Journal of Mathematical Research with Applications
Sept., 2022, Vol. 42, No. 5, pp. 463–475
DOI:10.3770/j.issn:2095-2651.2022.05.003
Http://jmre.dlut.edu.cn

Minimum and Maximum Resistance Status of Unicyclic Graphs

Meiqun CHENG, Bo ZHOU*
School of Mathematical Sciences, South China Normal University, Guangdong 510631*, P. R. China*

Abstract The resistance status of a vertex of a connected graph is the sum of the resistance distance between this vertex and any other vertices of the graph. The minimum (maximum, resp.) resistance status of a connected graph is the minimum (maximum, resp.) resistance status of all vertices of the graph. In this paper, we determine the extremal values and corresponding extremal graphs for the minimum (maximum, resp.) resistance status over all unicyclic graphs of fixed order, and we also discuss the dependence of the minimum (maximum, resp.) resistance status on the girth of unicyclic graphs.

Keywords minimum resistance status; maximum resistance status; resistance distance; unicyclic graph; extremal graph

MR(2020) Subject Classification 05C12; 05C35; 05C90

1. Introduction

We consider simple and undirected graphs. Let G be a connected graph of order n with vertex set $V(G)$. The distance between vertices u and v in G, denoted by $d_G(u,v)$, is the length of a shortest path connecting u and v in G. The status [1,2] (the transmission [3,4], the distance [5], or the total distance [6]) of a vertex u in G is defined as

$$s_G(u)=\sum_{v\in V(G)} d_G(u,v).$$

The minimum status of G, denoted by $s(G)$, is defined as [1,7,8]

$$s(G)=\min\{s_G(u): u\in V(G)\}.$$

Similarly, the maximum status of G, denoted by $S(G)$, is defined as [1,4,9]

$$S(G)=\max\{s_G(u): u\in V(G)\}.$$

In the case of a communication network, $s_G(u)$ may be interpreted as the contribution of the vertex u to the communicational cost of the network represented by the graph G (see [4]). The minimum status may be used as a centrality measure [10], and it received much attention [7,8,11–18]. The maximum status also received due attention [13–16]. We mention that, for

Received July 30, 2021; Accepted February 18, 2022
Supported by the National Natural Science Foundation of China (Grant No. 12071158).
* Corresponding author
E-mail address: 1677098467@qq.com (Meiqun CHENG); zhoubo@scnu.edu.cn (Bo ZHOU)

a connected graph G of order $n \geq 2$, $\frac{1}{n-1}s(G)$ and $\frac{1}{n-1}S(G)$ are called the proximity and the remoteness of G, respectively, see, e.g., [13–16]. Related works may also be found in [19, 20].

Let G be a connected graph. For $u, v \in V(G)$, the resistance distance between u and v in G, denoted by $r_G(u, v)$, is defined to be the effective resistance between the corresponding nodes of the electrical network obtained so that its nodes correspond to the vertices of G and each edge of G is replaced by a resistor of unit resistance, which is computed by the methods of the theory of resistive electrical networks based on Ohm's and Kirchhoff's laws. Work on the resistance distance dated back to Foster [21], who derived a simple sum rule for effective resistances between nearest neighbor pairs of vertices. Klein and Randić [22] identified the effective resistance between pairs of vertices as a distance function (or metric) on a graph. There are various interpretations of the resistance distance based on notions of matrix, flows, least square estimation, determinants, spanning forests, and a random walk [23, 24]. It is natural to investigate the counterpart of the minimum (maximum, resp.) status using resistance distance. This is what we do in this paper. The resistance status of a vertex u in G is defined as

$$r_G(u) = \sum_{v \in V(G)} r_G(u, v).$$

The minimum resistance status of G, denoted by $r(G)$, is defined as

$$r(G) = \min\{r_G(u) : u \in V(G)\}.$$

The maximum resistance status of G, denoted by $R(G)$, is defined as

$$R(G) = \max\{r_G(u) : u \in V(G)\}.$$

A graph G is resistance-regular [25] if $r(G) = R(G)$. It is well-known [22] that $r_G(u, v) \leq d_G(u, v)$ with equality if and only if there is a unique path connecting u and v in G. So, for trees, the resistance distance coincides to the ordinary distance. Thus, the ordinary minimum (maximum, resp.) status and the minimum (maximum, resp.) resistance status are equal for trees. So we study the minimum (maximum, resp.) resistance status of the connected graphs with cycles. These parameters based on resistance distance may serve as alternate measures for complex networks.

The rest of the paper is organized as follows. In the next section, we introduce some useful lemmas, which are used to prove the results of the subsequent sections. We give lower and upper bounds for minimum resistance status (maximum resistance status, resp.) of the unicyclic graphs and characterize the extremal graphs in Section 3 (Section 4, resp.).

2. Preliminaries

A vertex of degree one in a graph is called a leaf or a pendant vertex. Denote by S_n and P_n the star and the path of order n, respectively.

Let C_n be the cycle on $n \geq 3$ vertices, whose vertices are labeled consecutively by $v_1, \ldots, v_n$.

For $v_i, v_j \in V(C_n)$ with $i < j$, it is easy to see that

$$r_{C_n}(v_i, v_j) = \frac{1}{\frac{1}{j-i} + \frac{1}{n-(j-i)}} = \frac{(j-i)(n-(j-i))}{n},$$

which is strictly increasing for $j - i \le \lfloor \frac{n}{2} \rfloor$. Evidently, C_n is resistance-regular, as for $1 \le i \le n$, we have

$$r_{C_n}(v_i) = r_{C_n}(v_1) = \sum_{j=2}^{n} r_{C_n}(v_1, v_j) = \frac{n^2-1}{6}.$$

For a subset E_1 of edges of a graph G, $G - E_1$ denotes the subgraph obtained from G by deleting all the edges in E_1, and in particular, we write $G - uv$ for $G - \{uv\}$. For a subset E_2 of unordered vertex pairs of distinct vertices of G, if each element of E_2 is not an edge of G, then $G + E_2$ denotes the graph obtained from G by adding all elements of E_2 as edges, and in particular, we write $G + uv$ for $G + \{uv\}$. For convenience, for a tree G, we use $|G|$ for $|V(G)|$.

Let G be a unicyclic graph and C the unique cycle of G. Then $G - E(C)$ consists of $|C|$ components, each containing a vertex of C. If T is such a component containing vertex v of C, then we say T is a branch of G at v. If G is a unicyclic graph on n vertices with girth g, then we always denote by $C_g = v_1 \dots v_g v_1$ the cycle of G and denote by T_i the branch of G at v_i with $|T_i| = t_i$ for $i = 1, \dots, g$.

Let T be a nontrivial tree. For $u \in V(T)$, denote by $w_T(u)$ the maximum number of vertices of the components of $T - u$.

Let $h(k) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Lemma 2.1 ([12]) *Let T be a tree on $n \ge 2$ vertices and x a vertex of T. Then $s_T(x) = s(T)$ if and only if $w_T(x) \le \frac{n}{2}$.*

Lemma 2.2 ([11,13]) *For a tree T on n vertices with $v \in V(T)$,*

$$s_T(v) \le S(T) \le S(P_n) = h(n-1)$$

with equalities if and only if T is a path and v is a terminal vertex.

3. Minimum resistance status

First, we consider the minimum resistance status of unicyclic graphs.

Lemma 3.1 *Among all unicyclic graphs on n vertices with fixed girth g, the minimum resistance status is minimized by a unicyclic graph G, with the following property that any branch of G is a star with its center on the cycle.*

Proof Let C be the unique cycle of G. Assume that T is any branch of G at v on C. Suppose that T is not a star with center v. Let $G' = G - E(T) + \{vw : w \in V(T) \setminus \{v\}\}$. It suffices to show that $r(G) > r(G')$.

Let $V = V(T) \setminus \{v\}$ and $V' = V(G) \setminus V$. Obviously, $|V| = |T| - 1 \ge 2$. Assume that

$r(G) = r_G(u)$ with $u \in V(G)$. If $u \in V$, then

$$\begin{aligned} r_G(u) - r_{G'}(v) &= \sum_{w \in V \setminus \{u\}} d_G(u, w) + \sum_{w \in V'} (d_G(u, v) + r_G(v, w)) - \Big(|V| + \sum_{w \in V'} r_G(v, w)\Big) \\ &\geq \sum_{w \in V \setminus \{u\}} 1 + \sum_{w \in V'} 1 - |V| = |V| - 1 + |V'| - |V| = |V'| - 1 > 0, \end{aligned}$$

so $r_G(u) > r_{G'}(v)$, implying that

$$r(G) = r_G(u) > r_{G'}(v) \geq r(G').$$

If $u \in V'$, then

$$r_G(u) - r_{G'}(u) = \sum_{w \in V} d_G(v, w) - |V| > 0,$$

so $r(G) = r_G(u) > r_{G'}(u) \geq r(G')$. $\square$

Denote by $S_{n,g}$ the unicyclic graph obtained from a g-vertex cycle C_g by adding $n-g$ leaves adjacent to a common vertex. In particular, $S_{n,n}$ is just C_n.

Theorem 3.2 *Let G be a unicyclic graph on n vertices with fixed girth g. Then*

$$r(G) \geq n + \frac{g^2}{6} - g - \frac{1}{6}$$

with equality if and only if $G \cong S_{n,g}$.

Proof It is trivial if $g = n$. Suppose that $g < n$. Suppose that G is a unicyclic graphs on n vertices with girth g that minimizes the minimum resistance status.

By Lemma 3.1, each branch of G is a star with center on the cycle of G.

Assume that $r(G) = r_G(x)$. We claim that x lies on the cycle C_g of G. Otherwise, x is a leaf of G in some branch T at v on C_g. Let $V = V(T) \setminus \{v\}$ and $V' = V(G) \setminus V$. Then

$$r_G(x) - r_G(v) = \sum_{w \in V \setminus \{x\}} 2 + \sum_{w \in V'} 1 - |V| = 2(|V| - 1) + |V'| - |V| = n - 2 > 0,$$

so $r_G(x) > r_G(v)$, a contradiction to the choice of x. Thus, x lies on C_g. Assume that $x = v_1$. Then

$$\begin{aligned} r(G) &= \sum_{w \in V(G) \setminus V(C_g)} r_G(v_1, w) + r(C_g) \\ &= \sum_{j=1}^{g} \sum_{w \in V(T_j) \setminus \{v_j\}} (r_G(v_1, v_j) + 1) + r(C_g) \\ &= \sum_{j=1}^{g} (t_j - 1)(r_G(v_1, v_j) + 1) + r(C_g) \\ &= \sum_{j=2}^{g} (t_j - 1) r_G(v_1, v_j) + n - g + r(C_g), \end{aligned}$$

which is minimized if and only if $t_2 = \cdots = t_g = 1$. Thus G has exactly one nontrivial branch that is a star with center on the cycle, and

$$r(G) = n - g + r(C_g) = n - g + \frac{g^2 - 1}{6} = n + \frac{g^2}{6} - g - \frac{1}{6},$$

as desired. □

Theorem 3.3 *Let G be a unicyclic graph on n vertices, where $n \geq 3$. Then*

$$r(G) \geq n - \frac{5}{3}$$

with equality if and only if $G \cong S_{n,3}$.

Proof Let g be the girth of G. By Theorem 3.2, we have

$$r(G) \geq f(g) := n + \frac{g^2}{6} - g - \frac{1}{6} = n + \frac{(g-3)^2 - 10}{6}$$

with equality if and only if $G \cong S_{n,g}$.

Since $f(g)$ is strictly increasing for $g \geq 3$, we have

$$r(G) \geq f(g) \geq f(3) = n - \frac{5}{3}$$

with equalities if and only if $G \cong S_{n,g}$ with $g = 3$, i.e., $G \cong S_{n,3}$. □

Lemma 3.4 *Among all unicyclic graphs on n vertices with fixed girth g, the minimum resistance status is maximized by a unicyclic graph G, with the following property that each branch is a path with a terminal vertex on the cycle.*

Proof Let C be the unique cycle of G. Assume that T is any branch of G at v on C. Suppose that T is not a path with v being a terminal vertex. Let G' be the unicyclic graph obtained from G by replacing T by a path T' such that v is a terminal vertex and $V(T') = V(T)$. It suffices to show that $r(G) < r(G')$.

Let $V = V(T)$ and $V' = V(G) \setminus V$. Assume that $r(G') = r_{G'}(u)$ with $u \in V(G')$.

If $u \in V' \cup \{v\}$, then, as by comparing the structure of T and T' using Lemma 2.2, we have

$$s_T(v) < h(|T| - 1) = s_{T'}(v),$$

so

$$r_G(u) - r_{G'}(u) = \sum_{w \in V} d_G(v, w) - \sum_{w \in V} d_{G'}(v, w) = s_T(v) - s_{T'}(v) < 0,$$

implying that $r(G) \leq r_G(u) < r_{G'}(u) = r(G')$.

Next, suppose that $u \in V \setminus \{v\}$. Then

$$\begin{aligned} r_{G'}(u) &= \sum_{w \in V'} (d_{G'}(u, v) + r_{G'}(v, w)) + \sum_{w \in V} d_{G'}(u, w) \\ &= |V'| d_{G'}(u, v) + \sum_{w \in V'} r_G(v, w) + s_{T'}(u) \\ &\geq |V'| d_{G'}(u, v) + \sum_{w \in V'} r_G(v, w) + s(T'). \end{aligned} \tag{3.1}$$

Assume that $s(T) = s_T(x)$ with $x \in V(T)$. Then

$$\begin{aligned} r_G(x) &= \sum_{w \in V'} (d_G(x,v) + r_G(v,w)) + \sum_{w \in V} d_G(x,w) \\ &= |V'| d_G(x,v) + \sum_{w \in V'} r_G(v,w) + s(T). \end{aligned} \tag{3.2}$$

Case 1. $d_G(x,v) \le d_{G'}(u,v)$.

Note that $s(T) \le s(T')$ with equality if and only if T is also a path [13]. So, from (3.1) and (3.2), we have $r_{G'}(u) \ge r_G(x)$. If $r_{G'}(u) > r_G(x)$, then $r(G') > r(G)$.

Suppose that $r_{G'}(u) = r_G(x)$. Then $d_G(x,v) = d_{G'}(u,v)$, $s_{T'}(u) = s(T')$, and $s(T') = s(T)$. So T is a path but v is not its terminal vertex, and by Lemma 2.1, $\omega_T(x) = \lfloor \frac{|T|}{2} \rfloor = \omega_{T'}(u)$. If $|T|$ is odd, then $d_{G'}(u,v) = \frac{|T|-1}{2} > d_G(x,v)$, which is a contradiction. So $|T|$ is even. Then $d_{G'}(u,v) = \frac{|T|}{2} - 1, \frac{|T|}{2}$. So $d_G(x,v) = \frac{|T|}{2} - 1, \frac{|T|}{2}$. As v is not a terminal vertex of the path T, we have $d_G(x,v) \ne \frac{|T|}{2}$, so $d_G(x,v) = \frac{|T|}{2} - 1$. Let x' be the neighbor of x in the path from v to x in T. By Lemma 2.1, $s(T) = s_T(x')$. As above, we have

$$r_G(x') = |V'| d_G(x',v) + \sum_{w \in V'} r_G(v,w) + s(T) < r_G(x).$$

So $r_{G'}(u) = r_G(x) > r_G(x')$, implying that $r(G') > r(G)$.

Case 2. $d_G(x,v) > d_{G'}(u,v)$.

Choose a vertex z in the path connecting v and x in T such that $d_G(z,v) = d_{G'}(u,v) = k > 0$. Then

$$\begin{aligned} r_G(z) &= \sum_{w \in V'} (d_G(z,v) + r_G(v,w)) + \sum_{w \in V} d_G(z,w) \\ &= |V'| d_G(z,v) + \sum_{w \in V'} r_G(v,w) + s_T(z). \end{aligned}$$

Denote by z^* the neighbor of z in the path connecting v and z in T. Let H and H^* be the components of $T - zz^*$ containing z and z^*, respectively. Let $|H| = a$ and $|H^*| = b$. Note that $k \le b$ and $a + b = |T|$.

Let B be the component of $T - x$ containing z. Since $s_T(x) = s(T)$, we have $|B| \le \frac{|T|}{2}$ by Lemma 2.1. As H^* is a proper subtree of B, we have $b = |H^*| < |B| \le \frac{|T|}{2}$, $a = |T| - b > \frac{|T|}{2}$, and $k + 1 \le b + 1 \le \frac{|T|}{2} < a$.

Let H^{**} be the subtree of T induced by $V(H^*) \cup \{z\}$. By Lemma 2.2, we have

$$\sum_{w \in V(H)} d_T(z,w) = s_H(z) \le h(a-1) \quad \text{and} \quad \sum_{w \in V(H^*)} d_T(z,w) = s_{H^{**}}(z) \le h(b)$$

with equalities if and only if H and H^{**} are both paths with a terminal vertex z. Thus

$$\begin{aligned} s_T(z) &= \sum_{w \in V(H)} d_T(z,w) + \sum_{w \in V(H^*)} d_T(z,w) \\ &\le h(a-1) + h(b) \le h(a-1) + h(k) + \sum_{i=a}^{a+b-k-1} i \\ &= h(k) + h(|T| - k - 1) = s_{T'}(u), \end{aligned}$$

where equality holds in the first inequality if and only if H and H^{**} are both paths with a terminal vertex z, the second inequality follows as $k \le b$ and $k+1 < a$, and equality holds if and only if $b = k$, or equivalently, H^{**} is a path from v to z. Thus $s_T(z) < s_{T'}(u)$, as $s_T(z) = s_{T'}(u)$ implies that T is a path with a terminal vertex v, which is a contradiction. So $r(G) \le r_G(z) < r_{G'}(u) = r(G')$. □

Denote by $P_{n,g}$ by the n-vertex unicyclic graph obtained from the g-vertex cycle by attaching a pendant path of length $n-g$ to a vertex of this cycle.

Lemma 3.5 *For $n \ge 4$, we have $r(P_{n,3}) = \lfloor \frac{n^2}{4} \rfloor - \frac{5}{3}$.*

Proof Label the vertices of the unique branch of $P_{n,3}$ consecutively by $w_1, \dots, w_{n-2}$, where w_1 lies on the triangle. Then, for $1 \le x \le n-2$, we have

$$\begin{aligned} r_G(w_x) &= h(x-1) + h(n-2-x) + 2(x-1+\frac{2}{3}) \\ &= x^2 - (n-3)x + \frac{(n-1)(n-2)}{2} - \frac{2}{3}. \end{aligned}$$

Let $f(x)$ be the above expression for $r_G(w_x)$, where $1 \le x \le n-2$. Then $f(x)$ is minimized when $x = x_0 := \lceil \frac{n-3}{2} \rceil$. If n is odd, then $f(x_0) = f(\frac{n-3}{2}) = \frac{n^2-5}{4} - \frac{2}{3} = \lfloor \frac{n^2}{4} \rfloor - \frac{5}{3}$. If n is even, then $f(x_0) = f(\frac{n-2}{2}) = \frac{n^2-4}{4} - \frac{2}{3} = \lfloor \frac{n^2}{4} \rfloor - \frac{5}{3}$. The result follows. □

Lemma 3.6 *Let G be a connected graph with $e \in E(G)$ such that $G-e$ is connected. Then $r(G) \le r(G-e)$.*

Proof By Rayleigh's monotonicity law [26], the pairwise resistance distance does not decrease when edges are removed, so $r_G(w,z) \le r_{G-e}(w,z)$ for all $w, z \in V(G)$. Assume that $r(G) = r_G(u)$ and $r(G-e) = r_{G-e}(v)$ for $u, v \in V(G)$. Then $r(G) = r_G(u) \le r_G(v) \le r_{G-e}(v) = r(G-e)$, as desired. □

Theorem 3.7 *Let G be a unicyclic graph on n vertices. If $n = 3, 4$, then*

$$r(G) \le \frac{n^2-1}{6}$$

with equality if and only if $G \cong C_n$. If $n \ge 5$, then

$$r(G) \le \lfloor \frac{n^2}{4} \rfloor - \frac{5}{3}$$

with equality if and only if $G \cong P_{n,3}$.

Proof The result is trivial for $n = 3$. Suppose that $n \ge 4$.

If G is a cycle, then $r(G) = \frac{n^2-1}{6}$.

Suppose that G is not a cycle. Let Y_n be the tree of order n obtained from a path of order $n-2$ by attaching two pendant edges at an end vertex. By [18, Theorem 3.3], for any tree T of order n that is not isomorphic to P_n or Y_n, we have $s(T) \le \lfloor \frac{n^2}{4} \rfloor - 2$. If there is some edge e on the unique cycle of G such that $G - e \ncong P_n, Y_n$, so we have by Lemma 3.6 that

$$r(G) \le r(G-e) = s(G-e) \le \lfloor \frac{n^2}{4} \rfloor - 2 < \lfloor \frac{n^2}{4} \rfloor - \frac{5}{3}.$$

Suppose that $G-e\cong P_n, Y_n$ for any edge e on the unique cycle of G. Let g be the girth of G. Then $3\le g\le n-1$. Assume that $d_G(v_1)\ge 3$. If $g\ge 5$, then $G-v_3v_4\ncong P_n, Y_n$, which is a contradiction. So $g=3,4$.

Case 1. $g=3$.

Assume that $t_1\ge t_2\ge t_3$. If $t_3\ge 2$, then $G-v_1v_2\ncong P_n, Y_n$, which is a contradiction. So $t_3=1$. If $t_2\ge 2$ and $t_1\ge 3$, then $G-v_2v_3\ncong P_n, Y_n$, also a contradiction. We are left with two possibilities: (i) $t_1=t_2=2$, $n=5$, and by a direct calculation, $r(G)=4<\lfloor\frac{n^2}{4}\rfloor-\frac{5}{3}$; (ii) $t_2=1$, $G\cong P_{n,3}$ and by Lemma 3.5, $r(G)=\lfloor\frac{n^2}{4}\rfloor-\frac{5}{3}$.

Case 2. $g=4$.

Assume that $t_1\ge t_3$ and $t_1\ge t_2\ge t_4$. If $t_2,t_3\ge 2$, then $G-v_1v_2\ncong P_n, Y_n$, a contradiction. So $t_2=1$ or $t_3=1$. Suppose that $t_2=1$. Then $t_4=1$. If $t_1\ge 3$, then $G-v_2v_3\ncong P_n, Y_n$, a contradiction. So $t_1=2$. If $t_3=1$, then $n=5$, and $r(G)=\frac{7}{2}<\lfloor\frac{n^2}{4}\rfloor-\frac{5}{3}$. If $t_3=2$, then $n=6$, and $r(G)=\frac{11}{2}<\lfloor\frac{n^2}{4}\rfloor-\frac{5}{3}$. We are left with the case that $t_2\ge 2$ and $t_3=1$, so $G-v_2v_3\ncong P_n, Y_n$, which is a contradiction.

Combining Cases 1 and 2, we have $r(G)\le\lfloor\frac{n^2}{4}\rfloor-\frac{5}{3}$ with equality if and only if $G\cong P_{n,3}$.

Now the result follows by noting that $\lfloor\frac{n^2}{4}\rfloor-\frac{5}{3}<\frac{n^2-1}{6}$ for $n=4$ and $\lfloor\frac{n^2}{4}\rfloor-\frac{5}{3}>\frac{n^2-1}{6}$ for $n\ge 5$. $\square$

4. Maximum resistance status

Now we turn to consider the maximum resistance status of unicyclic graphs.

Lemma 4.1 *Among all unicyclic graphs on n vertices with fixed girth g, the maximum resistance status is maximized by a unicyclic graph G, with the following property that each branch is a path with a terminal vertex on the cycle.*

Proof Let C be the unique cycle of G. Assume that T is any branch of G at v on C. Suppose that T is not a path with v being a terminal vertex. Let G' be the unicyclic graph obtained from G by replacing T by a path T' such that v is a terminal vertex and $V(T')=V(T)$. It suffices to show that $R(G)<R(G')$.

Let $V=V(T)\setminus\{v\}$ and $V'=V(G)\setminus V$. Assume that $R(G)=r_G(u)$ with $u\in V(G)$.

If $u\in V'$, then

$$r_G(u)-r_{G'}(u)=\sum_{w\in V}d_G(v,w)-\sum_{w\in V}d_{G'}(v,w)=s_T(v)-h(|T|-1)<0,$$

so $R(G)=r_G(u)<r_{G'}(u)\le R(G')$.

If $u\in V$, then, denoting by z the leaf of G' in T', we have $d_G(u,v)<d_{G'}(z,v)$, so

$$\begin{aligned}r_G(u)-r_{G'}(z)=&\sum_{w\in V'\setminus\{v\}}(d_G(u,v)+r_G(v,w))+\sum_{w\in V(T)}d_G(u,w)-\\&\sum_{w\in V'\setminus\{v\}}(d_{G'}(z,v)+r_G(v,w))-\sum_{w\in V(T)}d_{G'}(z,w)\end{aligned}$$

$$= \sum_{w\in V'\setminus\{v\}} (d_G(u,v) - d_{G'}(z,v)) + \sum_{w\in V(T)} d_G(u,w) - h(|T|-1)$$
$$< \sum_{w\in V(T)} d_G(u,w) - h(|T|-1) \le 0,$$

implying that $R(G) = r_G(u) < r_{G'}(z) \le R(G')$. □

Theorem 4.2 *Let G be a unicyclic graph on n vertices with girth g. Then*

$$R(G) \le \frac{n^2-n}{2} - \frac{g^2}{3} + \frac{g}{2} - \frac{1}{6}$$

with equality if and only if $G \cong P_{n,g}$.

Proof The result is trivial if $g = n$, $n-1$. Suppose that $g \le n-2$. Suppose that G is a unicyclic graph on n vertices with girth g that maximizes the maximum resistance status.

By Lemma 4.1, T_i is a path with a terminal vertex v_i for each $i = 1,\dots,g$. Assume that $R(G) = r_G(x)$ with $x \in V(T_i)$ for some i. It is easy to see that if T_i is nontrivial, then x is a leaf of G.

Assume that $R(G) = r_G(x)$ with $x \in V(T_1)$. If T_1 is trivial, then $x = v_1$, and if T_1 is not trivial, then it is easy to see that x is the leaf of G in T_1. So

$$\begin{aligned}
R(G) &= h(t_1-1) + \sum_{j=2}^{g} \sum_{w\in V(T_j)} (t_1 - 1 + r_G(v_1,w)) \\
&= h(t_1-1) + \sum_{j=2}^{g} \sum_{w\in V(T_j)} (t_1 - 1 + r_G(v_1,v_j) + d_G(v_j,w)) \\
&= h(t_1-1) + \sum_{j=2}^{g} ((t_1 - 1 + r_G(v_1,v_j))t_j + h(t_j-1)) \\
&= h(t_1-1) + \sum_{j=2}^{g} (t_1-1)t_j + \sum_{j=2}^{g} r_G(v_1,v_j)t_j + \sum_{j=2}^{g} h(t_j-1) \\
&= h(t_1-1) + (t_1-1)(n-t_1) + \sum_{j=2}^{g} r_G(v_1,v_j)t_j + \sum_{j=2}^{g} h(t_j-1).
\end{aligned}$$

We view the above expression for $R(G)$ as a function of $t_1,\dots,t_g$. Suppose that $t_i \ge 2$ for some $i = 2,\dots,g$. Denote by $f(t_1,t_i)$ the expression for $R(G)$. For any $j = 2,\dots,g$, we have $r_G(v_1,v_j) = \frac{(j-1)(g-j+1)}{g} \le \frac{g}{4}$. Then

$$\begin{aligned}
f(t_1+1,t_i-1) - f(t_1,t_i) &= t_1 + t_1(n-t_1-1) - (t_1-1)(n-t_1) - r_G(v_1,v_i) - (t_i-1) \\
&= n - t_1 - (t_i-1) - r_G(v_1,v_i) \ge g - 1 - r_G(v_1,v_i) \\
&\ge g - 1 - \frac{g}{4} > 0,
\end{aligned}$$

so $f(t_1+1,t_i-1) > f(t_1,t_i)$, which is a contradiction to our choice of G. Thus, $t_2 = \cdots = t_g = 1$ and $t_1 = n-g+1$. Moreover,

$$R(G) = h(n-g) + (n-g)(g-1) + r(C_g)$$

$$= \frac{(n-g)(n+g-1)}{2} + \frac{g^2-1}{6}$$
$$= \frac{n^2-n}{2} - \frac{g^2}{3} + \frac{g}{2} - \frac{1}{6},$$

as desired. □

Theorem 4.3 *Let G be a unicyclic graph on n vertices, where $n \geq 3$. Then*

$$R(G) \leq \frac{n^2-n}{2} - \frac{5}{3}$$

with equality if and only if $G \cong P_{n,3}$.

Proof Denote by g the girth of G. By Theorem 4.2, we have

$$R(G) \leq f(g) := \frac{n^2-n}{2} - \frac{g^2}{3} + \frac{g}{2} - \frac{1}{6} = \frac{n^2-n}{2} - \frac{1}{3}(g - \frac{3}{4})^2 + \frac{1}{48}$$

with equality if and only if $G \cong P_{n,g}$. The result follows by noting that $f(g)$ is strictly decreasing for $g \geq 3$. □

Finally we consider lower bounds for the maximum resistance status.

Lemma 4.4 *Among all unicyclic graphs on n vertices with fixed girth g, the maximum resistance status is minimized by a unicyclic graph G, with the property that any branch of G is a star with its center on the cycle.*

Proof Let C be the unique cycle of G. Assume that T is any branch of G at v on C. Suppose that T is not a star with center v. Let $G' = G - E(T) + \{vw : w \in V(T) \setminus \{v\}\}$. It suffices to show that $R(G) > R(G')$.

Let $V = V(T) \setminus \{v\}$ and $V' = V(G) \setminus V$. Obviously, $|V| = |T| - 1 \geq 2$. Let z be a vertex in T that is farthest from v. Evidently, z is a leaf of T. Then

$$\begin{aligned} r_G(z) &= \sum_{w \in V \setminus \{z\}} d_G(z,w) + \sum_{w \in V'} (d_G(z,v) + r_G(v,w)) \\ &\geq 1 + 2(|V|-2) + \sum_{w \in V'} (2 + r_G(v,w)) \\ &= 2|V| - 3 + 2|V'| + \sum_{w \in V'} r_G(v,w). \end{aligned}$$

Assume that $R(G') = r_{G'}(u)$ with $u \in V(G)$. If $u \in V$, then

$$\begin{aligned} r_{G'}(u) &= \sum_{w \in V \setminus \{u\}} d_{G'}(u,w) + \sum_{w \in V'} (d_{G'}(u,v) + r_{G'}(v,w)) \\ &= 2(|V|-1) + |V'| + \sum_{w \in V'} r_G(v,w), \end{aligned}$$

so $r_G(z) - r_{G'}(u) \geq |V'| - 1 > 0$, implying that $R(G) \geq r_G(z) > r_{G'}(u) = R(G')$. Suppose that $u \in V'$. Observe that $d_G(v,w) > 1$ for some $w \in V$ and $r_G(u,w) = r_{G'}(u,w)$ for $w \in V' \setminus \{u\}$.

Then

$$\begin{aligned} r_G(u) - r_{G'}(u) &= \sum_{w\in V}(r_G(u,w) - r_{G'}(u,w)) + \sum_{w\in V'\setminus\{u\}}(r_G(u,w) - r_{G'}(u,w)) \\ &= \sum_{w\in V}(d_G(v,w) - d_{G'}(v,w)) = \sum_{w\in V}(d_G(v,w) - 1) > 0, \end{aligned}$$

so $R(G) \geq r_G(u) > r_{G'}(u) = R(G')$. □

Lemma 4.5 *Let G be a unicyclic graph on n vertices with girth 3, where $n \geq 4$. Then*

$$R(G) \geq 2n - \frac{11}{3}$$

with equality if and only if $G \cong S_{n,3}$.

Proof Let G be a unicyclic graph on n vertices with girth 3 that minimizes the maximum resistance status. By Lemma 4.4, each T_i is a star with center v_i on the triangle of G for $i = 1, 2, 3$. Assume that T_1 is nontrivial. Let z be a leaf of G in T_1. Then

$$\begin{aligned} r_G(z) &= \sum_{w\in V(T_1)} d_G(z,w) + \sum_{w\in V(T_2)\cup V(T_3)} r_G(z,w) \\ &= 1 + 2(t_1 - 2) + (1 + \frac{2}{3})\cdot 2 + (2 + \frac{2}{3})\cdot(t_2 + t_3 - 2) \\ &= 2(t_1 + t_2 + t_3) - 5 + \frac{2}{3}(t_2 + t_3) \\ &= 2n - 5 + \frac{2}{3}(t_2 + t_3) \geq 2n - \frac{11}{3} \end{aligned}$$

with equality if and only if $t_2 = t_3 = 1$, i.e., $G \cong S_{n,3}$. It is easy to see that $r_G(v_1) < r_G(z)$. By direct calculation, we have

$$r_G(v_2) = \frac{3^2 - 1}{6} + t_2 - 1 + (1 + \frac{2}{3})(t_1 + t_3 - 2) = n - 3 + \frac{2}{3}(t_1 + t_3).$$

Then

$$\begin{aligned} r_G(z) - r_G(v_2) &= 2n - 5 - (n-3) + \frac{2}{3}(t_2 + t_3 - t_1 - t_3) = n - 2 + \frac{2}{3}(t_2 - t_1) \\ &\geq n - 2 + \frac{2}{3}(1 - (n-2)) = \frac{n}{3} > 0, \end{aligned}$$

i.e., $r_G(v_2) < r_G(z)$. Similarly, $r_G(v_3) < r_G(z)$. So $R(G)$ cannot be achieved by v_1, v_2, v_3. Thus $R(G) \geq r_G(z) \geq 2n - \frac{11}{3}$ with equality if and only if $G \cong S_{n,3}$. □

Lemma 4.6 *Let G be a non-cycle unicyclic graph on n vertices with girth at least four. Then $R(G) > 2n - \frac{11}{3}$.*

Proof Let G be a non-cycle unicyclic graph on n vertices with girth at least four that minimizes the maximum resistance status. Let g be the girth of G. By Lemma 4.4, each branch T_i is a star with center v_i for $i = 1, \ldots, g$. Assume that T_1 is nontrivial. Let z be a leaf of T_1. Observe that

$\frac{g-1}{g} > \frac{2}{3}$ and $\frac{(j-1)(g-j+1)}{g} + d_G(v_j, w) \geq 1$ for $3 \leq j \leq g-1$ and $w \in V(T_j)$. Then

$$\begin{aligned}
r_G(z) = & \sum_{w \in V(T_1)} d_G(z,w) + \sum_{w \in V(T_2) \cup V(T_g)} r_G(z,w) + \sum_{j=3}^{g-1} \sum_{w \in V(T_j)} r_G(z,w) \\
= & 1 + 2(t_1 - 2) + (1 + \frac{g-1}{g}) \cdot 2 + (2 + \frac{g-1}{g}) \cdot (t_2 + t_g - 2) + \\
& \sum_{j=3}^{g-1} \sum_{w \in V(T_j)} \big(1 + \frac{(j-1)(g-j+1)}{g} + d_G(v_j, w)\big) \\
> & 1 + 2(t_1 - 2) + (1 + \frac{2}{3}) \cdot 2 + 2(t_2 + t_g - 2) + 2\sum_{j=3}^{g-1} t_j \\
= & 1 + \frac{10}{3} + 2(n-4) = 2n - \frac{11}{3},
\end{aligned}$$

as desired. □

Theorem 4.7 *Let G be a unicyclic graph on $n \geq 3$ vertices. If $n \leq 9$, then*

$$R(G) \geq \frac{n^2-1}{6}$$

with equality if and only if $G \cong C_n$. If $n \geq 10$, then

$$R(G) \geq 2n - \frac{11}{3}$$

with equality if and only if $G \cong S_{n,3}$.

Proof If G is a cycle, then $R(G) = \frac{n^2-1}{6}$. Otherwise, by Lemmas 4.5 and 4.6, $R(G) \geq 2n - \frac{11}{3}$ with equality if and only if $G \cong S_{n,3}$. The result follows by noting that $\frac{n^2-1}{6} < 2n - \frac{11}{3}$ for $n \leq 9$ and $\frac{n^2-1}{6} > 2n - \frac{11}{3}$ for $n \geq 10$. □

5. Concluding remarks

Aouchiche and Hansen studied in [14] the relationship between minimum (maximum, resp.) status and the girth of connected graphs. This paper extends the results in [14] by exploring the unicyclic graphs with given order and girth that minimize/maximize the minimum (maximum, resp.) resistance status. As consequences, we determine the minimum and maximum for the minimum (maximum, resp.) resistance status among unicyclic graphs of given order and characterize the extremal graphs.

Acknowledgements We thank the referees for constructive suggestions and comments.

References

[1] F. BUCKLEY, F. HARARY. *Distance in Graphs*. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1990.

[2] F. HARARY. *Status and contrastatus*. Sociometry, 1959, **22**(1): 23–43.

[3] L. ŠOLTÉS. *Transmission in graphs: a bound and vertex removing*. Math. Slovaca, 1991, **41**(1): 11–16.

[4] D. VUKIČEVIĆ, G. CAPOROSSI. *Network descriptors based on betweenness centrality and transmission and their extremal values.* Discrete Appl. Math., 2013, **161**(16-17): 2678–2686.

[5] R. C. ENTRINGER, D. E. JACKSON, D. A. SNYDER. *Distance in graphs.* Czechoslovak Math. J., 1976, **26**(2): 283–296.

[6] K. BALAKRISHNAN, B. BREŠAR, M. CHANGAT, et al. *Equal opportunity networks, distance-balanced graphs, and Wiener game.* Discrete Optim., 2014, **12**: 150–154.

[7] C. LIN, W. H. TSAI, J. L. SHANG, et al. *Minimum statuses of connected graphs with fixed maximum degree and order.* J. Combin. Optim., 2012, **24**(3): 147–161.

[8] R. RISSNER, R. E. BURKARD. *Bounds on the radius and status of graphs.* Networks, 2014, **64**(2): 76–83.

[9] K. PRAVAS, A. VIJAYAKUMAR. *The median problem on k-partite graphs.* Discuss. Math. Graph Theory, 2015, **35**(3): 439–446.

[10] J. GOLBECK. *Analyzing the Social Web.* Burlington, Morgan Kaufmann, MA, 2013.

[11] B. ZELINKA. *Medians and peripherians of trees.* Arch. Math. (Brno), 1968, **4**(2): 87–95.

[12] A. N. C. KANG, D. A. AULT. *Some properties of a centroid of a free tree.* Information Processing Lett., 1975/76, **4**(1): 18–20.

[13] M. AOUCHICHE, P. HANSEN. *Proximity and remoteness in graphs: results and conjectures.* Networks, 2011, **58**(2): 95–102.

[14] M. AOUCHICHE, P. HANSEN. *Proximity, remoteness and girth in graphs.* Discrete Appl. Math., 2017, **222**: 31–39.

[15] P. DANKELMANN. *Proximity, remoteness and minimum degree.* Discrete Appl. Math., 2015, **184**: 223–228.

[16] J. SEDLAR. *Remoteness, proximity and few other distance invariants in graphs.* Filomat, 2013, **27**(8): 1425–1435.

[17] Caixia LIANG, Bo ZHOU, Haiyan GUO. *Minimum status, matching and domination of graphs.* Comput. J., 2021, **64**(9): 1384–1392.

[18] Zhene PENG, Bo ZHOU. *Minimum status of trees with given parameters.* RAIRO Oper. Res., 2021, **55**(suppl.): S765–S785.

[19] Pu QIAO, Xingzhi ZHAN. *Pairs of a tree and a nontree graph with the same status sequence.* Discrete Math., 2020, **343**(2): 111662, 7 pp.

[20] Kexiang XU, S. KLAVŽAR. *Constructing new families of transmission irregular graphs.* Discrete Appl. Math., 2021, **289**: 383–391.

[21] R. M. FOSTER. *The average impedance of an electrical network.* Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 333–340. J. W. Edwards, Ann Arbor, Michigan, 1948.

[22] D. J. KLEIN, M. RANDIĆ. *Resistance distance.* J. Math. Chem., 1993, **12**(1-4): 81–95.

[23] R. B. BAPAT. *Resistance distance in graphs.* Math. Student, 1999, **68**(1-4): 87–98.

[24] A. GHOSH, S. BOYD, A. SABERI. *Minimizing effective resistance of a graph.* SIAM Rev., 2008, **50**(1): 37–66.

[25] Jiang ZHOU, Zhongyu WANG, Changjiang BU. *On the resistance matrix of a graph.* Electron. J. Combin., 2016, **23**(1): Paper 1.41, 10 pp.

[26] P. G. DOYLE, J. L. SNELL. *Random Walks and Electric Networks.* Mathematical Association of America, Washington, DC, 1984.