Journal of Mathematical Research with Applications
Sept., 2022, Vol. 42, No. 5, pp. 481–498
DOI:10.3770/j.issn:2095-2651.2022.05.005
Http://jmre.dlut.edu.cn

On Split Regular Hom-Leibniz-Rinehart Algebras

Shuangjian GUO[1,*], Xiaohui ZHANG[2], Shengxiang WANG[3]
1. *School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou* 550025*, P. R. China;*
2. *School of Mathematical Sciences, Qufu Normal University, Shandong* 273165*, P. R. China;*
3. *School of Mathematics and Finance, Chuzhou University, Anhui* 239000*, P. R. China*

Abstract In this paper, we introduce the notion of the Hom-Leibniz-Rinehart algebra as an algebraic analogue of Hom-Leibniz algebroid, and prove that such an arbitrary split regular Hom-Leibniz-Rinehart algebra L is of the form $L = U + \sum_{\gamma} I_{\gamma}$ with U a subspace of a maximal abelian subalgebra H and any I_{γ}, a well described ideal of L, satisfying $[I_{\gamma}, I_{\delta}] = 0$ if $[\gamma] \neq [\delta]$. In the sequel, we develop techniques of connections of roots and weights for split Hom-Leibniz-Rinehart algebras, respectively. Finally, we study the structures of tight split regular Hom-Leibniz-Rinehart algebras.

Keywords Hom-Leibniz-Rinehart algebra; root space; weight space; decomposition; simple ideal

MR(2020) Subject Classification 17A32; 17A60; 17B22; 17B60

1. Introduction

The notion of the Lie-Rinehart algebra plays an important role in many branches of mathematics. The idea of this notion goes back to the work of Jacobson to study certain field extensions. It also appears in some different names in several areas which includes differential geometry and differential Galois theory. In [1], Mackenzie provided a list of 14 different terms mentioned for this notion. Huebschmann viewed Lie-Rinehart algebras as an algebraic counterpart of Lie algebroids defined over smooth manifolds. His work on several aspects of this algebra has been developed systematically through a series of articles namely [2–5].

The notion of Hom-Lie algebras was first introduced by Hartwig, Larsson and Silvestrov in [6], who developed an approach to deformations of the Witt and Virasoro algebras based on σ-deformations [7]. In fact, Hom-Lie algebras include Lie algebras as a subclass, but the deformation of Lie algebras is twisted by a homomorphism.

Mandal and Mishra defined modules over a Hom-Lie-Rinehart algebra and studied a cohomology with coefficients in a left module. They presented the notion of extensions of Hom-Lie-Rinehart algebras and deduced a characterisation of low dimensional cohomology spaces in

Received February 21, 2022; Accepted June 26, 2022
Supported by the National Natural Science Foundation of China (Grant No. 12161013) and the Key Project of Guizhou University of Finance and Economics (Grant No. 2022KYZD05).
* Corresponding author
E-mail address: shuangjianguo@126.com (Shuangjian GUO)

terms of the group of automorphisms of certain abelian extensions and the equivalence classes of those abelian extensions in the category of Hom-Lie-Rinehart algebras in [8]. The concept of a Hom-Lie-Rinehart algebra has a geometric analogue which is nowadays called a Hom-Lie algebroid in [9, 10]. See also [11–15] for other works on Hom-Lie-Rinehart algebras.

The class of the split algebras is specially related to addition quantum numbers, graded contractions and deformations. For instance, for a physical system which displays a symmetry, it is interesting to know the detailed structure of the split decomposition, since its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Determining the structure of split algebras will become more and more meaningful in the area of research in mathematical physics, the structure of different classes of split algebras have been determined by the techniques of connections of roots. These techniques were introduced in [16] and have shown powerful to study the inner structure of different split and graded algebraic objects in [17–21]. Recently, the techniques of connections in a Rinehart setup are firstly developed in [22]. Later, we studied the structures of split regular Hom-Lie Rinehart algebras in [23] as a generalization of split Lie Rinehart algebras. The purpose of this paper is to consider the structure of split regular Hom-Leibniz-Rinehart algebras by the techniques of connections of roots based on some work in [21, 22].

This paper is organized as follows. In Section 3, we introduce the notion of the Hom-Leibniz-Rinehart algebra and prove that such an arbitrary split regular Hom-Leibniz-Rinehart algebras L is of the form $L = U + \sum_\gamma I_\gamma$ with U a subspace of a maximal abelian subalgebra H and any I_γ, a well described ideal of L, satisfying $[I_\gamma, I_\delta] = 0$ if $[\gamma] \neq [\delta]$. In Sections 4 and 5, we develop techniques of connections of roots and weights for split Hom-Leibniz-Rinehart algebras, respectively. In the last section, we study the structures of tight split regular Hom-Leibniz-Rinehart algebras.

2. Preliminaries

Let R denote a commutative ring with unity, $\mathbb{Z}$ the set of all integers and $\mathbb{N}$ the set of all nonnegative integers, all algebraic systems are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. And we recall some basic definitions and results related to our paper from [24, 25] and [8].

Definition 2.1 *Given an associative commutative algebra A, an A-module M and an algebra endomorphism $\phi : A \to A$, we call an R-linear map $\delta : A \to M$ a ϕ-derivation of A into M if it satisfies the required identity:*

$$\delta(ab) = \phi(a)\delta(b) + \phi(b)\delta(a), \text{ for any } a, b \in A.$$

Let us denote by $Der_\phi(A)$ the set of ϕ-derivations of A into itself.

Definition 2.2 *A Hom-Leibniz algebra L is a vector L, endowed with a bilinear product*

$$[\cdot, \cdot] : L \times L \to L,$$

and a homomorphism $\psi : L \to L$

$$[\psi(x), [y, z]] = [[x, y], \psi(z)] + [\psi(y), [x, z]] \text{ (Hom-Leibniz identity)}$$

holds for any $x, y, z \in L$.

If ψ is furthermore an algebra automorphism, that is, a linear bijective such that $\psi([x, y]) = [\psi(x), \psi(y)]$ for any $x, y \in L$, then L is called a regular Hom-Leibniz algebra.

Definition 2.3 *A representation of a Hom-Leibniz algebra $(L, [\cdot, \cdot], \psi)$ is a quadruple $(V, \psi_V, \rho^L, \rho^R)$, where V is a vector space, $\psi_V \in gl(V)$, $\rho^L, \rho^R : L \to gl(V)$ are linear maps such that the following equalities hold for all $x, y \in L$:*

(1) $\rho^L(\psi(x)) \circ \psi_V = \psi_V \circ \rho^L(x)$, $\rho^R(\psi(x)) \circ \psi_V = \psi_V \circ \rho^R(x)$;
(2) $\rho^L([x, y]) \circ \psi_V = \rho^L(\psi(x)) \circ \rho^L(y) - \rho^L(\psi(y)) \circ \rho^L(x)$;
(3) $\rho^R([x, y]) \circ \psi_V = \rho^L(\psi(x)) \circ \rho^R(y) - \rho^R(\psi(y)) \circ \rho^L(x)$;
(4) $\rho^R([x, y]) \circ \psi_V = \rho^L(\psi(x)) \circ \rho^R(y) + \rho^R(\psi(y)) \circ \rho^R(x)$.

Definition 2.4 *A Hom-Lie-Rinehart algebra over (A, ϕ) is a six tuple $(A, L, [\cdot, \cdot], \phi, \psi, \rho)$, where A is an associative commutative algebra, L is an A-module, $[\cdot, \cdot] : L \times L \to L$ is a skew symmetric bilinear map, $\phi : A \to A$ is an algebra homomorphism, $\psi : L \to L$ is a linear map satisfying $\psi([x, y]) = [\psi(x), \psi(y)]$, and the R-map $\rho : L \to Der_\phi(A)$ such that following conditions hold.*

(1) *The triple $(L, [\cdot, \cdot], \psi)$ is a Hom-Lie algebra.*
(2) $\psi(a \cdot x) = \phi(a) \cdot \psi(x)$ *for all $a \in A, x \in L$;*
(3) (ρ, ϕ) *is a representation of $(L, [\cdot, \cdot], \psi)$ on A;*
(4) $\rho(a \cdot x) = \phi(a) \cdot \rho(x)$ *for all $a \in A, x \in L$;*
(5) $[x, a \cdot y] = \phi(a) \cdot [x, y] + \rho(x)(a)\psi(y)$ *for all $a \in A, x, y \in L$.*

A Hom-Lie-Rinehart algebra $(A, L, [\cdot, \cdot], \phi, \psi, \rho)$ is said to be regular if the map $\phi : A \to A$ is an algebra automorphism and $\psi : L \to L$ is a bijective map.

3. Decomposition

In this section, we introduce the notion of the Hom-Leibniz-Rinehart algebra as an algebraic analogue of Hom-Leibniz algebroid. In a sequel, we introduce the class of split algebras in the framework of Hom-Leibniz-Rinehart algebras.

Definition 3.1 *A Hom-Leibniz-Rinehart algebra over (A, ϕ) is a seven tuple $(A, L, [\cdot, \cdot], \phi, \psi, \rho^L, \rho^R)$, where A is an associative commutative algebra, L is an A-module, $[\cdot, \cdot] : L \times L \to L$ is a linear map, $\phi : A \to A$ is an algebra homomorphism, $\psi : L \to L$ is a linear map satisfying $\psi([x, y]) = [\psi(x), \psi(y)]$, and the R-maps $\rho^L, \rho^R : L \to \mathrm{Der}_\phi(A)$ satisfying the following conditions.*

(1) *The triple $(L, [\cdot, \cdot], \psi)$ is a Hom-Leibniz algebra.*
(2) $\psi(a \cdot x) = \phi(a) \cdot \psi(x)$ *for all $a \in A, x \in L$;*
(3) (ρ^L, ρ^R, ϕ) *is a representation of $(L, [\cdot, \cdot], \psi)$ on A;*
(4) $\rho^L(a \cdot x) = \phi(a) \cdot \rho^L(x)$ *for all $a \in A, x \in L$;*
(5) $\rho^R(a \cdot x) = \phi(a) \cdot \rho^R(x)$ *for all $a \in A, x \in L$;*

(6) $[x, a\cdot y] = \phi(a)\cdot[x,y] + \rho^L(x)(a)\psi(y)$ *for all* $a\in A, x, y\in L$;

(7) $[a\cdot x, y] = \phi(a)\cdot[x,y] - \rho^R(x)(a)\psi(y)$ *for all* $a\in A, x, y\in L$.

We denote it by (L, A) or just by L if there is not any possible confusion. A Hom-Leibniz-Rinehart algebra $(A, L, [\cdot,\cdot], \phi, \psi, \rho^L, \rho^R)$ is said to be regular if the map $\phi: A\to A$ is an algebra automorphism and $\psi: L\to L$ is a bijective map.

Example 3.2 A Leibniz-Rinehart algebra L over A with the linear map $[\cdot,\cdot]: L\times L\to L$ and the R-maps $\rho^L, \rho^R: L\to \mathrm{Der}(A)$ is a Hom-Leibniz-Rinehart algebra $(A, L, [\cdot,\cdot], \phi, \psi, \rho^L, \rho^R)$, where $\psi = \mathrm{Id}_L, \phi = \mathrm{Id}_A$ and $\rho: L\times L\to \mathrm{Der}_\phi(A) = \mathrm{Der}(A)$.

Example 3.3 A Hom-Leibniz algebra $(L, [\cdot,\cdot], \psi)$ structure over an R-module L gives the Hom-Leibniz-Rinehart algebra $(A, L, [\cdot,\cdot], \phi, \psi, \rho^L, \rho^R)$ with $A = R$, the algebra morphism $\phi = \mathrm{id}_R$ and the trivial action of L on R.

If we consider a Leibniz-Rinehart algebra L over A along with an endomorphism

$$(\phi, \psi): (A, L)\to (A, L)$$

in the category of Leibniz-Rinehart algebras, then we get a Hom-Leibniz-Rinehart algebra $(A, L, [\cdot,\cdot]_\psi, \phi, \psi, \rho^L_\phi, \rho^R_\phi)$ as follows:

(1) $[x,y]_\psi = \psi([x,y])$ for any $x, y\in L$;

(2) $\rho^L_\phi(x)(a) = \phi(\rho^L(x)(a))$ for all $a\in A, x\in L$;

(3) $\rho^R_\phi(x)(a) = \phi(\rho^R(x)(a))$ for all $a\in A, x\in L$.

Definition 3.4 *A Hom-Leibniz algebroid is a tuple* $(\xi, [\cdot,\cdot], \phi, \psi, \rho^L, \rho^R)$, *where* $\xi: A\to M$ *is a vector bundle over a smooth manifold* M, $\phi: M\to M$ *is a smooth map,* $[-,-]: \Gamma(A)\times\Gamma(A)\to\Gamma(A)$ *is a bilinear map, the maps* $\rho^L, \rho^R: \phi^! A\to \phi^! TM$ *is called the anchor and* $\psi: \Gamma(A)\to\Gamma(A)$ *is a linear map such that following conditions are satisfied.*

(1) *The triplet* $(\Gamma(A), [-,-], \psi)$ *is a Hom-Lie algebra;*

(2) $\psi(fX) = \phi^*(F)\psi(X)$ *for all* $X\in\Gamma(A)$, $f\in C^\infty(M)$;

(3) (ρ^L, ρ^R, ϕ^*) *is a representation of* $(\Gamma(A), [-,-], \psi)$ *on* $C^\infty(M)$;

(4) $[X, fY] = \phi^*(f)[X,Y] + \rho^L(X)[f]\psi(Y)$ *for all* $X, Y\in\Gamma(A), f\in C^\infty(M)$;

(5) $[fX, Y] = \phi^*(f)[X,Y] - \rho^R(X)[f]\psi(Y)$ *for all* $X, Y\in\Gamma(A), f\in C^\infty(M)$.

Example 3.5 A Hom-Leibniz algebroid provides a Hom-Leibniz-Rinehart algebra $(C^\infty(M), \Gamma(A), [\cdot,\cdot], \phi^*, \psi, \rho^L, \rho^R,)$, where $\Gamma(A)$ is the space a section of the underline vector bundle $A\to M$ and $\phi^*: C^\infty(M)\to C^\infty(M)$ is canonically defined by the smooth map $\phi: M\to M$.

Example 3.6 Let $(A, L, [\cdot,\cdot]_L, \phi, \psi_L, \rho_L, \rho_R)$ and $(A, M, [\cdot,\cdot]_M, \phi, \psi_M, \rho^L_M, \rho^R_M)$ be two Hom-Leibniz-Rinehart algebras over (A, ϕ). We consider

$$L\times_{\mathrm{Der}_\phi A} M = \{(l, m)\in L\times M: \rho^L_L(l) = \rho^L_M(m), \rho^R_L(l) = \rho^R_M(m)\}.$$

Then $(A, L\times_{\mathrm{Der}_\phi A} M, [\cdot,\cdot], \phi, \psi, \widetilde{\rho}_L, \widetilde{\rho}_R)$ is a Hom-Leibniz-Rinehart algebra, where

(1) The linear bracket $[\cdot,\cdot]$ is given by

$$[(l_1,m_1),(l_2,m_2)] := ([l_1,l_2],[m_1,m_2]),$$

for any $l_1,l_2 \in L$ and $m_1,m_2 \in M$.

(2) The map $\psi : L \times_{\mathrm{Der}_\phi A} M \to L \times_{\mathrm{Der}_\phi A} M$ is given by

$$\psi(l,m) := (\psi_L(l),\psi_M(m)),$$

for any $l \in L$ and $m \in M$.

(3) The action of $L \times_{\mathrm{Der}_\phi A} M$ on A is given by

$$\widetilde{\rho}_L(l,m)(a) := \rho_L^L(l)(a) = \rho_M^L(m)(a),$$
$$\widetilde{\rho}_R(l,m)(a) := \rho_L^R(l)(a) = \rho_M^R(m)(a),$$

for any $l \in L, m \in M$ and $a \in A$.

Next we define homomorphisms of Hom-Leibniz-Rinehart algebras.

Definition 3.7 *Let $(A, L, [\cdot,\cdot]_L, \phi, \psi_L, \rho_L^L, \rho_L^R)$ and $(B, L', [\cdot,\cdot]_{L'}, \psi, \psi_{L'}, \rho_{L'}^L, \rho_{L'}^R)$ be two Hom-Leibniz-Rinehart algebras, then a Hom-Leibniz-Rinehart algebra homomorphism is defined as a pair of maps (g, f), where the map $g : A \to B$ is an R-algebra homomorphism and $f : L \to L'$ is an R-linear map such that following identities hold:*

(1) $f(a \cdot x) = g(a) \cdot f(x)$, for all $x \in L$ and $a \in A$;

(2) $f([x,y]_L) = [f(x), f(y)]_{L'}$, for all $x, y \in L$;

(3) $f(\psi_L(x)) = \psi_{L'}(f(x))$, for all $x \in L$;

(4) $g(\phi(a)) = \psi(g(a))$, for all $a \in A$;

(5) $g(\rho_L^L(x)(a)) = \rho_{L'}^L(f(x))(g(a))$;

(6) $g(\rho_L^R(x)(a)) = \rho_{L'}^R(f(x))(g(a))$, for all $x \in L$ and $a \in A$.

Definition 3.8 *A subalgebra (S, A) of (L, A) is called a* Hom-Leibniz subalgebra, *if (S, A) satisfies $AS \subset S$ such that S acts on A via the composition $S \hookrightarrow L \to Der_\phi(A)$. A Hom-Leibniz subalgebra (I, A) of (L, A) is called an* ideal, *if I is a Hom-Leibniz ideal of L such that $\rho^L(I)(A)L \subseteq I$ and $\rho^R(I)(A)L \subseteq I$.*

The ideal J generated by

$$\{[x,y] + [y,x] : x, y \in L\}$$

plays an important role in mathematics since it determines the non-super Lie character of L. From Hom-Leibniz identity, it is straightforward to check that this ideal satisfies

$$[L, J] = 0. \tag{3.1}$$

Let us introduce the class of split algebras in the framework of Hom-Leibniz algebras from [21]. Denote by H a maximal abelian subalgebra of a Hom-Leibniz algebra L. For a linear functional

$$\gamma : H \to \mathbb{K},$$

we define the root space of L associated to γ as the subspace

$$L_\gamma := \{v_\alpha \in L : [h, \psi(v_\gamma)] = \alpha(h)\psi(v_\gamma), \text{ for any } h \in H\}.$$

The elements $\gamma : H \to \mathbb{K}$ satisfying $L_\gamma \neq 0$ are called roots of L with respect to H and we denote $\Gamma := \{\gamma \in H^* \setminus \{0\} : L_\gamma \neq 0\}$. We call that L is a split regular Hom-Leibniz algebra with respect to H if

$$L = H \oplus \bigoplus_{\gamma\in\Gamma} L_\gamma.$$

We also say that Γ is the root system of L.

Definition 3.9 *A split regular Hom-Leibniz-Rinehart algebra (with respect to an MASA H of the regular Hom-Leibniz algebra L, here MASA means maximal abelian subalgebra) is a regular Hom-Leibniz-Rinehart algebra (L, A) in which the Hom-Leibniz algebra L contains a splitting Cartan subalgebra H and the algebra A is a weight module (with respect to H) in the sense that A can be written as the direct sum $A = A_0 \oplus (\bigoplus_{\alpha\in\Lambda} A_\alpha)$ with $\phi(A_\alpha) \subset A_\alpha$, where*

$$A_\alpha := \{a_\alpha \in A | \rho^L(h)(a_\alpha) = \alpha(h)\phi(a_\alpha),\ \forall h \in H\},$$

for a linear functional $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^*\backslash\{0\} : A_\alpha \neq 0\}$ denotes the weights system of A. The linear subspace A_α, for $\alpha \in \Lambda$, is called the weight space of A associate to α, *the element $\alpha \in \Lambda \cup \{0\}$ are called* weights *of A.*

Lemma 3.10 *Let (L, A) be a Leibniz-Rinehart algebra, where $L = H \oplus (\bigoplus_{\alpha\in\Gamma} L_\gamma),\ A = A_0 \oplus (\bigoplus_{\alpha\in\Lambda} A_\alpha)$, $\psi : L \to L, \phi : A \to A$ are two automorphisms such that $\psi(H) = H, \phi(A_0) = A_0$ and $\phi(A_\alpha) \subset A_\alpha$. By Example 3.3, we know that (L, A) is a regular Hom-Leibniz-Rinehart algebra. Then we have*

$$L = H \oplus \Big(\bigoplus_{\alpha\in\Gamma} \mathfrak{L}_{\alpha\psi^{-1}}\Big),\quad A = A_0 \oplus \Big(\bigoplus_{\alpha\in\Lambda} A_{\alpha\phi^{-1}}\Big),$$

which makes the regular Hom-Leibniz-Rinehart algebra (L, A) be the roots system $\Gamma' = \{\alpha\psi^{-1} : \alpha \in \Gamma\}$ and weights system $\Lambda' := \{\alpha\phi^{-1} : \alpha \in \Lambda\}$.

The following lemma is analogous to the results of [23].

Lemma 3.11 *For any $\gamma, \xi \in \Gamma \cup \{0\}$ and $\alpha, \beta \in \Lambda \cup \{0\}$, the following assertions hold.*

(1) $L_0 = H$;

(2) $\psi(L_\gamma) = L_{\gamma\psi^{-1}}$ and $\psi^{-1}(L_\gamma) = L_{\gamma\psi}$;

(3) If $[L_\gamma, L_\xi] \neq 0$, then $\gamma\psi^{-1} + \xi\psi^{-1} \in \Gamma \cup \{0\}$ and $[L_\gamma, L_\xi] \subset L_{\gamma\psi^{-1}+\xi\psi^{-1}}$;

(4) If $A_\alpha A_\beta \neq 0$, then $\alpha + \beta \in \Lambda \cup \{0\}$ and $A_\alpha A_\beta \subset A_{\alpha+\beta}$;

(5) If $A_\alpha L_\gamma \neq 0$, then $\alpha + \gamma \in \Gamma \cup \{0\}$ and $A_\alpha L_\gamma \subset L_{\alpha+\gamma}$;

(6) If $\rho^L(L_\gamma)A_\alpha \neq 0$, then $\alpha + \gamma \in \Lambda \cup \{0\}$ and $\rho(L_\gamma)A_\alpha \subset A_{\alpha+\gamma}$.

4. Connections of roots

In what follows, L denotes a split regular Hom-Leibniz-Rinehart algebra and

$$L = H \oplus \Big(\bigoplus_{\gamma\in\Gamma} L_\gamma\Big),\quad A = A_0 \oplus \Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big).$$

Given a linear functional $\gamma : H \to \mathbb{K}$, we denote by $-\gamma : H \to \mathbb{K}$ the element in H^* defined by $(-\gamma)(h) := -\gamma(h)$ for all $h \in H$. We also denote $-\Gamma := \{-\gamma : \gamma \in \Gamma\}$. In a similar way we can define $-\Lambda := \{-\alpha : \alpha \in \Lambda\}$. Finally, we denote $\pm\Gamma := \Gamma \cup -\Gamma$ and $\pm\Lambda := \Lambda \cup -\Lambda$.

Definition 4.1 *Let $\gamma, \xi \in \Gamma$. We say that γ is connected to ξ if*

- *Either $\xi = \epsilon\gamma\psi^z$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{1, -1\}$.*
- *Either there exists a family $\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \subset \pm\Lambda \cup \pm\Gamma$, with $n \geq 2$, such that*

(1) $\zeta_1 \in \{\gamma\psi^k | k \in \mathbb{Z}\}$.

(2)
$$\begin{aligned}
&\zeta_1\psi^{-1} + \zeta_2\psi^{-1} \in \pm\Gamma,\\
&\zeta_1\psi^{-2} + \zeta_2\psi^{-2} + \zeta_3\psi^{-1} \in \pm\Gamma,\\
&\zeta_1\psi^{-3} + \zeta_2\psi^{-3} + \zeta_3\psi^{-2} + \zeta_4\psi^{-1} \in \pm\Gamma,\\
&\cdots\\
&\zeta_1\psi^{-i} + \zeta_2\psi^{-i} + \zeta_3\psi^{-i+1} + \cdots + \zeta_{i+1}\psi^{-1} \in \pm\Gamma,\\
&\cdots\\
&\zeta_1\psi^{-n+2} + \zeta_2\psi^{-n+2} + \zeta_3\psi^{-n+3} + \cdots + \zeta_{n-1}\psi^{-1} \in \pm\Gamma.
\end{aligned}$$

(3) $\zeta_1\psi^{-n+1} + \zeta_2\psi^{-n+1} + \zeta_3\psi^{-n+2} + \cdots + \zeta_n\psi^{-1} \in \{\pm\xi\psi^{-m} | m \in \mathbb{Z}\}$.

We will also say that $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ is a connection from γ to ξ.

The proof of the next result is analogous to the one of [21].

Proposition 4.2 *The relation $\sim$ in Γ is an equivalence relation, where $\gamma \sim \xi$ if and only if γ is connected to ξ.*

By Proposition 4.2 we can consider the quotient set

$$\Gamma/\sim = \{[\gamma] : \gamma \in \Gamma\}$$

with $[\gamma]$ being the set of nonzero roots which are connected to γ. Our next goal is to associate an ideal $I_{[\gamma]}$ to $[\gamma]$. Fix $[\gamma] \in \Gamma/\sim$, we start by defining

$$L_{0,[\gamma]} := \Big(\bigoplus_{\xi\in[\gamma], -\xi\in\Lambda} A_{-\xi}L_\xi\Big) + \Big(\bigoplus_{\xi\in[\gamma]} [L_{-\xi}, L_\xi]\Big).$$

Now we define

$$L_{[\gamma]} := \bigoplus_{\xi\in[\gamma]} L_\xi.$$

Finally, we denote by $I_{[\gamma]}$ the direct sum of the two subspaces above:

$$I_{[\gamma]} := L_{0,[\gamma]} \oplus L_{[\gamma]}.$$

Proposition 4.3 *For any $[\gamma] \in \Lambda/\sim$, the following assertions hold.*

(1) $[I_{[\gamma]}, I_{[\gamma]}] \subset I_{[\gamma]}$;

(2) $\psi(I_{[\gamma]}) = I_{[\gamma]}$;

(3) $AI_{[\gamma]} \subset I_{[\gamma]}$;

(4) $\rho^L(I_{[\gamma]})(A)L \subset I_{[\gamma]}$;

(5) *For any $[\gamma] \neq [\delta]$, we have* $[I_{[\gamma]}, I_{[\delta]}] = 0$.

Proof (1) First we check that $[I_{[\gamma]}, I_{[\gamma]}] \subset I_{[\gamma]}$, we can write

$$\begin{aligned}[I_{[\gamma]}, I_{[\gamma]}] &= [L_{0,[\gamma]} \oplus L_{[\gamma]}, L_{0,[\gamma]} \oplus L_{[\gamma]}] \\ &\subset [L_{0,[\gamma]}, L_{[\gamma]}] + [L_{[\gamma]}, L_{0,[\gamma]}] + [L_{[\gamma]}, L_{[\gamma]}]. \end{aligned} \tag{4.1}$$

Given $\delta \in [\gamma]$, we have $[L_{0,[\gamma]}, L_\delta] \subset L_\delta \subset L_{[\gamma]}$. By a similar argument, we get $[L_\delta, L_{0,[\gamma]}] \subset L_{[\gamma]}$.

Next we consider $[L_{[\gamma]}, L_{[\gamma]}]$. If we take $\delta, \eta \in [\gamma]$ such that $[L_\delta, L_\eta] \neq 0$, then $[L_\delta, L_\eta] \subset L_{\delta+\eta}$. If $\delta\psi^{-1} + \eta\psi^{-1} = 0$, we get $[L_\delta, L_{-\delta}] \subset L_{0,[\gamma]}$. Suppose that $\delta\psi^{-1} + \eta\psi^{-1} \in \Gamma$. We infer that $\{\delta, \eta\}$ is a connection from δ to $\delta\psi^{-1} + \eta\psi^{-1}$. The transitivity of $\sim$ now gives that $\delta\psi^{-1} + \eta\psi^{-1} \in [\gamma]$ and so $[L_\delta, L_\eta] \subset L_{[\gamma]}$. Hence

$$[L_{[\gamma]}, L_{[\gamma]}] \in I_{[\gamma]}. \tag{4.2}$$

From (4.1) and (4.2), we get $[I_{[\gamma]}, I_{[\gamma]}] \subset I_{[\gamma]}$.

(2) It is easy to check that $\psi(I_{[\gamma]}) = I_{[\gamma]}$.

(3) and (4) are similar to [23].

(5) We will study the expression $[I_{[\gamma]}, I_{[\delta]}]$. Notice that

$$\begin{aligned}[I_{[\gamma]}, I_{[\delta]}] &= [L_{0,[\gamma]} \oplus L_{[\gamma]}, L_{0,[\delta]} \oplus L_{[\delta]}] \\ &\subset [L_{0,[\gamma]}, L_{[\delta]}] + [L_{[\gamma]}, L_{0,[\delta]}] + [L_{[\gamma]}, L_{[\delta]}]. \end{aligned} \tag{4.3}$$

First we consider $[L_{[\gamma]}, L_{[\delta]}]$ and suppose that there exist $\gamma_1 \in [\gamma], \delta_1 \in [\delta]$ such that $[L_{\gamma_1}, L_{\delta_1}] \neq 0$. As necessarily $\gamma_1\psi^{-1} \neq -\delta_1\psi^{-1}$, then $\gamma_1\psi^{-1}+\delta_1\psi^{-1} \in \Gamma$. So $\{\gamma_1, \delta_1, -\gamma_1\psi^{-1}\}$ is a connection between γ_1 and δ_1. By the transitivity of the connection relation we see $\gamma \in [\delta]$, a contradiction. Hence $[L_{\gamma_1}, L_{\delta_1}] = 0$, and so

$$[L_{[\gamma]}, L_{[\delta]}] = 0. \tag{4.4}$$

By the definition of $L_{0,[\gamma]}$, we have

$$[L_{0,[\gamma]}, L_{[\delta]}] = \Big[\Big(\sum_{\gamma_1 \in [\gamma], -\gamma_1 \in \Lambda} A_{-\gamma_1} L_{\gamma_1}\Big) + \Big(\sum_{\gamma_1 \in [\gamma]} [L_{-\gamma_1}, L_{\gamma_1}]\Big), L_{[\delta]}\Big].$$

Suppose that there exist $\gamma_1 \in [\gamma]$ and $\delta_1 \in [\delta]$ such that

$$[L_{\delta_1}, [L_{\gamma_1}, L_{-\gamma_1}]] = 0.$$

Suppose that $[L_{\delta_1}, [L_{\gamma_1}, L_{-\gamma_1}]] \neq 0$, then Hom-Leibniz identity gives

$$\begin{aligned} 0 \neq &[\psi\psi^{-1}(L_{\delta_1}), [L_{\gamma_1}, L_{-\gamma_1}]] \\ \subset &[[\psi^{-1}(L_{\delta_1}), L_{\gamma_1}], \psi(L_{-\gamma_1})] + [[\psi^{-1}(L_{\delta_1}), L_{-\gamma_1}], \psi(L_{\gamma_1})]. \end{aligned}$$

Hence

$$[\psi^{-1}(L_{\delta_1}), L_{\gamma_1}] + [\psi^{-1}(L_{\delta_1}), L_{-\gamma_1}] \neq 0,$$

which contradicts (4.4). Therefore, $[L_{\delta_1}, [L_{\gamma_1}, L_{-\gamma_1}]] = 0$.

For the expression $[A_{-\gamma_1} L_{\gamma_1}, L_{[\delta]}]$, suppose there exists $\delta_1 \in [\delta]$ such that $[A_{-\gamma_1} L_{\gamma_1}, L_{\delta_1}] \neq 0$. By Definition 3.1, we have

$$[A_{-\gamma_1} L_{\gamma_1}, L_{\delta_1}] = [L_{\delta_1}, A_{-\gamma_1} L_{\gamma_1}] \subset \phi(A_{-\gamma_1})[L_{\delta_1}, L_{\gamma_1}] + \rho^L(L_{\delta_1})(A_{-\gamma_1}) L_{\delta_1\psi^{-1}}.$$

By the discussion above, we get $[L_{\delta_1}, L_{\gamma_1}] = 0$. Since $[A_{-\gamma_1}L_{\gamma_1}, L_{[\delta]}] \neq 0$, it follows that $0 \neq \rho^L(L_{\delta_1})(A_{-\gamma_1})L_{\delta_1\psi^{-1}} \subset A_{\delta_1-\gamma_1}L_{\delta_1\psi^{-1}}$. Thus $A_{\delta_1-\gamma_1} \neq 0$ and $\delta_1 - \gamma_1 \in \Lambda \cup \{0\}$. $\delta_1 \sim \gamma_1$, a contradiction. So $[A_{-\gamma_1}L_{\gamma_1}, L_{[\delta]}] = 0$. Therefore, we have $[L_{0,[\gamma]}, L_{[\delta]}] = 0$. In a similar way we can prove $[L_{[\gamma]}, L_{0,[\delta]}] = 0$, we conclude $[I_{[\gamma]}, I_{[\delta]}] = 0$. $\square$

Definition 4.4 *A Hom-Leibniz-Rinehart algebra (L, A) is simple if $[L, L] \neq 0$, $AA \neq 0$, $AL \neq 0$ and its only ideals are $\{0\}$, J, L and the kernel of ρ.*

Theorem 4.5 *The following assertions hold.*

(1) For any $[\gamma] \in \Gamma/\sim$, the linear space $I_{[\gamma]} = L_{0,[\gamma]} + L_{[\gamma]}$ of L associated to $[\gamma]$ is an ideal of L.

(2) If L is simple, then there exists a connection from γ to δ for any $\gamma, \delta \in \Gamma$ and

$$H = \Big(\sum_{\gamma\in\Gamma,-\gamma\in\Lambda} A_{-\gamma}L_\gamma\Big) + \Big(\sum_{\gamma\in\Gamma}[L_{-\gamma}, L_\gamma]\Big).$$

Proof (1) Since $[I_{[\gamma]}, H] + [H, I_{[\gamma]}] \subset I_{[\gamma]}$, by Proposition 3.3, we have

$$[I_{[\gamma]}, L] = \Big[I_{[\gamma]}, H \oplus \Big(\bigoplus_{\xi\in[\gamma]} L_\xi\Big) \oplus \Big(\bigoplus_{\delta\notin[\gamma]} L_\delta\Big)\Big] \subset I_{[\gamma]}$$

and

$$[L, I_{[\gamma]},] = \Big[H \oplus \Big(\bigoplus_{\xi\in[\gamma]} L_\xi\Big) \oplus \Big(\bigoplus_{\delta\notin[\gamma]} L_\delta\Big), I_{[\gamma]}\Big] \subset I_{[\gamma]}.$$

Furthermore,

$$[I_{[\gamma]}, L]+[L, I_{[\gamma]}] = \Big[I_{[\gamma]}, H\oplus\Big(\bigoplus_{\xi\in[\gamma]} L_\xi\Big)\oplus\Big(\bigoplus_{\delta\notin[\gamma]} L_\delta\Big)\Big]+\Big[H\oplus\Big(\bigoplus_{\xi\in[\gamma]} L_\xi\Big)\oplus\Big(\bigoplus_{\delta\notin[\gamma]} L_\delta\Big), I_{[\gamma]}\Big] \subset I_{[\gamma]}.$$

As we also have $\psi(I_{[\gamma]}) = I_{[\gamma]}$. So we show that $I_{[\gamma]}$ is a Hom-Leibniz ideal of L. We also have that $I_{[\gamma]}$ is an A-module, then we conclude $I_{[\gamma]}$ is an ideal of L.

(2) The simplicity of L implies $I_{[\gamma]} \in \{J, L, \ker\rho^L\}$. If some $\gamma \in \Gamma$ is such that $I_{[\gamma]} = L$, then $[\gamma] = \Gamma$. Otherwise, if $I_{[\gamma]} = J$ for any $\alpha \in \Gamma$, then $[\gamma] = [\delta]$ for any $\gamma, \delta \in \Gamma$, and so $[\gamma] = \Gamma$. Otherwise, if $I_{[\gamma]} = \ker\rho$ for any $\gamma \in \Gamma$, then $[\gamma] = [\delta]$ for any $\gamma, \delta \in \Gamma$, and so $[\gamma] = \Gamma$. Thus $H = (\sum_{\gamma\in\Gamma,-\gamma\in\Lambda} A_{-\gamma}L_\gamma) + (\sum_{\gamma\in\Gamma}[L_{-\gamma}, L_\gamma])$. $\square$

Theorem 4.6 *We have*

$$L = U + \sum_{[\gamma]\in\Lambda/\sim} I_{[\gamma]},$$

where U is a linear complement in H of $(\sum_{\gamma\in\Gamma,-\gamma\in\Lambda} A_{-\gamma}L_\gamma) + (\sum_{\gamma\in\Gamma}[L_{-\gamma}, L_\gamma])$ and any $I_{[\gamma]}$ is one of the ideals of L described in Theorem 4.4, satisfying $[I_{[\gamma]}, I_{[\delta]}] = 0$ if $[\gamma] \neq [\delta]$.

Proof $I_{[\gamma]}$ is well defined and is an ideal of L and it is clear that

$$L = H \oplus \sum_{[\gamma]\in\Gamma} L_{[\gamma]} = U + \sum_{[\gamma]\in\Gamma/\sim} I_{[\gamma]}.$$

Finally, Proposition 4.3 gives us $[I_{[\gamma]}, I_{[\delta]}] = 0$ if $[\gamma] \neq [\delta]$. $\square$

Definition 4.7 *The annihilator of a Hom-Leibniz-Rinehart algebra L is the set*

$$Z(L) := \{v \in L : [v, L] + [L, v] = 0 \text{ and } \rho(v) = 0\}.$$

Corollary 4.8 *If $Z(L) = 0$ and $H = (\sum_{\gamma\in\Gamma,-\gamma\in\Lambda} A_{-\gamma}L_\gamma) + (\sum_{\gamma\in\Gamma}[L_{-\gamma}, L_\gamma])$. Then L is the direct sum of the ideals given in Theorem 4.5,*

$$L = \bigoplus_{[\gamma]\in\Gamma/\sim} I_{[\gamma]}.$$

Furthermore, $[I_{[\gamma]}, I_{[\delta]}] = 0$ if $[\gamma] \neq [\delta]$.

Proof Since $H = (\sum_{\gamma\in\Gamma,-\gamma\in\Lambda} A_{-\gamma}L_\gamma)+(\sum_{\gamma\in\Gamma}[L_{-\gamma}, L_\gamma])$, it follows that $L = \sum_{[\gamma]\in\Gamma/\sim} I_{[\gamma]}$. To verify the direct character of the sum, take some $v \in I_{[\gamma]} \cap (\sum_{[\delta]\in\Gamma/\sim,[\delta]\neq[\gamma]} I_{[\delta]})$. Since $v \in I_{[\gamma]}$, the fact $[I_{[\gamma]}, I_{[\delta]}] = 0$ when $[\gamma] \neq [\delta]$ gives us

$$\Big[v, \sum_{[\delta]\in\Gamma/\sim,[\delta]\neq[\gamma]} I_{[\delta]}\Big] + \Big[\sum_{[\delta]\in\Gamma/\sim,[\delta]\neq[\gamma]} I_{[\delta]}, v\Big] = 0.$$

In a similar way, $v \in \sum_{[\delta]\in\Gamma/\sim,[\delta]\neq[\gamma]} I_{[\delta]}$ implies $[v, I_{[\gamma]}] + [I_{[\gamma]}, v] = 0$. It is easy to obtain that $\rho^L(v) = 0$. That is $v \in Z(L)$ and so $v = 0$. $\square$

5. Decompositions of A

We will discuss the weight spaces and decompositions of A similar to [23] and omit the proof.

Definition 5.1 *Let $\alpha, \beta \in \Lambda$ we say that α and β are connected if*

- *Either $\beta = \varepsilon\alpha$ for some $\varepsilon \in \{1, -1\}$;*
- *Either there exists a family $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset \pm\Lambda \cup \pm\Gamma$, with $n \geq 2$, such that*

(1) $\sigma_1 = \alpha$.

(2) $\sigma_1 + \sigma_2 \in \pm\Lambda \cup \pm\Gamma$,

$\sigma_1 + \sigma_2 + \sigma_3 \in \pm\Lambda \cup \pm\Gamma$,

$\cdots$

$\sigma_1 + \sigma_2 \cdots + \sigma_{n-1} \in \pm\Lambda \cup \pm\Gamma$,

(3) $\sigma_1 + \sigma_2 \cdots + \sigma_n \in \{\beta, -\beta\}$.

We will also say that $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is a connection *from α to β.*

Proposition 5.2 *The relation $\approx$ in Λ is an equivalence relation, where $\alpha \approx \beta$ if and only if α is connected to β.*

By Proposition 5.2, we can consider the quotient set

$$\Lambda/\approx := \{[\alpha] | \alpha \in \Lambda\},$$

where $[\alpha]$ denotes the set of nonzero weights of A which are connected to α. In the following we will associate an adequate ideal $\mathcal{A}_{[\alpha]}$ to any $[\alpha]$. For a fixed $\alpha \in \Lambda$, we define

$$A_{0,[\alpha]} := \Big(\sum_{\beta\in[\alpha],-\beta\in\Lambda} \rho^L(L_{-\beta})(A_\beta) + \Big(\sum_{\beta\in[\alpha]} A_{-\beta}, A_\beta\Big)\Big) \subset A_0, \quad A_{[\alpha]} := \bigoplus_{\beta\in[\alpha]} A_\beta.$$

Then we denote by $\mathcal{A}_{[\alpha]}$ the direct sum of the two subspaces above,

$$\mathcal{A}_{[\alpha]} := A_{0,[\alpha]} \oplus A_{[\alpha]}.$$

Proposition 5.3 *For any $\alpha, \beta \in \Lambda$, the following assertions hold.*

(1) $\mathcal{A}_{[\alpha]}\mathcal{A}_{[\alpha]} \subset \mathcal{A}_{[\alpha]}$.

(2) If $[\alpha] \neq [\beta]$, then $\mathcal{A}_{[\alpha]}\mathcal{A}_{[\beta]} = 0$.

Theorem 5.4 *Let A be a commutative and associative algebra associated to a Hom-Leibniz-Rinehart algebra L. Then the following assertions hold.*

(1) For any $[\alpha] \in \Lambda/\approx$, the linear space $\mathcal{A}_{[\alpha]} = A_{0,[\alpha]} \oplus A_{[\alpha]}$ of A associated to $[\alpha]$ is an ideal of A.

(2) If A is simple, then all weights of Λ are connected. Furthermore,

$$A_0 = \sum_{-\alpha\in\Gamma,\alpha\in\Lambda} \rho^L(L_{-\alpha})(A_\alpha) + \Big(\sum_{\alpha\in\Lambda} A_{-\alpha}A_\alpha\Big).$$

Theorem 5.5 *Let A be a commutative and associative algebra associated to a Hom-Leibniz-Rinehart algebra L. Then*

$$A = V + \sum_{[\alpha]\in\Lambda/\approx} \mathcal{A}_{[\alpha]},$$

where V is a linear complement in A_0 of $\sum_{-\alpha\in\Gamma,\alpha\in\Lambda} \rho^L(L_{-\alpha})(A_\alpha) + (\sum_{\alpha\in\Lambda} A_{-\alpha}A_\alpha)$ and any $A_{[\alpha]}$ is one of the ideals of A described in Theorem 4.4 (1), satisfying $\mathcal{A}_{[\alpha]}\mathcal{A}_{[\beta]} = 0$, whenever $[\alpha] \neq [\beta]$.

Let us denote by $Z(A)$ the center of A, that is, $Z(A) := \{a \in L | aA = 0\}$.

Corollary 5.6 *Let (L, A) be a Hom-Leibniz-Rinehart algebra. If $Z(A) = 0$ and*

$$A_0 = \sum_{-\alpha\in\Gamma,\alpha\in\Lambda} \rho^L(L_{-\alpha})(A_\alpha) + \Big(\sum_{\alpha\in\Lambda} A_{-\alpha}A_\alpha\Big),$$

then A is the direct sum of the ideals given in Theorem 4.5, that is,

$$A = \sum_{[\alpha]\in\Lambda/\approx} \mathcal{A}_{[\alpha]},$$

satisfying $\mathcal{A}_{[\alpha]}\mathcal{A}_{[\beta]} = 0$, whenever $[\alpha] \neq [\beta]$.

6. The simple components

In this section we focus on the simplicity of split regular Hom-Leibniz-Rinehart algebra (L, A) by centering our attention in those of maximal length. From now on we always assume that Λ is symmetric in the sense that $\Lambda = -\Lambda$.

Lemma 6.1 *Let (L, A) be a split regular Hom-Leibniz-Rinehart and I an ideal of L. Then $I = (I \cap H) \oplus (I \cap \bigoplus_{\gamma\in\Gamma} L_\gamma)$.*

Proof Since (L, A) is split, we get $L = H \oplus (\bigoplus_{\gamma\in\Gamma} L_\gamma)$. By the assumption that I is an ideal of L, it is clear that I is a submodule of L. Since a submodule of a weight module is again a weight module. Thus I is a weight module and therefore, $I=(I\cap H)\oplus(I\cap\bigoplus_{\gamma\in\Gamma}L_\gamma)$. □

Lemma 6.2 *Let (L,A) be a split regular Hom-Leibniz-Rinehart algebra with $Z(L)=0$ and I an ideal of L. If $I\subset H$, then $I=\{0\}$.*

Proof Since $I\subset H$, $[I,H]+[H,I]\subset[H,H]=0$. It follows that $[I,L]+[L,I]=[I,\bigoplus_{\gamma\in\Gamma}L_\gamma]+[\bigoplus_{\gamma\in\Gamma}L_\gamma,I]\subset H\cap(\bigoplus_{\gamma\in\Gamma}L_\gamma)=0$. So $I\subset Z(L)=0$. □

Observe that if L is of maximal length, then we have

$$I=(I\cap H)\oplus\Big(\bigoplus_{\gamma\in\Gamma^I}L_\gamma\Big), \tag{6.1}$$

where $\Gamma^I=\{\gamma\in\Gamma:I\cap L_\gamma\neq0\}$.

In particular, in case $I=J$, we get

$$J=(J\cap H)\oplus\Big(\bigoplus_{\gamma\in\Gamma^J}L_\gamma\Big) \tag{6.2}$$

with $\Gamma^J=\{\gamma\in\Gamma:J\cap L_\gamma\neq0\}=\{\gamma\in\Gamma:0\neq L_\gamma\subset J\}$.

From here, we can write

$$\Gamma=\Gamma^J\cup\Gamma^{\neg J}, \tag{6.3}$$

where $\Gamma^{\neg J}=\{\gamma\in\Gamma:L_\gamma\neq0 \text{ and } J\cap L_\gamma=0\}$. Therefore, we can write

$$L=H\oplus\Big(\bigoplus_{\gamma\in\Gamma^J}L_\gamma\Big)\oplus\Big(\bigoplus_{\delta\in\Gamma^{\neg J}}L_\delta\Big). \tag{6.4}$$

Let us introduce the notion of root-multiplicativity in the framework of split regular Hom-Leibniz-Rinehart algebras of maximal length, in a similar way to the ones for split regular Hom-Lie Rinehart algebras in [23].

Definition 6.3 *A split regular Hom-Leibniz-Rinehart algebra (L,A) is called* root-multiplicative *if the following conditions hold:*

(1) Given $\gamma,\delta\in\Gamma^{\neg J}$ such that $\gamma\psi^{-1}+\delta\psi^{-1}\in\Gamma$, then $[L_\gamma,L_\delta]\neq0$.

(2) Given $\gamma\in\Gamma^J,\delta\in\Gamma^{\neg J}$ such that $\gamma\psi^{-1}+\delta\psi^{-1}\in\Gamma^J$, then $[L_\gamma,L_\delta]\neq0$.

(3) Given $\alpha\in\Lambda,\gamma\in\Gamma$ such that $\alpha+\gamma\in\Gamma$, then $A_\alpha L_\gamma\neq0$.

(4) If $\alpha+\beta\in\Lambda$, then $A_\alpha A_\beta\neq0$.

Definition 6.4 *A split regular Hom-Leibniz-Rinehart (L,A) is called of maximal length if dim L_γ=dim $A_\alpha=1$ for any $\gamma\in\Gamma$ and $\alpha\in\Lambda$.*

Proposition 6.5 *Suppose $H=(\sum_{\gamma\in\Gamma^{\neg J},-\gamma\in\Lambda}A_{-\gamma}L_\gamma)+(\sum_{\gamma\in\Gamma^{\neg J}}[L_{-\gamma},L_\gamma])$, $Z_{\rm Lie}(L)=0$ and root-multiplicative. If $\Gamma^{\neg J}$ has all of its roots $\neg J$-connected, then any ideal I of L such that $I\nsubseteq H\oplus J$, then $I=L$.*

Proof By (6.1) and (6.3), we can write

$$I=(I\cap H)\oplus\Big(\bigoplus_{\gamma\in\Gamma^{\neg J,I}}L_\gamma\Big),$$

where $\Gamma^{\neg J,I} = \Gamma^{\neg J} \cap \Gamma^{I}$ and $\Gamma^{J,I} = \Gamma^{J} \cap \Gamma^{I}$. Since $I \nsubseteq H \oplus J$, there exists $\gamma_0 \in \Gamma^{\neg J}$ such that

$$0 \neq L_{\gamma_0} \subset I. \tag{6.5}$$

By Lemma 3.11, $\psi(L_{\gamma_0}) = L_{\gamma_0\psi^{-1}}$. Eq. (6.5) gives us $\psi(L_{\gamma_0}) \subset \psi(I) = I$. So $L_{\gamma_0\psi^{-1}} \subset I$. Similarly, we get

$$L_{\gamma_0\psi^{-n}} \subset I, \text{ for } n \in \mathbb{N}. \tag{6.6}$$

For any $\delta \in \Gamma^{\neg J}$, $\beta \notin \pm\gamma_0\psi^{-n}$, for $n \in \mathbb{N}$, the fact that γ_0 and γ are $\neg J$-connected gives us a $\neg J$-connection $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset \Gamma^{\neg J}$ from γ_0 to δ such that

$$\begin{aligned}
&\gamma_1 = \gamma_0 \in \Gamma^{\neg J}, \gamma_k \in \Gamma^{\neg J}, \text{ for } k = 2, \ldots, n,\\
&\gamma_1\psi^{-1} + \gamma_2\psi^{-1} \in \Gamma^{\Upsilon},\\
&\qquad\qquad\qquad \cdots\\
&\gamma_1\psi^{-n+1} + \gamma_2\psi^{-n+1} + \gamma_3\psi^{-n+2} + \cdots + \gamma_{n-1}\psi^{-2} + \gamma_n\psi^{-1} \in \Gamma^{\Upsilon},\\
&\gamma_1\psi^{-n+1} + \gamma_2\psi^{-n+1} + \gamma_3\psi^{-n+2} + \cdots + \gamma_i\psi^{-n+i-1} + \cdots +\\
&\quad \gamma_{n-1}\psi^{-2} + \gamma_n\psi^{-1} \in \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}.
\end{aligned}$$

Taking into account $\gamma_1 = \gamma_0 \in \Gamma^{\neg J}$, if $\gamma_2 \in \Lambda$ (resp., $\gamma_2 \in \Gamma^{\neg J}$), the root-multiplicativity and maximal length of (L, A) allow us to assert

$$0 \neq A_{\gamma_1} L_{\gamma_2} = L_{\gamma_1+\gamma_2} \text{ (resp., } 0 \neq [L_{\gamma_1}, L_{\gamma_2}] = L_{\gamma_1\psi^{-1}+\gamma_2\psi^{-1}}).$$

By (6.6), we have

$$0 \neq L_{\gamma_1\psi^{-1}+\gamma_2\psi^{-1}} \subset I.$$

We can discuss in a similar way from $\gamma_1\psi^{-1} + \gamma_2\psi^{-1} \in \Gamma^{\neg J}, \gamma_3 \in \Lambda \cup \Gamma^{\neg J}$ and $\gamma_1\psi^{-2} + \gamma_2\psi^{-2} + \gamma_3\psi^{-1} \in \Gamma^{\neg J}$ to get

$$0 \neq [L_{\gamma_1\psi^{-1}+\gamma_2\psi^{-1}}, L_{\gamma_3}] = L_{\gamma_1\psi^{-2}+\gamma_2\psi^{-2}+\gamma_3\psi^{-1}}.$$

Thus we have

$$0 \neq L_{\gamma_1\psi^{-2}+\gamma_2\psi^{-2}+\gamma_3\psi^{-1}} \subset I.$$

Following this process with the $\neg J$-connection $\{\gamma_1, \gamma_2, ..., \gamma_n\}$, we obtain that

$$0 \neq L_{\gamma_1\psi^{-n+1}+\gamma_2\psi^{-n+1}+\gamma_3\psi^{-n+2}+\cdots+\gamma_n\psi^{-1}} \subset I.$$

It follows that either

$$L_{\delta\psi^{-m}} \subset I \text{ or } L_{-\delta\psi^{-m}} \subset I \tag{6.7}$$

for any $\delta \in \Gamma^{\neg J}, m \in \mathbb{N}$. Moreover, we have

$$\delta\psi^{-m} \in \Gamma^{\neg J}. \tag{6.8}$$

Since $H = (\sum_{\gamma\in\Gamma^{\neg J}, -\gamma\in\Lambda} A_{-\gamma}L_{\gamma}) + (\sum_{\gamma\in\Gamma^{\neg J}} [L_{-\gamma}, L_{\gamma}])$, by (6.7) and (6.8), we get

$$H \subset I. \tag{6.9}$$

Now, for any $\Upsilon \in \{J, \neg J\}$, given any $\delta \in \Gamma^J$, the facts $\delta \neq 0, H \subset I$ and the maximal length of (L, A) show that

$$L_\delta = [L_{\delta\psi}, H] \subset I.$$

The decomposition of L in (6.4) finally gives us $H = I$. $\square$

Another interesting notion related to a split Hom-Leibniz-Rinehart algebra of maximal length (L, A) is Lie annihilator. Write $L = H \oplus (\bigoplus_{\gamma\in\Gamma^{\neg J}} L_\gamma) \oplus (\bigoplus_{\delta\in\Gamma^J} L_\delta)$.

Definition 6.6 *The Lie-annihilator of a split Hom-Leibniz-Rinehart algebra of maximal length (L, A) is the set*

$$Z_{\mathrm{Lie}}(L) := \Big\{v \in L : \Big[v, H \oplus \Big(\bigoplus_{\gamma\in\Gamma^{\neg J}} L_\gamma\Big)\Big] + \Big[H \oplus \Big(\bigoplus_{\gamma\in\Gamma^{\neg J}} L_\gamma\Big), v\Big] = 0 \text{ and } \rho(v) = 0\Big\}.$$

Observe that $Z(L) \subset Z_{\mathrm{Lie}}(L)$.

In the following, we will discuss the relation between the decompositions of L and A of a Hom-Leibniz-Rinehart algebra (L, A).

Definition 6.7 *A split regular Hom-Leiniz-Rinehart algebra (L, A) is tight if $Z_{\mathrm{Lie}}(L) = 0$, $Z(A) = 0$, $AA = A$, $AL = L$ and*

$$H = \Big(\sum_{\gamma\in\Gamma^{\neg J}, -\gamma\in\Lambda} A_{-\gamma}L_\gamma\Big) + \Big(\sum_{\gamma\in\Gamma^{\neg J}} [L_{-\gamma}, L_\gamma]\Big),$$
$$A_0 = \Big(\sum_{-\alpha\in\Gamma^{\neg J}, \alpha\in\Lambda} \rho(L_{-\alpha})(A_\alpha)\Big) + \Big(\sum_{\alpha\in\Lambda} A_{-\alpha}A_\alpha\Big).$$

Remark 6.8 Let (L, A) be a tight split regular Hom-Leibniz-Rinehart algebra, then

$$L = \sum_{[\gamma]\in\Gamma^{\neg J}/\sim} I_{[\gamma]}, \quad A = \sum_{[\alpha]\in\Lambda/\approx} \mathcal{A}_{[\alpha]}$$

with any $I_{[\gamma]}$ an ideal of L verifying $[I_{[\gamma]}, I_{[\delta]}] = 0$ if $[\gamma] \neq [\delta]$ and any $\mathcal{A}_{[\alpha]}$ an ideal of A satisfying $\mathcal{A}_{[\alpha]}\mathcal{A}_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$.

Proposition 6.9 *Let (L, A) be a tight split regular Hom-Leibniz-Rinehart algebra, then for any $[\gamma] \in \Gamma^{\neg J}/\sim$ there exists a unique $[\alpha] \in \Lambda/\approx$ such that $\mathcal{A}_{[\alpha]}I_{[\gamma]} = 0$.*

Proof Similar to Proposition 4.2 in [21]. $\square$

Theorem 6.10 *Let (L, A) be a tight split regular Hom-Leibniz-Rinehart algebra, then*

$$L = \sum_{i\in\Gamma^{\neg J}/I} L_i, \quad A = \sum_{j\in K} A_j$$

with any L_i a nonzero ideal of L and any A_j a nonzero ideal of A. Furthermore, for any $i \in I$ there exists a unique $\tilde{j} \in K$ such that $A_{\tilde{j}}L_i = 0$.

Theorem 6.11 *Let (L, A) be a tight split regular Hom-Leibniz-Rinehart algebra of maximal length and root multiplicative. If Γ^J, $\Gamma^{\neg J}$ are symmetric and $\Gamma^{\neg J}$ has all of its roots $\neg J$-connected, then any ideal I of L such that $I \subseteq J$ satisfies either $I = J$ or $J = I \oplus I'$ with I' an* *ideal of L.*

Proof By (6.1), we can write

$$I = (I \cap H) \oplus \Big(\bigoplus_{\gamma \in \Gamma^{JI}} L_\gamma \Big), \tag{6.10}$$

and with $\Gamma^{JI} \subset \Gamma^J$. For any $\gamma \notin \Gamma^J$, we have

$$[J \cap H, L_\gamma] + [L_\gamma, J \cap H] \subset L_\gamma \subset J.$$

Hence, in case $[J \cap H, L_\gamma] + [L_\gamma, J \cap H] \neq 0$ we have $\gamma \in \Gamma^J$, a contradiction. Hence $[J \cap H, L_\gamma] + [L_\gamma, J \cap H] = 0$, and so

$$J \cap H \subset Z_{\mathrm{Lie}}(L). \tag{6.11}$$

Taking into account $I \cap H \subset J \cap H = 0$, we also write

$$I = \bigoplus_{\delta \in \Gamma^{JI}} L_\delta$$

with $\Gamma^{JI} \subset \Gamma^J$. Hence, we can take some $\delta_0 \in \Gamma^I$ such that

$$0 \neq L_{\delta_0} \subset I.$$

Now, we can argue with the root-multiplicativity and the maximal length of L as in Proposition 6.5 to conclude that given any $\delta \in \Gamma^J$, there exists a $\neg J$-connection $\{\delta_1, \delta_2, \ldots, \delta_r\}$ from δ_0 to δ such that

$$0 \neq [[...[L_{\delta_0}, L_{\delta_2}], ...], L_{\delta_r}] \in L_{\delta\psi^{-m}}, \text{ for } m \in \mathbb{N}$$

and so

$$L_{\epsilon\delta\psi^{-m}} \subset I, \quad \text{for some } \epsilon \in \pm 1, m \in \mathbb{N}. \tag{6.12}$$

Note that $\delta \in \Gamma^J$ indicates $L_\delta \in J$. By Lemma 3.11, $\psi(L_\delta) = L_{\delta\psi^{-1}}$. Since L is of maximal length, we have $\psi(L_\delta) \subset \psi(J) = J$. So $L_{\delta\psi^{-1}} \subset I$. Similarly, we get

$$L_{\delta\psi^{-m}} \in J, \text{ for } m \in \mathbb{N}. \tag{6.13}$$

Hence we can argue as above with the root-multiplicativity and maximal length of L from δ instead of δ_0, to get that in case $\epsilon\delta_0\phi^{-m} \in \Gamma^J$ for some $\epsilon \in \pm 1$, then $0 \neq L_{\epsilon\delta_0\psi^{-m}} \in I$.

The decomposition of J in (6.12) finally gives us $I = J$.

Now suppose there is not any $\delta_0 \in \Gamma^{JI}$ such that $-\delta_0 \in \Gamma^{JI}$. Then we have

$$\Gamma^J = \Gamma^{JI} \dot{\cup} - \Gamma^{JI}, \tag{6.14}$$

where $-\Gamma^{JI} := \{-\gamma | \gamma \in \Gamma^{JI}\}$. Define

$$I' := \Big(\sum_{-\gamma \in -\Gamma^{JI}, \gamma \in \Lambda} A_\gamma L_{-\gamma} \Big) \oplus \Big(\bigoplus_{-\gamma \in -\Gamma^{JI}} L_{-\gamma} \Big). \tag{6.15}$$

First, we claim that I' is a Hom-Leibniz-ideal of L. In fact, By Lemma 3.11, $\psi(L_{-\gamma}) \subset L_{-\gamma\psi^{-1}}$, $-\gamma\psi^{-1} \in -\Gamma^{JI}$ and $\psi(A_\gamma L_{-\gamma}) \subset \psi(L_0) \subset L_0$ if $A_\gamma L_{-\gamma} \neq 0$ (otherwise is trivial). So $\psi(I') \subset I'$.

Since $A_\gamma L_{-\gamma} \subset L_0$, by (6.15), we have

$$[L, I'] = \Big[H \oplus \Big(\bigoplus_{\delta\in\Gamma^{\neg J}} L_\delta\Big), \Big(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma}\Big) \oplus \Big(\bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma}\Big)\Big] \subset$$

$$\Big[\bigoplus_{\delta\in\Gamma^{\neg J}} L_\delta, \Big(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma}\Big)\Big] + \Big[\bigoplus_{\delta\in\Gamma} L_\delta, \bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma}\Big] + \sum_{-\gamma\in-\Gamma^{JI}} L_{-\gamma}. \quad (6.16)$$

For the expression $[\bigoplus_{\delta\in\Gamma^{\neg J}} L_\delta, (\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma})]$ in (6.16), if some $[L_\delta, A_\gamma L_{-\gamma}] \neq 0$, we have that in case $\delta = -\gamma$, $[L_{-\gamma}, A_\gamma L_{-\gamma}] \subset L_{-\gamma\psi^{-1}} \subset I'$, and in case $\delta = \gamma$, since I is a Hom-Leibniz-ideal of L, $-\gamma \notin \Gamma^{JI}$ implies $[L_{-\gamma}, A_{-\gamma} L_\gamma] = 0$. By the maximal length of L and the symmetry of Γ, we have $[L_\gamma, A_\gamma L_{-\gamma}] = 0$. Suppose $\delta \notin \{\gamma, -\gamma\}$. By Definition 3.3,

$$[L_\delta, A_\gamma L_{-\gamma}] \subset \phi(A_\gamma)[L_\delta, L_{-\gamma}] + \rho^L(L_\delta)(A_\gamma)L_{-\gamma}.$$

Since (L, A) is regular, $\phi(A_\gamma) \subset A_\gamma$. As $[L_\delta, A_\gamma L_{-\gamma}] \neq 0$, we get

$$A_\gamma[L_\delta, L_{-\gamma}] \neq 0 \text{ or } \rho^L(L_\delta)(A_\gamma)L_{-\gamma} \neq 0.$$

By the maximal length of L, either $A_\gamma[L_\delta, L_{-\gamma}] = L_{\gamma+(\delta-\gamma)\psi^{-1}}$ or $\rho^L(L_\delta)(A_\gamma)L_{-\gamma} = L_\delta$. In both cases, since $\gamma \in \Gamma^{JI}$, by the root-multiplicativity of L, we have $L_{-\delta} \subset I$ and therefore $-\delta \in \Gamma^{JI}$. That is, $L_\delta \subset I'$. So $[\bigoplus_{\delta\in\Gamma^{\neg J}} L_\delta, (\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma})] \subset I'$.

For the expression $[\bigoplus_{\delta\in\Gamma^{\neg J}} L_\delta, \bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma}]$ in (6.16), if some $[L_\delta, L_{-\gamma}] \neq 0$, then

$$[L_\delta, L_{-\gamma}] = L_{(\delta-\gamma)\psi^{-1}}.$$

On the one hand, let $\delta - \gamma \neq 0$. Since $\gamma \in \Gamma^{JI}$, by the root-multiplicativity of L, we have $[L_\gamma, L_{-\delta}] = L_{(\gamma-\delta)\psi^{-1}} \subset I$. So $(\delta-\gamma)\psi^{-1} \in \Gamma_I$ and therefore, $L_{(\delta-\gamma)\psi^{-1}} \subset I'$. On the other hand, let $\delta - \gamma = 0$. Suppose $[L_\gamma, L_{-\gamma}] \neq 0$, since $\gamma \in \Gamma^{JI}$, we get $[L_\gamma, L_{-\gamma}] \subset I$. Thus $L_{-\gamma} = [[L_\gamma, L_{-\gamma}], L_{-\gamma\psi}] \subset I$. According to the discussion above, $\gamma, -\gamma \in \Gamma^{JI}$, a contradiction with (6.14). So $[\bigoplus_{\delta\in\Gamma^{\neg J}} L_\delta, \bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma}] \subset I'$.

Secondly, we claim that $\rho^L(I')(A)L \subset I'$. In fact, by Definition 3.1, we have

$$\rho(I')(A)L \subset [I', AL] + A[I', L].$$

Since I' is a Hom-Leibniz-ideal of L, we get $[I', AL] \subset I', [I', L] \subset I'$. So it suffices to verify that $AI' \subset I'$. For this, we calculate

$$AI' = \Big(A_0 \oplus \Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big)\Big)\Big(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma}\Big) \oplus \Big(\bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma}\Big) \subset$$

$$I' + \Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big)\Big(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma}\Big) + \Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big)\Big(\bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma}\Big). \quad (6.17)$$

For the expression $(\bigoplus_{\alpha\in\Lambda} A_\alpha)(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma})$ in (6.17), if some $A_\alpha(A_\gamma L_{-\gamma}) \neq 0$, we have that in case $\alpha = -\gamma$, clearly, $A_\alpha(A_\gamma L_{-\gamma}) = A_{-\gamma}(A_\gamma L_{-\gamma}) \subset L_{-\gamma} \subset I'$. In case of $\alpha = \gamma$, since $-\gamma \notin \Gamma_I$, we get $A_{-\gamma}(A_{-\gamma} L_\gamma) = 0$. By the the maximal length of L, we have $A_\alpha(A_\gamma L_{-\gamma}) = A_\gamma(A_\gamma L_{-\gamma}) = 0$. Suppose that $\alpha \notin \{\gamma, -\gamma\}$, by the the maximal length of L, we have $A_\alpha(A_\gamma L_{-\gamma}) = L_\alpha$. Since $\gamma \in \Gamma^{JI}$, by the root-multiplicativity of L, we have $L_{-\gamma} \subset I$, that is, $-\alpha \in \Gamma^{JI}$. So $\alpha \in -\Gamma^{JI}$ and $L_\alpha \subset I'$. Thus $(\bigoplus_{\alpha\in\Lambda} A_\alpha)(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma}) \subset I'$.

For the expression $(\bigoplus_{\alpha\in\Lambda} A_\alpha)(\bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma})$ in (5.17), if some $A_\alpha L_{-\gamma}\neq 0$, in case $\alpha-\gamma\in\Gamma^{JI}$, by the root-multiplicativity of L, we have $A_{-\alpha}L_\gamma\neq 0$. Again by the maximal length of L, we have $A_{-\alpha}L_\gamma = L_{-\alpha+\gamma}$. So $-\alpha+\gamma\in\Gamma^{JI}$, a contradiction. Thus $\alpha-\gamma\in-\Gamma^{JI}$ and therefore, $(\bigoplus_{\alpha\in\Lambda} A_\alpha)(\bigoplus_{-\gamma\in-\Gamma^{JI}} L_{-\gamma})\subset I'$.

By the discussion above, we have shown that $\rho(I')(A)L\subset I'$ and therefore I' is an ideal of (L,A).

Finally, we will verify that $L=I\oplus I'$ with ideals I,I'. Since $[I',I]=0$, by the commutativity of H, we get $\sum_{\gamma\in\Gamma}[L_\gamma,L_{-\gamma}]=0$, so H must have the form

$$H=\Big(\sum_{\gamma\in\Gamma^{JI},-\gamma\in\Lambda} A_\gamma L_{-\gamma}\Big)\oplus\Big(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma}\Big). \tag{6.18}$$

In order to show that the sum in (6.18) is direct, we take any $h\in(\sum_{\gamma\in\Gamma^{JI},-\gamma\in\Lambda} A_\gamma L_{-\gamma})\oplus(\sum_{-\gamma\in-\Gamma^{JI},\gamma\in\Lambda} A_\gamma L_{-\gamma})$. Suppose $h\neq 0$, then $h\notin Z(L)$. Since L is split, there is $v_\delta\in L_\delta,\delta\in\Gamma$ satisfying $[h,v_\delta]=\delta(h)\psi(v_\delta)\neq 0$. By Proposition 4.3, $0\neq\delta(h)\psi(v_\delta)\in L_{\delta\psi^{-1}}$. While $L_{\delta\psi^{-1}}\subset I\cap I'=0$, a contradiction. So $h=0$, as required. And this finishes the proof. □

Corollary 6.12 *Let (L,A) be a tight split regular Hom-Leibniz-Rinehart algebra of maximal length and root-multiplicative. If Γ^J, $\Gamma^{\neg J}$ are symmetric and $\Gamma^{\neg J}$ has all of its roots $\neg J$-connected, and Λ has all its nonzero weights connected. Then*

$$L=\bigoplus_{i\in I} L_i,\quad A=\bigoplus_{j\in K} A_j,$$

where any L_i is a simple ideal of L having all of its nonzero roots connected satisfying $[L_i,L_{i'}]=0$ for any $i'\in I$ with $i\neq i'$, and any A_j is a simple ideal of A satisfying $A_jA_{j'}=0$ for any $j'\in K$ such that $j'\neq j$.

Furthermore, for any $i\in I$ there exists a unique $\overline{j}\in K$ such that $A_{\overline{j}}L_i\neq 0$. We also have that any L_i is a split regular Hom-Leibniz-Rinehart algebra over $A_{\overline{j}}$.

Proof It is analogous to Theorem 5.7 in [23]. □

Acknowledgements The authors are grateful to the referees for carefully reading the manuscript and for many valuable comments which largely improved the article.

References

[1] K. MACKENZIE. *General Theory of Lie Groupoids and Lie Algebroids*. Cambridge University Press, Cambridge, 2005.

[2] J. HUEBSCHMANN. *Poisson cohomology and quantization*. J. Reine Angew. Math., 1990, **408**: 57–113.

[3] J. HUEBSCHMANN. *Lie-Rinehart algebras, Gerstenhaber algebras and Batalin-Vilkovisky algebras*. Ann. Inst. Fourier, 1998, **48**: 425–440.

[4] J. HUEBSCHMANN. *Duality for Lie-Rinehart algebras and the modular class*. J. Reine Angew. Math., 1999, **510**: 103–159.

[5] J. HUEBSCHMANN. *Lie-Rinehart Algebras, Descent, and Quantization*. Amer. Math. Soc., Providence, RI, 2004.

[6] J. T. HARTEIG, D. LARSSON, S. D. SILVERSTROV. *Deformations of Lie algebras using σ-derivations*. J. Algebra, 2006, **295**: 314–361.

[7] Naihong HU. *q-Witt algebras, q-Lie algebras, q-holomorph structure and representations*. Algebra Colloq., 1999, **6**(1): 51–70.

[8] A. MANDAL, S. K. MISHRA. *Hom-Lie-Rinehart algebras*. Comm. Algebra, 2018, **46**(9): 3722–3744.

[9] Liqiang CAI, Jiefeng LIU, Yunhe SHENG. *Hom-Lie algebroids, Hom-Lie bialgebroids and Hom-Courant algebroids*. J. Geom. Phys., 2017, **121**: 15–32.

[10] C. LAURENT-GENGOUX, J. TELES. *Hom-Lie algebroids*. J. Geom. Phys., 2013, **68**: 69–75.

[11] J. L. CASTIGLIONI, X. GARCIA-MARTINEZ, M. LADRA. *Universal central extensions of Lie-Rinehart algebras*. J. Algebra Appl., 2018, **17**(7): 1850134, 30 pp.

[12] A. MANDAL, S. K. MISHRA. *Deformation of Hom-Lie-Rinehart algebras*. Comm. Algebra, 2020, **48**(4): 1653–1670.

[13] A. MANDAL, S. K. MISHRA. *On Hom-Gerstenhaber algebras and Hom-Lie algebroids*. J. Geom. Phys., 2018, **133**: 287–302.

[14] A. MANDAL, S. K. MISHRA. *Universal central extensions and non-abelian tensor product of Hom-Lie-Rinehart algebras*. Colloq. Math., 2021, **166**(1): 23–52.

[15] Tao ZHANG, Fengying HAN, Yanhui BI. *Crossed modules for Hom-Lie-Rinehart algebras*. Colloq. Math., 2018, **152**(1): 1–14.

[16] A. J. CALDERÓN. *On split Lie algebras with symmetric root systems*. Proc. Indian Acad. Sci. Math. Sci., 2008, **118**(3): 351–356.

[17] M. J. ARAGÓN, A. J. CALDERÓN. *Split regular Hom-Lie algebras*. J. Lie Theory, 2015, **25**(3): 875–888.

[18] A. J. CALDERÓN, J. M. SÁNCHEZ. *On split Leibniz algebras*. Linear Algebra Appl., 2012, **436**(6): 1648–1660.

[19] A. J. CALDERÓN, J. M. SÁNCHEZ. *On the structure of split Lie color algebras*. Linear Algebra Appl., 2012, **436** (2): 307–315.

[20] A. J. CALDERÓN, J. M. SÁNCHEZ. *The structure of split regular Hom-Lie algebras*. J. Geom. Phy., 2016, **110**: 296–305.

[21] Yan CAO, Liangyun CHEN, Bin SUN. *On split regular Hom-Leibniz algebras*. J. Algebra Appl., 2018, **17** (10): 1850185, 18 pp.

[22] A. HELENA, B. ELISABETE, A. J. CALDERÓN, et al. *Split Lie-Rinehart algebras*. J. Algebra Appl., 2021, **20**(9): 2150164, 22pp.

[23] Shengxiang WANG, Xiaohui ZHANG, Shuangjian GUO. *On the structure of split regular Hom-Lie Rinehart algebras*. Colloq. Math., 2020, **160**(2): 165–182.

[24] A. MAKHLOUF, S. SILVESTROV. *Hom-algebra structures*. J. Gen. Lie Theory Appl., 2008, **2**(2): 51–64.

[25] A. MAKHLOUF, S. SILVESTROV. *Notes on formal deformations of Hom-associative and Hom-Lie algebras*. Forum Math., 2010, **22**(4): 715–759.