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Finite Groups with p-Supersolvable Normalizers of p-Subgroups

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Abstract In the literature, p-nilpotency of the normalizers of p-subgroups has an important influence on finite p-nilpotent groups. In this paper, we extend the p-nilpotency to p-supersolvability and choose every normal p-subgroups H of P such that $|H| = p^d$ and explore p-supersolvability of G by the conditions of weakly $\mathcal{M}$-supplemented properties of H and p-supersolvability of the normalizer $N_G(H)$, where $1 \leq p^d < |P|$. Also, we study the p-nilpotency of G under the assumptions that $N_G(P)$ is p-nilpotent and the weakly $\mathcal{M}$-supplemented condition on a subgroup K such that $K_p \trianglelefteq K$ and $P' \leq K_p \leq \Phi(P)$, K_p is a Sylow p-subgroup K. To some extent, our main results can be regarded as generalizations of the Frobenius theorem.

Keywords normalizer; weakly $\mathcal{M}$-supplemented subgroup; p-supersolvability; p-nilpotency

MR(2020) Subject Classification 20D10; 20D20

1. Introduction

Throughout this paper, all groups are finite, terminology and notation are standard. In particular, $|G|$ is the order of G and p is a prime divisor of $|G|$. Let H_G denote the core of a subgroup H in G and let $M < \cdot G$ denote M is a maximal subgroup of G. Let $A \rtimes B$ denote the semidirect product of groups A and B, where B is an operator group of A. In [1], let $\mathcal{K}(P)$ denote a set of subgroups K of G satisfying $K_p \trianglelefteq K$ and $P' \leq K_p \leq \Phi(P)$ for $P \in \mathrm{Syl}_p(G)$ and $K_p \in \mathrm{Syl}_p(K)$.

As all we know, the normalizers of p-subgroups have an important influence on the structure of finite groups. Based on the famous Frobenius theorem [2, Satz IV.5.8] and Glauberman-Thompson normal p-complement theorem [3, VIII, Theorem 3.1], there are many research works which were related to this topic [4–9].

In 2011, Miao and Lempken [10] introduced the concept of weakly $\mathcal{M}$-supplemented subgroups which is a generalization of $\mathcal{M}$-supplemented subgroups ([11, Definition 1.1], or [12, Defi-

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nition 1.3]) and c-normal subgroups [13, Definition 1.1], and they obtained some new criteria on the structure of finite groups.

Definition 1.1 ([10, Definition 1.1]) *A subgroup T of a group G is said to be weakly $\mathcal{M}$-supplemented in G, if there exists a subgroup B of G provided that (1) $G = TB$, and (2) if T_1/T_G is a maximal subgroup of T/T_G, then $T_1B = BT_1 < G$, where T_G is the largest normal subgroup of G contained in T.*

As a next step of the previous contributions, we obtain the following results.

Theorem 1.2 *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. Assume that $1 \leq p^d < |P|$. If every normal subgroup H of P with $|H| = p^d$ is weakly $\mathcal{M}$-supplemented in G and $N_G(H)$ is p-supersolvable, then G is p-supersolvable.*

Remark 1.3 In Theorem 1.2, when "$N_G(H)$ is p-supersolvable" is replaced by "$N_G(H)$ is p-nilpotent", we immediately prove that G is p-nilpotent. Also, we list an example of a group satisfying Theorem 1.2. Assume that $G = S_3 \times C_3$, P is an elementary abelian Sylow 3-subgroup of G. Furthermore, we can prove that G is 3-supersolvable, H is weakly $\mathcal{M}$-supplemented in G and $N_G(H)$ is 3-supersolvable for every normal subgroup H of P with $|H| = 3$.

Corollary 1.4 *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. Assume that $1 < p^d < |P|$. If every subgroup H of P with $|H| = p^d$ is weakly $\mathcal{M}$-supplemented in G and $N_G(H)$ is p-nilpotent, then G is p-nilpotent.*

Corollary 1.5 *Let G be a group and P a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. If every maximal subgroup P_1 of P is weakly $\mathcal{M}$-supplemented in G and $N_G(P_1)$ is p-supersolvable, then G is p-supersolvable.*

Theorem 1.6 *Let G be a group, P a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. If there is an element $K \in \mathcal{K}(P)$ which is weakly $\mathcal{M}$-supplemented in G and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Remark 1.7 We list an example of a group satisfying Theorem 1.6. Assume that $G = A_4 \times C_9$, P is a Sylow 3-subgroup of G and $K = Q \times H$ where Q is a Sylow 2-subgroup of G and $H = \Phi(P)$ is a normal subgroup of P with $|H| = 3$. Furthermore, we prove that G is 3-nilpotent by the structure of the alternating group A_4, K is normal in G and K is also weakly $\mathcal{M}$-supplemented in G.

Corollary 1.8 *Let p be an odd prime divisor of $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and supposing that P has a subgroup D such that $1 < D < P$, and every subgroup E of P with order $|D|$ is weakly $\mathcal{M}$-supplemented in G, then G is p-nilpotent.*

2. Proofs

For the sake of convenience, we list some known results which will be useful in the sequel.

Lemma 2.1 ([10, Lemma 2.1]) *Let G be a group. Then*

(1) If T is weakly $\mathcal{M}$-supplemented in G, $T \leq M \leq G$, then T is weakly $\mathcal{M}$-supplemented in M.

(2) Let $N \trianglelefteq G$ and $N \leq T$. Then T is weakly $\mathcal{M}$-supplemented in G if and only if T/N is weakly $\mathcal{M}$-supplemented in G/N.

(3) Let π be a set of primes. Let K be a normal π'-subgroup and H be a π-subgroup of G. If T is weakly $\mathcal{M}$-supplemented in G, then TK/K is weakly $\mathcal{M}$-supplemented in G/K.

(4) Let R be a solvable minimal normal subgroup of G and R_1 be a maximal subgroup of R. If R_1 is weakly $\mathcal{M}$-supplemented in G, then R is a cyclic group of prime order.

(5) Let P be a p-subgroup of G, where p is a prime divisor of $|G|$. If P is weakly $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $|G : P_1B| = p$ for every maximal subgroup P_1 of P containing P_G.

Lemma 2.2 ([2, Satz I.6.6]) *Let $H \leq G$. If the index $|G : H| = n$, then G/H_G is isomorphic to a subgroup of S_n.*

Lemma 2.3 ([14, Lemma 3.6.10]) *Let G be a group, K be a normal subgroup of G and P be a p-subgroup of G. Then $N_{G/K}(PK/K) = N_G(P_1)K/K$, where $P_1 \in \mathrm{Syl}_p(PK)$.*

Lemma 2.4 ([2, Hilfssatz III.3.3]) *Assume that $N \trianglelefteq G$ and $U \leq G$. Then*

(1) If $N \leq \Phi(U)$, then $N \leq \Phi(G)$.

(2) $\Phi(N) \leq \Phi(G)$.

Lemma 2.5 ([2, Satz IV.4.7]) *If P is a Sylow p-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.6 ([15, Lemma 2.8]) *Let G be a p-supersolvable group. If $O_{p'}(G) = 1$, then G is supersolvable.*

Proof of Theorem 1.2 Assume that the assertion is false and let G be a minimal counterexample. Set $\delta = \{H \trianglelefteq P||H| = p^d, 1 \leq p^d < |P|\}$.

(1) G is not nonabelian simple.

If G is nonabelian simple, then G is isomorphic to a subgroup of S_p by Lemmas 2.1 and 2.2 and $|P| = p$. Hence $p^d = 1$ and $N_G(H) = G$ is p-supersolvable by the hypothesis, a contradiction. Furthermore, by the hypothesis, for every element $H \in \delta$, H is not normal in G and there is a subgroup B of G such that $G = HB$, $H_iB < G$ where $H_i < \cdot H$. Then $(H_iB)_G \neq 1$ by arguing as above.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$ and set $\overline{G} = G/O_{p'}(G)$. By the hypothesis and Lemma 2.1, every normal subgroup $\overline{H}$ of $\overline{P}$ with $|\overline{H}| = p^d$ is weakly $\mathcal{M}$-supplemented in $\overline{G}$. Also, $N_{\overline{G}}(\overline{H}) = \overline{N_G(H)}$ is p-supersolvable by Lemma 2.3. Hence $\overline{G}$ satisfies the hypothesis and $\overline{G}$ is p-supersolvable by the choice of G. Then G is p-supersolvable, a contradiction.

(3) $G = PN$ for every minimal normal subgroup N of G.

If there exists a minimal normal subgroup N of G such that $PN < G$, then PN is p-supersolvable by the choice of G, Lemmas 2.1 and 2.3. Furthermore, N is p-supersolvable and N is a p-subgroup by (2). By the hypothesis and the choice of G, $|N| \neq p^d$. Then we assert that $|N| < p^d$. Otherwise, $|N| > p^d$ and there is an element $H \in \delta$ such that $H < N$. Furthermore, there exists a subgroup B of G such that $G = HB$, $H_iB < G$ and $G = NH_iB$, where $H_i < \cdot H$. Hence $N \cap H_iB = 1$ and $|N| = p$ by Lemma 2.1. Then $p^d = 1$ and $N_G(H) = G$ is p-supersolvable by the hypothesis, a contradiction. Since $|N| < p^d$, G/N is p-supersolvable by the choice of G, Lemmas 2.1 and 2.3. Furthermore, there is the unique minimal normal subgroup N of G contained in $O_p(G)$, $|N| \neq p$ and $N \nleq \Phi(G)$.

Then there exists a maximal subgroup M of G such that $N \rtimes M = G$ by Lemma 2.4. Clearly, $P = N(P \cap M)$ and $P \cap M \neq 1$. Then we may choose a maximal subgroup P_1 of P containing $S = P \cap M$ and $P_1 = P_1 \cap (NS) = (P_1 \cap N)S$. Furthermore, $1 \neq P_1 \cap N \trianglelefteq P$ and we may pick an element $R \in \delta$ such that $1 \neq P_1 \cap N \leq R \leq P_1$. Then $N \nleq R$, $R_G = 1$ and $R = R \cap (P_1 \cap N)S = (P_1 \cap N)(R \cap S)$. Since R is weakly $\mathcal{M}$-supplemented in G, there is a subgroup T of G such that $G = RT$, $R_iT < G$, where $R_i < \cdot R$. Then we may choose a maximal subgroup R_1 of R such that $R_1 \geq R \cap S$ and $R_1 = R_1 \cap R = R_1 \cap (P_1 \cap N)(R \cap S) = (R_1 \cap N)(R_1 \cap S)$. Furthermore, $NR_1T = G$ and $N \cap R_1T = 1$ since $R_1T < G$. Then $|N| = p$ by Lemma 2.1, a contradiction.

(4) The final contradiction.

By (3), $O_p(G) = 1$ and $N = O^p(G)$. Now we may assume that N_p is a Sylow p-subgroup of N. If $|N_p| \geq p^d$, then there is an element $H \in \delta$ such that $H \leq N_p$. Furthermore, there exists a subgroup B of G such that $G = HB$, $H_iB < G$ since $H_G = 1$, and $G = NH_iB$, where $H_i < \cdot H$. Since $(H_iB)_G \neq 1$ by the proof of (1), $N \cap (H_iB)_G = 1$ and $(H_iB)_G$ is a p-subgroup, a contradiction. If $|N_p| < p^d$, then there is an element $E \in \delta$ such that $N_p < E$. Furthermore, $E_G = 1$ and $N_p \nleq \Phi(E)$ by Lemma 2.5. Then $E = N_pE_1$ and there exists a subgroup W of G such that $G = EW$ and $E_1W < G$ since $E_G = 1$, where $E_1 < \cdot E$. Hence $G = EW = N_pE_1W = NE_1W$. Since $O_p(G) = 1$ and $(E_1W)_G \neq 1$ by the proof of (1), $N \cap (E_1W)_G = N$ and $G = EW = N_pE_1W = NE_1W = E_1W < G$, a contradiction. □

Proof of Corollary 1.4 Here, we may assume that $O_{p'}(G) = 1$. By Theorem 1.2 and Lemma 2.6, G is supersolvable and $P \trianglelefteq G$. Furthermore, there is a normal subgroup H of G such that $H \trianglelefteq P$ and $|H| = p^d$. Then $N_G(H) = G$ is p-nilpotent. □

Proof of Theorem 1.6 By the hypothesis, K is weakly $\mathcal{M}$-supplemented in G and there exists a subgroup B of G such that $G = KB$, $K_iB < G$ where $K_i < \cdot K$ containing K_G. Then we assert that $K_p \neq 1$. Otherwise, $K_p = 1$ and P is abelian by the definition of K. Hence G is p-nilpotent by Burnside Theorem.

First, we assume that $K_G \neq 1$. Then $P \cap K_G \leq P \cap K \leq \Phi(P)$ and K_G is p-nilpotent. Next, let T be the Hall p'-subgroup of K_G. If $T \neq 1$, then we can prove that $(PT/T)' \leq$

$P'T/T \leq K_pT/T \leq \Phi(P)T/T \leq \Phi(PT/T)$ and $K/T \in \mathcal{K}(PT/T)$. Furthermore, G/T satisfies the hypothesis by Lemma 2.1 and Lemma 2.3. Hence G/T is p-nilpotent by induction on $|G|$ and G is p-nilpotent. If $T = 1$, then K_G is a p-subgroup, $K_G \leq \Phi(P)$ and $K_G \leq \Phi(G)$ by Lemma 2.4. Furthermore, G/K_G satisfies the hypothesis by Lemmas 2.1 and 2.3, G/K_G is p-nilpotent by induction on $|G|$. Hence G is p-nilpotent.

We assume that $K_G = 1$. Then there is a maximal subgroup K_1 of K such that $K_{p'} \leq K_1$ where $K_{p'}$ is a Hall p'-subgroup of K. Since $K_p \trianglelefteq K$, $L = K_p \cap K_1 \trianglelefteq K_1$ and $K_1 = LK_{p'}$. Next, we assume that M is the subgroup generalized by $K_{p'}$ and B. Then $G = KB = K_pK_{p'}B = K_pM$ and $G = M$ since $K_p \leq \Phi(P)$. By the fact that $K_1B < G$, $K_1B = LK_{p'}B = LM = G$, a contradiction. $\square$

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