Δ_n be the positive solution of the following equation

$$2\exp\{-\frac{nx^2}{2(\sigma^2 + 6(1+H)x)}\} = \delta_n. \tag{3.2}$$

Then

$$\Delta_n = \frac{6(1+H)}{n}\ln(2/\delta_n) + \sqrt{\frac{36(1+H)^2}{n^2}\ln^2(2/\delta_n) + \frac{2}{n}\sigma^2\ln(2/\delta_n)}. \tag{3.3}$$

By (3.1), we obtain

$$\mathbb{P}(Z_{n_0,n} > \frac{\Delta_n\sqrt{n}}{\sigma}) = \mathbb{P}(\mu < \frac{1}{n}\ln(\frac{Z_{n_0+n}}{Z_{n_0}}) - \Delta_n) \le \delta_n.$$

The last inequality implies that

$$\mathbb{P}(\mu \ge \frac{1}{n}\ln(\frac{Z_{n_0+n}}{Z_{n_0}}) - \Delta_n) \ge 1 - \delta_n.$$

This completes the proof of Proposition 3.1. □

Similarly, when X is bounded, we have the following estimate for μ.

Proposition 3.2 *Assume that there exist positive constants* H_1 *and* H_2 *such that*

$$H_1 \le X - \mu \le H_2.$$

Let $\delta_n \in [2\exp\{-\frac{n}{2}(H_2 - H_1)^2\}, 1]$ *and*

$$\Delta_n = (H_2 - H_1)\sqrt{\frac{2}{n}\ln(2/\delta_n)}.$$

Then $[A_n, +\infty)$, *with*

$$A_n = \frac{1}{n}\ln(\frac{Z_{n_0+n}}{Z_{n_0}}) - \Delta_n,$$

is a $1-\delta_n$ *confidence interval for* μ.

Proof By Corollary 2.7, we have

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu > x) \le 2\exp\{-\frac{x^2}{2n^{-1}(H_2 - H_1)^2}\}. \tag{3.4}$$

Let Δ_n be the positive solution of the following equation

$$\delta_n = 2\exp\{-\frac{nx^2}{2(H_2 - H_1)^2}\}. \tag{3.5}$$

Then

$$\Delta_n = (H_2 - H_1)\sqrt{\frac{2}{n}\ln(2/\delta_n)}. \tag{3.6}$$

It is easy to see that

$$\mathbb{P}(\mu \geq \frac{1}{n}\ln(\frac{Z_{n_0+n}}{Z_{n_0}}) - \Delta_n) \geq 1 - \delta_n.$$

This completes the proof of Proposition 3.2. □

3.2. Construction of confidence intervals for Z_{n+n_0}

When the parameters μ and Z_{n_0} are known, we can use Theorem 2.1 to preview Z_{n+n_0}.

Proposition 3.3 *Assume that there exists a positive constant H such that*

$$\mathbb{E}(X-\mu)^k \leq \frac{1}{2}k!H^{k-2}\mathbb{E}(X-\mu)^2 \quad \textit{for all} \quad k \geq 3.$$

Let $\delta_n \in (0,1)$ and

$$\Delta_n = \frac{6(1+H)}{n}\ln(2/\delta_n) + \sqrt{\frac{36(1+H)^2}{n^2}\ln^2(2/\delta_n) + \frac{2}{n}\sigma^2\ln(2/\delta_n)}.$$

Then $[1, A_n]$, with

$$A_n = Z_{n_0}\exp\{n(\mu+\Delta_n)\},$$

is a $1-\delta_n$ confidence interval for Z_{n+n_0}.

Proof With arguments similar to that of (3.1)–(3.3), we have

$$\mathbb{P}(Z_{n_0,n} > \frac{\Delta_n\sqrt{n}}{\sigma}) = \mathbb{P}(Z_{n_0+n} > Z_{n_0}\exp\{n(\mu+\Delta_n)\}) \leq \delta_n.$$

Then

$$\mathbb{P}(Z_{n_0+n} \leq Z_{n_0}\exp\{n(\mu+\Delta_n)\}) \geq 1-\delta_n.$$

This completes the proof of Proposition 3.3. □

Similarly, when X is bounded, the parameters μ and Z_{n_0} are known, we can use Corollary 2.7 to preview Z_{n+n_0}.

Proposition 3.4 *Assume that there exist positive constants H_1 and H_2 such that*

$$H_1 \leq X - \mu \leq H_2.$$

Let $\delta_n \in [2\exp\{-\frac{n}{2}(H_2-H_1)^2\}, 1]$ and

$$\Delta_n = (H_2 - H_1)\sqrt{\frac{2}{n}\ln(2/\delta_n)}.$$

Then $[1, A_n]$, with

$$A_n = Z_{n_0}\exp\{n(\mu+\Delta_n)\},$$

is a $1-\delta_n$ confidence interval for Z_{n+n_0}.

Proof Again by arguments similar to that of (3.4)–(3.6), we get

$$\mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu > \Delta_n) = \mathbb{P}(Z_{n_0+n} > Z_{n_0}\exp\{n(\mu+\Delta_n)\}) \leq \delta_n.$$

It is easy to see that

$$\mathbb{P}(Z_{n_0+n} \le Z_{n_0} \exp\{n(\mu + \Delta_n)\}) \ge 1 - \delta_n.$$

This completes the proof of Proposition 3.4. □

4. Proofs of Theorems

Denote by

$$W_n = \frac{Z_n}{\Pi_n}, \quad n \ge 0, \tag{4.1}$$

the normalized population size. As we all know, the sequence $(W_n)_{n\ge 0}$ is a positive martingale both under the quenched law $\mathbb{P}_\xi$ and under the annealed law $\mathbb{P}$ with respect to the natural filtration

$$\mathcal{F}_0 = \sigma\{\xi\}, \ \mathcal{F}_n = \sigma\{\xi, N_{k,i}, 0 \le k \le n-1, i \ge 1\}, \quad n \ge 1.$$

According to Doob's convergence theorem and Fatou's lemma, the limit $W = \lim_{n\to\infty} W_n$ exists $\mathbb{P}$-a.s. and $\mathbb{E}W \le 1$. Evidently, formula (4.1) implies the following decomposition:

$$\ln Z_n = \sum_{i=1}^{n} X_i + \ln W_n, \tag{4.2}$$

where $X_i = \ln m_{i-1}$ $(i \ge 1)$ are i.i.d. random variables depending only on the environment ξ. Consequently, the asymptotic behavior of $\ln Z_n$ is primarily affected by the associated random walk

$$S_n = \sum_{i=1}^{n} X_i.$$

In the sequel, we denote

$$\eta_{n,i} = \frac{X_i - \mu}{\sigma\sqrt{n}}, \quad i = 1, \dots, n_0 + n,$$

and

$$W_{n_0,n} = \frac{W_{n_0+n}}{W_{n_0}}.$$

Then it is easy to see that $\sum_{i=1}^{n} \mathbb{E}\eta_{n,n_0+i}^2 = 1$ and $\sum_{i=1}^{n} \eta_{n,n_0+i}$ is a sum of i.i.d. random variables.

Proof of Theorem 2.1 We first give a proof of the inequality for $0 \le x < \frac{\sigma\sqrt{n}}{2}$. Clearly, it holds for all $x \ge 0$,

$$\mathbb{P}(Z_{n_0,n} \ge x) = \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} + \frac{\ln W_{n_0,n}}{\sigma\sqrt{n}} \ge x\Big) \le I_1 + I_2, \tag{4.3}$$

where

$$I_1 = \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} \ge (x - \frac{x^2}{\sigma\sqrt{n}})\Big) \ \text{ and } \ I_2 = \mathbb{P}(\frac{\ln W_{n_0,n}}{\sigma\sqrt{n}} \ge \frac{x^2}{\sigma\sqrt{n}}). \tag{4.4}$$

Next, we give some estimations for I_1 and I_2. Using Bernstein's inequality [24] for i.i.d. random variables, we obtain for all $0 \leq x < \frac{\sigma\sqrt{n}}{2}$,

$$I_1 \leq \exp\{-\frac{x^2(1-\frac{x}{\sigma\sqrt{n}})^2}{2(1+\frac{H}{\sigma\sqrt{n}}x(1-\frac{x}{\sigma\sqrt{n}}))}\}$$
$$\leq \exp\{-\frac{x^2}{2(1+6(1+H)\frac{x}{\sigma\sqrt{n}})}\}. \tag{4.5}$$

By Markov's inequality and the fact that $\mathbb{E}W_n = 1$, we have for all $0 \leq x < \frac{\sigma\sqrt{n}}{2}$,

$$I_2 = \mathbb{P}(W_{n_0,n} \geq \exp\{x^2\}) \leq \exp\{-x^2\}\mathbb{E}W_{n_0,n} = \exp\{-x^2\}$$
$$\leq \exp\{-\frac{x^2}{2(1+6(1+H)\frac{x}{\sigma\sqrt{n}})}\}. \tag{4.6}$$

Combining (4.3), (4.5) and (4.6), we obtain for all $0 \leq x < \frac{\sigma\sqrt{n}}{2}$,

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq 2\exp\{-\frac{x^2}{2(1+6(1+H)\frac{x}{\sigma\sqrt{n}})}\}.$$

When $x > \frac{\sigma\sqrt{n}}{2}$, it holds

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq I_3 + I_4, \tag{4.7}$$

where

$$I_3 = \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} \geq \frac{x}{2}\Big) \quad \text{and} \quad I_4 = \mathbb{P}(\frac{\ln W_{n_0,n}}{\sigma\sqrt{n}} \geq \frac{x}{2}). \tag{4.8}$$

Again by Bernstein's inequality for i.i.d. random variables, we obtain for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$I_3 \leq \exp\{-\frac{(x/2)^2}{2(1+\frac{H}{\sigma\sqrt{n}}\frac{x}{2})}\} \leq \exp\{-\frac{x^2}{2(1+6(1+H)\frac{x}{\sigma\sqrt{n}})}\}. \tag{4.9}$$

Again by Markov's inequality and the fact that $\mathbb{E}W_n = 1$, we have for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$I_4 = \mathbb{P}(W_{n_0,n} \geq \exp\{\frac{x\sigma\sqrt{n}}{2}\}) \leq \exp\{-\frac{x\sigma\sqrt{n}}{2}\}\mathbb{E}W_{n_0,n}$$
$$= \exp\{-\frac{x\sigma\sqrt{n}}{2}\} \leq \exp\{-\frac{x^2}{2(1+6(1+H)\frac{x}{\sigma\sqrt{n}})}\}. \tag{4.10}$$

Thus, for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq 2\exp\{-\frac{x^2}{2(1+6(1+H)\frac{x}{\sigma\sqrt{n}})}\}.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 From (4.3) and (4.4), using the inequality of Fan, Grama and Liu [20] for i.i.d. random variables, we get for all $0 \leq x < \frac{\sigma\sqrt{n}}{2}$,

$$I_1 = \mathbb{P}\Big(\sum_{i=1}^{n}(X_{n_0+i} - \mu) \geq \sigma\sqrt{n}(x - \frac{x^2}{\sigma\sqrt{n}})\Big)$$

$$\le 2\exp\{-\frac{(\sigma\sqrt{n}(x-\frac{x^2}{\sigma\sqrt{n}}))^2}{2(u_n+(\sigma\sqrt{n}(x-\frac{x^2}{\sigma\sqrt{n}}))^{2-\alpha})}\}$$
$$= 2\exp\{-\frac{x^2(1-\frac{x}{\sigma\sqrt{n}})^2}{2(\frac{u_n}{\sigma^2 n}+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha}(1-\frac{x}{\sigma\sqrt{n}})^{2-\alpha})}\}$$
$$\le 2\exp\{-\frac{x^2}{8(u+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha})}\}, \tag{4.11}$$

where

$$u_n = n\mathbb{E}[(X-\mu)^2\exp\{((X-\mu)^+)^\alpha\}]$$

and I_i $(i=1,2,3,4)$ are as defined in the proof of Theorem 2.1. By Markov's inequality and the fact that $\mathbb{E}W_n = 1$, we have for all $0 \le x < \frac{\sigma\sqrt{n}}{2}$,

$$I_2 \le \exp\{-x^2\}. \tag{4.12}$$

Combining (4.3), (4.11) and (4.12), we obtain for all $0 \le x < \frac{\sigma\sqrt{n}}{2}$,

$$\mathbb{P}(Z_{n_0,n} \ge x) \le 2\exp\{-\frac{x^2}{8(u+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha})}\} + \exp\{-x^2\}$$
$$\le 3\exp\{-\frac{x^2}{8(u+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha})}\},$$

which gives the desired inequality for $0 \le x < \frac{\sigma\sqrt{n}}{2}$. With arguments similar to that of (4.11) and (4.12), we get for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$I_3 = \mathbb{P}\Big(\sum_{i=1}^n (X_{n_0+i}-\mu) \ge \sigma\sqrt{n}\frac{x}{2}\Big) \le 2\exp\{-\frac{(\frac{\sigma\sqrt{n}x}{2})^2}{2(u_n+(\frac{\sigma\sqrt{n}x}{2})^{2-\alpha})}\}$$
$$= 2\exp\{-\frac{x^2}{8(\frac{u_n}{\sigma^2 n}+(\sigma\sqrt{n})^{-\alpha}(\frac{x}{2})^{2-\alpha})}\} \le 2\exp\{-\frac{x^2}{8(u+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha})}\} \tag{4.13}$$

and

$$I_4 \le \exp\{-\frac{x\sigma\sqrt{n}}{2}\}. \tag{4.14}$$

Combining (4.7), (4.13) and (4.14), we obtain for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$\mathbb{P}(Z_{n_0,n} \ge x) \le 2\exp\{-\frac{x^2}{8(u+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha})}\} + \exp\{-\frac{x\sigma\sqrt{n}}{2}\}$$
$$\le 3\exp\{-\frac{x^2}{8(u+(\sigma\sqrt{n})^{-\alpha}x^{2-\alpha})}\}.$$

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3 From (4.3) and (4.4), using Fuk-Nagaev's inequality [25] for i.i.d. random variables, we obtain for all $0 < x \le \frac{\sigma\sqrt{n}}{2}$,

$$I_1 \le \exp\{-\frac{x^2(1-\frac{x}{\sigma\sqrt{n}})^2}{\frac{1}{2}V^2}\} + \frac{2^{-(p+1)}nC_p}{(\sqrt{n}(x-\frac{x^2}{\sigma\sqrt{n}}))^p}$$
$$\le \exp\{-\frac{x^2}{2V^2}\} + \frac{C_p}{2n^{(p-2)/2}x^p}, \tag{4.15}$$

where V^2 and C_p are defined in the theorem and the definition of I_i $(i = 1, 2, 3, 4)$ are shown in the proof of Theorem 2.1. By (4.12), we have for all $0 < x \leq \frac{\sigma\sqrt{n}}{2}$,

$$I_2 \leq \exp\{-\frac{1}{2}x^2\} \leq \frac{C_p}{2n^{(p-2)/2}x^p}. \tag{4.16}$$

Combining (4.15) and (4.16), we have for all $0 < x < \frac{\sigma\sqrt{n}}{2}$,

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq \exp\{-\frac{x^2}{2V^2}\} + \frac{C_p}{n^{(p-2)/2}x^p},$$

which gives the desired inequality. When $x > \frac{\sigma\sqrt{n}}{2}$, it holds

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq I_3 + I_4.$$

Again by Fuk-Nagaev's inequality [25] for i.i.d. random variables, we obtain for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$\begin{aligned} I_3 &\leq \exp\{-\frac{(\frac{x}{2})^2}{\frac{1}{2}V^2}\} + \frac{2^{-(p+1)}nC_p}{(\sqrt{n}\frac{x}{2})^p} \\ &\leq \exp\{-\frac{x^2}{2V^2}\} + \frac{C_p}{2n^{(p-2)/2}x^p}. \end{aligned} \tag{4.17}$$

By an argument similar to that of (4.14), we have for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$I_4 \leq \exp\{-\frac{x\sigma\sqrt{n}}{2}\} \leq \frac{C_p}{2n^{(p-2)/2}x^p}. \tag{4.18}$$

Combining (4.17) and (4.18), we get for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$\mathbb{P}(Z_{n_0,n} \geq x) \leq \exp\{-\frac{x^2}{2V^2}\} + \frac{C_p}{n^{(p-2)/2}x^p}.$$

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4 Recall that $\ln Z_n = S_n + \ln W_n$. It is easy to see that for all $x > 0$,

$$\begin{aligned} \mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \geq x) &\leq \mathbb{P}(\frac{\sum_{i=1}^n (X_{n_0+i} - \mu)}{n} \geq \frac{x}{2}) + \mathbb{P}(\frac{\ln W_{n_0,n}}{n} \geq \frac{x}{2}) \\ &=: K_1 + K_2. \end{aligned} \tag{4.19}$$

Using Markov's inequality and von Bahr-Esseen's inequality [21] for i.i.d. random variables, we obtain for all $x > 0$,

$$\begin{aligned} K_1 &\leq \frac{2^p}{x^p}\mathbb{E}\Big|\sum_{i=1}^n \frac{X_{n_0+i} - \mu}{n}\Big|^p \leq \frac{2^{p+1}}{x^p}\sum_{i=1}^n \mathbb{E}|\frac{X_{n_0+i} - \mu}{n}|^p \\ &= \frac{2^{p+1}n\mathbb{E}|X - \mu|^p}{x^p n^p} = 2^{p+1}\mathbb{E}|X - \mu|^p \frac{1}{x^p n^{p-1}}. \end{aligned} \tag{4.20}$$

By an argument similar to that of (4.14), we have for all $x > 0$,

$$K_2 \leq \exp\{-\frac{1}{2}nx\} \leq (2p)^p e^{-p}\frac{1}{x^p n^{p-1}}. \tag{4.21}$$

Combining (4.20) and (4.21), we obtain the desired inequality. □

Proof of Theorem 2.5 We first give a proof for (2.11). From (4.3) and (4.4), using Hoeffding's inequality [26] for i.i.d. random variables and the relation among the bounds of Hoeffding and Bernstein [22], we get for all $0 \le x < \frac{\sigma\sqrt{n}}{2}$,

$$I_1 \le \exp\{-\frac{x(1-\frac{x}{\sigma\sqrt{n}})}{H}[(1+\frac{\sigma^2}{H\frac{\sigma}{\sqrt{n}}x(1-\frac{x}{\sigma\sqrt{n}})})\ln(1+\frac{H\frac{\sigma}{\sqrt{n}}x(1-\frac{x}{\sigma\sqrt{n}})}{\sigma^2})-1]\}$$
$$\le \exp\{-\frac{\frac{\sigma^2 n}{H^2}x^2(1-\frac{x}{\sigma\sqrt{n}})^2}{2(\frac{\sigma^2 n}{H^2}+\frac{1}{3}\frac{\sigma\sqrt{n}}{H}x(1-\frac{x}{\sigma\sqrt{n}}))}\},$$

where I_i $(i=1,2,3,4)$ are as defined in the proof of Theorem 2.1. After some calculations, we get

$$I_1 \le \exp\{-\frac{x}{2H}[(1+\frac{2\sigma\sqrt{n}}{Hx})\ln(1+\frac{Hx}{2\sigma\sqrt{n}})-1]\}$$
$$\le \exp\{-\frac{x^2}{8(1+\frac{Hx}{6\sigma\sqrt{n}})}\}. \tag{4.22}$$

By Markov's inequality, the fact that $\mathbb{E}W_n = 1$ and the following inequality

$$\frac{x}{2H}[(1+\frac{2\sigma\sqrt{n}}{Hx})\ln(1+\frac{Hx}{2\sigma\sqrt{n}})-1] \le \frac{x}{2H}\cdot\frac{Hx}{2\sigma\sqrt{n}} = \frac{x^2}{4\sigma\sqrt{n}},$$

we have for all $0 \le x < \frac{\sigma\sqrt{n}}{2}$,

$$I_2 \le \exp\{-x^2\} \le \exp\{-\frac{x^2}{4\sigma\sqrt{n}}\} \le \exp\{-\frac{x}{2H}[(1+\frac{2\sigma\sqrt{n}}{Hx})\ln(1+\frac{Hx}{2\sigma\sqrt{n}})-1]\}$$
$$\le \exp\{-\frac{x^2}{8(1+\frac{Hx}{6\sigma\sqrt{n}})}\}. \tag{4.23}$$

Combining (4.22) and (4.23), we obtain the desired inequality for all $0 \le x < \frac{\sigma\sqrt{n}}{2}$.

Next, we give a proof for (2.12). When $x > \frac{\sigma\sqrt{n}}{2}$, it holds

$$\mathbb{P}(Z_{n_0,n} \ge x) \le I_3 + I_4.$$

Again by Hoeffding's inequality [26] for i.i.d. random variables and the relation among the bounds of Hoeffding and Bernstein [22], we obtain for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$I_3 \le \exp\{-\frac{x}{2H}[(1+\frac{2\sigma\sqrt{n}}{Hx})\ln(1+\frac{Hx}{2\sigma\sqrt{n}})-1]\}$$
$$\le \exp\{-\frac{x^2}{8(1+\frac{Hx}{6\sigma\sqrt{n}})}\}. \tag{4.24}$$

By an argument similar to that of (4.10), we have for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$I_4 \le \exp\{-\frac{x\sigma\sqrt{n}}{2}\}. \tag{4.25}$$

Combining (4.24) and (4.25), we get for all $x > \frac{\sigma\sqrt{n}}{2}$,

$$\mathbb{P}(Z_{n_0,n} \ge x) \le \exp\{-\frac{x}{2H}[(1+\frac{2\sigma\sqrt{n}}{Hx})\ln(1+\frac{Hx}{2\sigma\sqrt{n}})-1]\} + \exp\{-\frac{x\sigma\sqrt{n}}{2}\}$$

$$\le \exp\{-\frac{x^2}{8(1+\frac{Hx}{6\sigma\sqrt{n}})}\} + \exp\{-\frac{x\sigma\sqrt{n}}{2}\},$$

which gives the desired inequality. □

Proof of Theorem 2.6 From (4.19), using Rio's inequality [23] for i.i.d. random variables, we get for all $x \in [0, 2(H_2 - H_1))$,

$$\begin{aligned} K_1 &= \mathbb{P}(\frac{\sum_{i=1}^n (X_{n_0+i} - \mu)}{H_2 - H_1} \ge n \cdot \frac{x}{2(H_2 - H_1)}) \\ &\le \exp\{-n \max(\psi_1(x), \psi_2(x))\} \\ &\le (1 - \frac{x}{2(H_2 - H_1)})^{\frac{nx}{H_2 - H_1}(1 - \frac{x}{4(H_2 - H_1)})}, \end{aligned} \tag{4.26}$$

where $\psi_1(x)$ and $\psi_2(x)$ are defined as (2.13) and K_i $(i = 1, 2)$ are as defined in the proof of Theorem 2.4. Next, by Markov's inequality and the fact that $\mathbb{E}W_n = 1$, we have for all $x \ge 0$,

$$K_2 \le \exp\{-\frac{1}{2}nx\}. \tag{4.27}$$

Combining (4.26) and (4.27), we obtain for all $x \in [0, 2(H_2 - H_1))$,

$$\begin{aligned} \mathbb{P}(\frac{1}{n}\ln\frac{Z_{n+n_0}}{Z_{n_0}} - \mu \ge x) &\le \exp\{-n \max(\psi_1(x), \psi_2(x))\} + \exp\{-\frac{1}{2}nx\} \\ &\le (1 - \frac{x}{2(H_2 - H_1)})^{\frac{nx}{H_2 - H_1}(1 - \frac{x}{4(H_2 - H_1)})} + \exp\{-\frac{1}{2}nx\}. \end{aligned}$$

This completes the proof of Theorem 2.6. □

Acknowledgements The author would like to thank Xiequan Fan for his helpful suggestions.

References

[1] W. L. SMITH, W. E. WILKINSON. *On branching processes in random environments.* Ann. Math. Statist., 1969, **40**: 814–827.

[2] V. I. AFANASYEV, C. BÖINGHOFF, G. KERSTING, et al. *Limit theorems for weakly subcritical branching processes in random environment.* J. Theoret. Probab., 2012, **25**(3): 703–732.

[3] V. I. AFANASYEV, C. BÖINGHOFF, G. KERSTING. *Conditional limit theorems for intermediately subcritical branching processes in random environment.* Ann. Inst. Henri Poincaré Probab. Stat., 2014, **50**(2): 602–627.

[4] V. A. VATUTIN. *A Refinement of Limit Theorems for the Critical Branching Processes in Random Environment.* Springer, Berlin, 2010.

[5] C. BÖINGHOFF, G. KERSTING. *Upper large deviations of branching processes in a random environemnt - offspring distributions with geomertrically bounded tails.* Stochastic Process. Appl., 2010, **120**(10): 2064–2077.

[6] V. BANSAYE, J. BERESTYCKI. *Large deviations for branching processes in random environment.* Markov Process. Related Fields, 2009, **15**(4): 493–524.

[7] Chunmao HUANG, Quansheng LIU. *Moments, moderate and large deviations for a branching process in a random environment.* Stochastic Process. Appl., 2012, **122**(2): 522–545.

[8] V. BANSAYE, C. BÖINGHOFF. *Upper large deviations for branching processes in random environment with heavy tails.* Electron. J. Probab., 2011, **16**(69): 1900–1933.

[9] M. V. KOZLOV. *On large deviations of branching processes in a random environment: geometric distribution of descendants.* Discrete Math. Appl., 2006, **16**(2): 155–174.

[10] M. NAKASHIMA. *Lower deviations of branching processes in random environment with geometrical offspring distributions.* Stochastic Process. Appl., 2013, **123**(9): 3560–3587.

[11] C. BÖINGHOFF. *Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions*. Stochastic Process. Appl., 2014, **124**(11): 3553–3577.

[12] Yanqing WANG, Quansheng LIU. *Limit theorems for a supercritical branching process with immigration in a random environment*. Sci. China Math., 2017, **60**(12): 2481–2502.

[13] I. GRAMA, Quansheng LIU, M. MIQUCU. *Berry-Esseen's bound and Cramér's large deviations for a supercritical branching process in a random environment*. Stochastic Process. Appl., 2017, **127**(4): 1255–1281.

[14] Xiequan FAN, Haijuan HU, Quansheng LIU. *Uniform Cramér moderate deviations and Berry-Esseen bounds for a supercritical branching process in a random environment*. Front. Math. China, 2020, **15**(5): 891–914.

[15] S. V. NAGAEV, V. VAKHTEL. *Probability inequalities for a critical Galton-Watson process*. Theory Probab. Appl., 2006, **20**(2): 225–247.

[16] V. H. DE lA PEÑA. *A general class of exponential inequalities for martingales and ratios*. Ann. Probab., 1999, **27**(1): 537–564.

[17] Xiequan FAN, I. GRAMA, Quansheng LIU. *Cramér large deviation expansions for martingales under Bernstein's condition*. Stochastic Process. Appl., 2013, **123**(11): 3919–3942.

[18] A. A. BOROVKOV. *Estimates for the distribution of sums and maxima of sums of random variables when the Cramér condition is not satisfied*. Siberian Math. J, 2000, **41**(5): 811–848.

[19] J. DEDECKER, P. DOUKHAN, Xiequan FAN. *Deviation inequalities for separately Lipschitz functionals of composition of random functions*. J. Math. Anal. Appl, 2019, **479**(2): 1549–1568.

[20] Xiequan FAN, I. GRAMA, Quansheng LIU. *Deviation inequalities for martingales with applications*. J. Math. Anal. Appl., 2017, **448**(1): 538–566.

[21] B. VON BAHR, C. G. ESSEEN. *Inequlities for the rth absolute moment of a sum of random variables,* $1 \leqq r \leqq 2$. Ann. Math. Stat., 1965, **36**(1): 299–303.

[22] Xiequan FAN, I. GRAMA, Quansheng LIU. *Hoeffding's inequality for supermartingales*. Stochastic Process. Appl., 2012, **122**(10): 3545–3559.

[23] E. RIO. *On McDiarmid's concentration inequality*. Electron. Commun. Probab., 2013, **18**(44): 1–11.

[24] S. N. BERNSTEIN. *The Theory of Probabilities*. Leningrad, Moscow, 1946.

[25] S. V. NAGAEV. *Large deviations of sums of independent random variables*. Ann. Probab., 1979, **7**(5): 745–789.

[26] W. HOEFFDING. *Probalility inequalities for sums of bounded random variables*. J. Amer. Statist. Assoc., 1963, **58**: 13–30.