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Deleting Vertices and Interlacing of A_α Eigenvalues of a Graph

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Abstract Let G be simple graph with vertex set V and edge set E. In this paper, we establish an interlacing inequality between the A_α eigenvalues of G and its subgraph $G-U$, where $U\subseteq V$. Moreover, as an application, this interlacing property can be used to deduce some A_α spectral conditions concerning the independence number, cover number, Hamiltonian property and spanning tree of a graph, respectively.

Keywords A_α eigenvalue; interlacing inequality; independence number; cover number; Hamiltonian properties; spanning tree

MR(2020) Subject Classification 05C50

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v\in V(G)$, let $d_G(v)$ and $N_G(v)$ (or $d(v)$ and $N(v)$ for short) be the degree and the set of neighbors of v, respectively. Clearly, $d_G(v)=|N_G(v)|$. We use $G-v$ to denote the graph obtained by deleting v from G. Similarly, for any subset U of $V(G)$, $G-U$ is the graph obtained by deleting the vertices in U from G. A cycle C (or a path P) in a graph G is called a Hamiltonian cycle (or a Hamiltonian path) of G if C (or P) contains all the vertices of G. A graph G is called Hamiltonian (or traceable) if G has a Hamiltonian cycle (or path). A graph G is called Hamilton-connected if for each pair of vertices in G there is a Hamiltonian path between them. A cycle C in G is said to be dominating if $V(G)-V(C)$ is independent. The connectivity, independence number and cover number of G are denoted by $\kappa(G)$, $\alpha'(G)$ and $\beta(G)$, respectively. For any undefined notions, see Bondy and Murty [1].

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The Laplacian and signless Laplacian matrices of G are defined as $L(G)=D(G)-A(G)$ and $Q(G)=D(G)+A(G)$, respectively. For any $\alpha\in[0,1]$, Nikiforov [2] defined the $A_\alpha(G)$-matrix of G as $A_\alpha(G)=\alpha D(G)+(1-\alpha)A(G)$. In particular, $A_0(G)=A(G)$, $A_{\frac{1}{2}}(G)=\frac{1}{2}Q(G)$ and $A_1(G)=D(G)$. The eigenvalues, Laplacian eigenvalues, signless Laplacian

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and A_α eigenvalues of G are the eigenvalues of $A(G)$, $L(G)$, $Q(G)$ and $A_\alpha(G)$, denoted by $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$, $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$, $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$ and $\sigma_1(G) \geq \sigma_2(G) \geq \cdots \geq \sigma_n(G)$, respectively. When only one graph G is under consideration, we sometimes write ρ_i, μ_i, q_i and σ_i instead of $\rho_i(G)$, $\mu_i(G)$, $q_i(G)$ and $\sigma_i(G)$ for $1 \leq i \leq n$, respectively. The eigenvalues of $A(G)$, $L(G)$, $Q(G)$ and $A_\alpha(G)$ have been studied extensively. We refer the reader to Brouwer and Haemers [3] and Cvetković et al. [4] for literature in this area.

The eigenvalues of an $n \times n$ real symmetric matrix M are denoted by $\lambda_i(M)$, where we always assume the eigenvalues to be arranged in nonincreasing order: $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$. The following is a classical result.

Theorem 1.1 ([3]) *Let M be an $n \times n$ real symmetric matrix. For an integer m with $1 \leq m \leq n$, let N be an $m \times m$ principal submatrix of M. Then for $i = 1, 2, \ldots, m$,*

$$\lambda_i(M) \geq \lambda_i(N) \geq \lambda_{i+n-m}(M).$$

Let G be a graph of order n, and let $H = G - U$, where $U \subset V(G)$ with $|U| = k$. Theorem 1.1 gives an interlacing property of the eigenvalues of G and the eigenvalues of $G - U$, that is $\rho_i(G) \geq \rho_i(H) \geq \rho_{i+k}(G)$ for $i = 1, 2, \ldots, n-k$. In particular, when $k = 1$, then $\rho_1(G) \geq \rho_1(H) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(H) \geq \rho_n(G)$. Theorem 1.1 does not directly apply to the (signless) Laplacian matrix (or A_α-matrix) of G and H since the principal submatrices of a (signless) Laplacian matrix (or A_α-matrix) may no longer be the (signless) Laplacian matrix (or A_α-matrix) of a subgraph. However, the following result due to Wu et al. [5] reflects an interlacing property for the Laplacian eigenvalues of G and H.

$$\mu_i(G) - \omega_1 \geq \mu_i(H) \geq \mu_{i+k}(G) - \omega_2, \quad i = 1, 2, \ldots, n-k, \tag{1.1}$$

where $\omega_1 = \min_{v \in V \setminus U} |N_G(v) \cap U|$ and $\omega_2 = \max_{v \in V \setminus U} |N_G(v) \cap U|$.

In particular, when $k = 1$, then

$$\omega_1 = \min_{v \in V \setminus U} |N_G(v) \cap U| = 0$$

and

$$\omega_2 = \max_{v \in V \setminus U} |N_G(v) \cap U| = 1.$$

(1.1) implies that

$$\mu_i(G) \geq \mu_i(H) \geq \mu_{i+1}(G) - 1, \quad i = 1, 2, \ldots, n-1,$$

which was obtained by Lotker in [6]. Moreover, for signless Laplacian eigenvalues, Wang and Belardo [7] also established the interlacing property for signless Laplacian eigenvalues of G and H when $k = 1$ as follows.

$$q_i(G) \geq q_i(H) \geq q_{i+1}(G) - 1, \quad i = 1, 2, \ldots, n-1. \tag{1.2}$$

Motivated by the above mentioned recent results, in this paper, we further study the interlacing property for $A_\alpha(G)$ eigenvalues of G and H, and establish the following result.

Theorem 1.2 *Let G be a graph of order n. For any $U \subseteq V(G)$ with $|U| = k$, let $H = G - U$. Then we have*

$$\sigma_i(G) - \alpha\omega_1 \geq \sigma_i(H) \geq \sigma_{i+k}(G) - \alpha\omega_2, \;\; i = 1, 2, \ldots, n-k, \tag{1.3}$$

where $\omega_1 = \min_{v \in V \setminus U} |N_G(v) \cap U|$ and $\omega_2 = \max_{v \in V \setminus U} |N_G(v) \cap U|$.

Remark 1.3 For $\alpha = 0$, Theorem 1.2 becomes the interlacing property for the adjacency matrix; for $k = 1$, note that $\omega_1 = \min_{v \in V \setminus U} |N_G(v) \cap U| = 0$ and $\omega_2 = \max_{v \in V \setminus U} |N_G(v) \cap U| = 1$. Then Theorem 1.2 becomes

$$\sigma_i(G) \geq \sigma_i(H) \geq \sigma_{i+1}(G) - \alpha, \;\; i = 1, 2, \ldots, n-1.$$

Recall that $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$. We then have the following interlacing property for the signless Laplacian eigenvalues of G and H, which is a direct consequence of Theorem 1.2.

Corollary 1.4 *Let G be a graph of order n. For any $U \subseteq V(G)$ with $|U| = k$, let $H = G - U$. Then we have*

$$q_i(G) - \omega_1 \geq q_i(H) \geq q_{i+k}(G) - \omega_2, \;\; i = 1, 2, \ldots, n-k, \tag{1.4}$$

where $\omega_1 = \min_{v \in V \setminus U} |N_G(v) \cap U|$ and $\omega_2 = \max_{v \in V \setminus U} |N_G(v) \cap U|$.

Clearly, (1.2) is a special case of Corollary 1.4 when $k = 1$.

The rest of this paper is organized as follows: The proof of Theorem 1.2 is presented in Section 2. In Section 3, as an application of Theorem 1.2, we use Theorem 1.2 to deduce some A_α spectral conditions concerning the independence number, cover number, Hamiltonian property and spanning tree of a graph, respectively.

2. Proof of Theorem 1.2

In order to give the proof of Theorem 1.2, the following preliminary results on real symmetric matrices are needed. The following corollary immediately follows from Theorem 1.1.

Corollary 2.1 *Let G be a graph of order n. For any $v \in V(G)$, let $W_v(G)$ be the principal submatrix of $A_\alpha(G)$ obtained by deleting the row and the column corresponding to the vertex v. Then*

$$\sigma_1(G) \geq \lambda_1(W_v(G)) \geq \sigma_2(G) \geq \cdots \geq \lambda_{n-1}(W_v(G)) \geq \sigma_n(G).$$

Lemma 2.2 ([8]) *Let $K = M + N$, where K, M and N are three Hermitian matrices of order n. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ be the eigenvalues of K and M, γ_1 and γ_n be the largest and smallest eigenvalues of N, respectively. Then for each $i = 1, 2, \ldots, n$, we have*

$$\eta_i + \gamma_n \leq \lambda_i \leq \eta_i + \gamma_1.$$

Lemma 2.3 ([8]) *Let M, N be two $n \times n$ real symmetric matrices. Then for each integer $i = 1, 2, \ldots, n$, we have*

$$\max_{r+s=n+i}\{\lambda_r(M) + \lambda_s(N)\} \leq \lambda_i(M+N) \leq \min_{r+s=1+i}\{\lambda_r(M) + \lambda_s(N)\}.$$

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2 For any $U \subseteq V(G)$ with $|U| = k$, let $W_U(G)$, $L_U(G)$ and $A_U(G)$ be the principal submatrices of $A_\alpha(G)$, $L(G)$ and $A(G)$ obtained by removing the rows and columns of $A_\alpha(G)$, $L(G)$ and $A(G)$ that correspond to the vertices in U, respectively. Note that $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G) = \alpha L(G) + A(G)$. Then we have $W_U(G) = \alpha L_U(G) + A_U(G)$ and $A_\alpha(G-U) = \alpha L(G-U) + A(G-U)$. Let $D_U(G) = W_U(G) - A_\alpha(G-U)$. Then we have $D_U(G) = W_U(G) - A_\alpha(G-U) = \alpha[L_U(G) - L(G-U)]$ since $A_U(G) = A(G-U)$. Note that $D_U(G)$ is a diagonal matrix whose diagonal entry corresponding to v is $\alpha|N_G(v) \cap U|$. Then by Theorem 1.1, Lemmas 2.2 and 2.3, for each $i = 1, 2, \ldots, n-k$, we have that

$$\begin{aligned}
\lambda_i(A_\alpha(G-U)) &= \lambda_i(W_U(G) - D_U(G)) \\
&\le \min_{r+s=1+i}\{\lambda_r(W_U(G)) + \lambda_s(-D_U(G))\} \\
&\le \lambda_i(W_U(G)) + \lambda_1(-D_U(G)) \\
&= \lambda_i(W_U(G)) - \lambda_{n-k}(D_U(G)) \\
&\le \sigma_i(G) - \min_{v\in V\setminus U} \alpha|N_G(v) \cap U| = \sigma_i(G) - \alpha\omega_1, \\
\lambda_i(A_\alpha(G-U)) &= \lambda_i(W_U(G) - D_U(G)) \\
&\ge \max_{r+s=n-k+i}\{\lambda_r(W_U(G)) + \lambda_s(-D_U(G))\} \\
&\ge \lambda_i(W_U(G)) + \lambda_{n-k}(-D_U(G)) \\
&= \lambda_i(W_U(G)) - \lambda_1(D_U(G)) \\
&\ge \sigma_{i+k}(G) - \max_{v\in V\setminus U} \alpha|N_G(v) \cap U| = \sigma_{i+k}(G) - \alpha\omega_2.
\end{aligned}$$

This completes the proof of Theorem 1.2. □

3. Applications

As an application of Theorem 1.2, in this section, we present some results concerning the $A_\alpha(G)$ eigenvalues of a graph and its structural parameters. We begin with the following result.

Theorem 3.1 *Let G be a connected graph of order n with cover number $\beta(G)$. Then*

(1) For $i = 1, 2, \ldots, n-\beta(G)$, we have $\sigma_i(G) \ge \alpha$;

(2) For $i = \beta(G)+1, \ldots, n$, we have $\sigma_i(G) \le \alpha\beta(G)$.

Proof Let U be a minimum vertex cover of G with $|U| = \beta(G) = k$. Note that $\sigma_i(G-U) = 0$ for $i = 1, 2, \ldots, n-k$, $\omega_1 = \min_{v\in V\setminus U} |N_G(v) \cap U| \ge 1$ and $\omega_2 = \max_{v\in V\setminus U} |N_G(v) \cap U| \le k$. Then Theorem 1.2 implies that

$$\sigma_i(G) \ge \alpha\omega_1 = \alpha \min_{v\in V\setminus U} |N_G(v) \cap U| \ge \alpha \text{ and}$$
$$\sigma_{i+k}(G) \le \alpha\omega_2 = \alpha \max_{v\in V\setminus U} |N_G(v) \cap U| \le k\alpha.$$

This completes the proof. □

Recall that $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$. The following corollary immediately follows from Theorem 3.1.

Corollary 3.2 *Let G be a connected graph of order n with cover number $\beta(G)$. Then*

(1) For $i = 1, 2, \ldots, n-\beta(G)$, we have $q_i(G) \geq 1$;

(2) For $i = \beta(G)+1, \ldots, n$, we have $q_i(G) \leq \beta(G)$.

Recall that $A_0(G) = A(G)$ when $\alpha = 0$. In what follows, we consider the case $0 < \alpha \leq 1$. We begin with the following relation between the A_α eigenvalues of G and its independence number.

Theorem 3.3 *Let G be a graph of order n with independence number α'. Then*

$$\alpha' + \frac{1}{\alpha}\sigma_{n-\alpha'+1} \leq n.$$

Proof Suppose that $I = \{v_1, v_2, \ldots, v_{\alpha'}\}$ be an independent set of G and $N = V(G) \setminus I = \{u_1, u_2, \ldots, u_s\}$, where $s = n - \alpha'$. Then by Theorem 1.2, we have

$$\begin{aligned} \sigma_s(G-u_1) &\geq \sigma_{s+1}(G) - \alpha, \\ \sigma_{s-1}(G-u_1-u_2) &\geq \sigma_s(G-u_1) - \alpha, \\ \sigma_{s-2}(G-u_1-u_2-u_3) &\geq \sigma_{s-1}(G-u_1-u_2) - \alpha, \\ &\vdots \\ \sigma_{s-(s-1)}(G-u_1-u_2-\cdots-u_s) &\geq \sigma_{s-(s-2)}(G-u_1-u_2-\cdots-u_{s-1}) - \alpha. \end{aligned}$$

Summing up the inequalities above, we then have

$$\sigma_1(G-u_1-u_2-\cdots-u_s) \geq \sigma_{s+1}(G) - s\alpha = \sigma_{s+1}(G) - (n-\alpha')\alpha.$$

Note that $\sigma_1(G-u_1-u_2-\cdots-u_s) = 0$ since there is no edge in the graph $G-u_1-u_2-\cdots-u_s$. Thus $\sigma_{s+1}(G) \leq (n-\alpha')\alpha$. Namely, $\alpha' + \frac{1}{\alpha}\sigma_{n-\alpha'+1} \leq n$, as desired. □

Similarly, we have the following.

Corollary 3.4 *Let G be a graph of order n with independence number α'. Then*

$$\alpha' + q_{n-\alpha'+1} \leq n.$$

We now continue to use Theorem 3.3 to establish the following results on the Hamiltonian properties and spanning trees of graphs, respectively.

Theorem 3.5 *Let G be a graph of order n with connectivity κ.*

(1) If $n \leq \kappa + \frac{1}{\alpha}\sigma_{n-\kappa}$, then G is Hamiltonian;

(2) If $n \leq \kappa + \frac{1}{\alpha}\sigma_{n-\kappa-1} + 1$, then G is traceable;

(3) If $n \leq \kappa + \frac{1}{\alpha}\sigma_{n-\kappa+1} - 1$, then G is Hamilton-connected.

Similarly, we have the following on the signless Laplacian eigenvalues of a graph.

Corollary 3.6 *Let G be a graph of order n with connectivity κ.*

(1) If $n \leq \kappa + q_{n-\kappa}$, then G is Hamiltonian;

(2) If $n \leq \kappa + q_{n-\kappa-1} + 1$, then G is traceable;

(3) If $n \leq \kappa + q_{n-\kappa+1} - 1$, then G is Hamilton-connected.

We now give an example to illustrate that our results in Theorem 3.5 are best possible.

Example 3.7 Recall that $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$. We now consider $\alpha = \frac{1}{2}$ in the following.

(i) Let G be the non-Hamiltonian complete bipartite graph $K_{r,r+1}$ ($r \geq 2$). Notice that $\kappa(G) = r$ and $\frac{1}{\alpha}\sigma_{n-\kappa} = q_{n-\kappa} = \mu_{n-\kappa} = r$. Thus $n-1 = 2r \leq \kappa + \frac{1}{\alpha}\sigma_{n-\kappa}$. Therefore (1) in Theorem 3.5 is best possible.

(ii) Let G be the non-traceable complete bipartite graph $K_{r,r+2}$ ($r \geq 1$). Notice that $\kappa(G) = r$ and $\frac{1}{\alpha}\sigma_{n-\kappa-1} = q_{n-\kappa-1} = \mu_{n-\kappa-1} = r$. Thus $n-1 = 2r+1 \leq \kappa + \frac{1}{\alpha}\sigma_{n-\kappa-1} + 1$. Therefore (2) in Theorem 3.5 is best possible.

(iii) Let G be the non-Hamiltonian complete bipartite graph $K_{r,r}$ ($r \geq 3$). Notice that $\kappa(G) = r$ and $\frac{1}{\alpha}\sigma_{n-\kappa+1} = q_{n-\kappa+1} = \mu_{n-\kappa+1} = r$. Thus $n-1 = 2r-1 \leq \kappa + \frac{1}{\alpha}\sigma_{n-\kappa+1} - 1$. Therefore (3) in Theorem 3.5 is best possible.

Theorem 3.8 *Let G be a 2-connected triangle-free graph of order n with connectivity κ. If $n \leq 2\kappa + \frac{1}{\alpha}\sigma_{n-2\kappa+2} - 2$, then every longest cycle in G is dominating.*

Theorem 3.9 *Let G be a graph of order n with connectivity κ, where $n \geq \kappa + k$ and $k \geq 2$ is an integer. If $n \leq (\kappa + k - 1) + \frac{1}{\alpha}\sigma_{n-(\kappa+k-1)}$, then G has a spanning tree with at most k leaves.*

Similarly, we have the following corollaries.

Corollary 3.10 *Let G be a 2-connected triangle-free graph of order n with connectivity κ. If $n \leq 2\kappa + q_{n-2\kappa+2} - 2$, then every longest cycle in G is dominating.*

Corollary 3.11 *Let G be a graph of order n with connectivity κ, where $n \geq \kappa + k$ and $k \geq 2$ is an integer. If $n \leq (\kappa + k - 1) + q_{n-(\kappa+k-1)}$, then G has a spanning tree with at most k leaves.*

In order to prove Theorems 3.5, 3.8 and 3.9, we need the following results.

Theorem 3.12 ([9]) *Let G be a graph of order n with connectivity κ and independence number α'.*

(1) If $\alpha' \leq \kappa$, then G is Hamiltonian;

(2) If $\alpha' \leq \kappa + 1$, then G is traceable;

(3) If $\alpha' \leq \kappa - 1$, then G is Hamilton-connected.

Theorem 3.13 ([10]) *Let G be a 2-connected triangle-free graph of order n with connectivity κ and independence number α'. If $\alpha' \leq 2\kappa - 2$, then every longest cycle in G is dominating.*

Theorem 3.14 ([11]) *Let G be a graph of order n with independence number α' and $k \geq 2$ be an integer. If $\alpha' \leq k + \kappa - 1$, then G has a spanning tree with at most k leaves.*

Now we give the proofs of Theorems 3.5, 3.8 and 3.9, respectively.

Proof of Theorem 3.5 Let G be a graph satisfying the conditions in Theorem 3.5.

(1) If $\alpha' \leq \kappa$, then Theorem 3.12 (1) implies that G is Hamiltonian. So we assume that $\alpha' \geq \kappa + 1$, then there exists an independent set S in G such that $|S| = \kappa + 1$. Applying Theorem

3.3, we have that $\kappa + 1 + \frac{1}{\alpha}\sigma_{n-\kappa} = \kappa + 1 + \frac{1}{\alpha}\sigma_{n-(\kappa+1)+1} = |S| + \frac{1}{\alpha}\sigma_{n-|S|+1} \leq n$, which is a contradiction.

(2) If $\alpha' \leq \kappa + 1$, then Theorem 3.12 (2) implies that G is traceable. So we assume that $\alpha' \geq \kappa+2$, then there exists an independent set S in G such that $|S| = \kappa+2$. Applying Theorem 3.3, we have that $\kappa + 2 + \frac{1}{\alpha}\sigma_{n-\kappa-1} = \kappa + 2 + \frac{1}{\alpha}\sigma_{n-(\kappa+2)+1} = |S| + \frac{1}{\alpha}\sigma_{n-|S|+1} \leq n$, which is a contradiction.

(3) If $\alpha' \leq \kappa - 1$, then Theorem 3.12 (3) implies that G is Hamilton-connected. So we assume that $\alpha' \geq \kappa$, then there exists an independent set S in G such that $|S| = \kappa$. Applying Theorem 3.3, we have that $\kappa + \frac{1}{\alpha}\sigma_{n-\kappa+1} = |S| + \frac{1}{\alpha}\sigma_{n-|S|+1} \leq n$, which is a contradiction. □

Proof of Theorem 3.8 Let G be a graph satisfying the conditions in Theorem 3.8. If $\alpha' \leq 2\kappa-2$, then Theorem 3.13 implies that every longest cycle in G is dominating. So we assume that $\alpha' \geq 2\kappa - 1$, then there exists an independent set S in G such that $|S| = 2\kappa - 1$. Applying Theorem 3.3, we have that $2\kappa - 1 + \frac{1}{\alpha}\sigma_{n-2\kappa+2} = 2\kappa - 1 + \frac{1}{\alpha}\sigma_{n-(2\kappa-1)+1} = |S| + \frac{1}{\alpha}\sigma_{n-|S|+1} \leq n$, which is a contradiction. □

Proof of Theorem 3.9 Let G be a graph satisfying the conditions in Theorem 3.9. If $\alpha' \leq \kappa + k - 1$, then Theorem 3.14 implies that G has a spanning tree with at most k leaves. So we assume that $\alpha' \geq \kappa + k$, then there exists an independent set S in G such that $|S| = \kappa + k$. Applying Theorem 3.3, we have that $\kappa + k + \frac{1}{\alpha}\sigma_{n-(\kappa+k-1)} = \kappa + k + \frac{1}{\alpha}\sigma_{n-(\kappa+k)+1} = |S| + \frac{1}{\alpha}\sigma_{n-|S|+1} \leq n$, which is a contradiction. □

4. Concluding remarks

Let G be a graph of order n, and let $H = G - U$, where $U \subset V(G)$ with $|U| = k$. In this paper, we deduce an interlacing inequality between the A_α eigenvalues of G and H, which we refer to as the vertex version of the interlacing property. In fact, the following results due to Mohar [12] and Chen et al. [13] reflect an edge version of the interlacing property.

Theorem 4.1 ([12,13]) *Let G be a graph of order n, and let $G' = G - e$, where $e \in E(G)$. Then*

$$\mu_i(G) \geq \mu_i(G') \geq \mu_{i+1}(G), \text{ for each } i = 1, 2, \ldots, (n-1).$$
$$\theta_{i-1}(G) \geq \theta_i(G') \geq \theta_{i+1}(G), \text{ for each } i = 1, 2, \ldots, (n-1),$$

where $\theta_0(G) = 2$, $\theta_{n+1}(G) = 0$ and $\theta_1(G) \geq \theta_2(G) \geq \cdots \geq \theta_{n-1}(G) \geq \theta_n(G) = 0$ are the normalized Laplacian eigenvalues of G.

More generally, for any graph G of order n and $F \subset E(G)$ with $|F| = k$, we would like to know there is an edge version of the interlacing property on A_α eigenvalues of G and its subgraph $G - F$.

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