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Some Results for a Family of Harmonic Univalent Functions

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Abstract In this paper, we introduce a class of the univalent sense-preserving harmonic functions associated with Ruscheweyh derivatives. By establishing the extremal theory, we obtain the sharp coefficients bounds, sharp growth theorems and sharp distortion theorems for the class. The radius equation between this class and a known class of harmonic functions is given. Also, we investigate the results of modified-Hadamard product for this class.

Keywords extreme points; growth theorem; harmonic functions; modified-Hadamard product; Ruscheweyh derivatives

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1. Introduction

A complex valued continuous function $f = u + iv$ is harmonic in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ if both u and v are harmonic in $\mathbb{D}$. In fact, any complex-valued harmonic function f in $\mathbb{D}$ can be uniquely represented as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. Moreover, Lewy [1] showed that harmonic function f is locally univalent and sense-preserving in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ if and only if $|h'(z)| > |g'(z)|$, $z \in \mathbb{U}$. Let $\mathcal{S}$ be the class of univalent analytic function in $\mathbb{U}$ and let $\mathcal{H}$ be the family of continuous complex valued functions which are harmonic in $\mathbb{U}$, where

$$\mathcal{H} = \Big\{f : f = h + \overline{g}, f = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, h(0) = f_z(0) - 1 = 0\Big\}. \tag{1.1}$$

In particular, if $b_1 = g'(0) = 0$, then $\mathcal{H} \equiv \mathcal{H}_0$. It is obvious that $\mathcal{H}_0 \subset \mathcal{H}$. Denote by $\mathcal{S}_{\mathcal{H}}$, the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{U}$. Also, let $\mathcal{S}_{\mathcal{H}_0} = \{f \in \mathcal{S}_{\mathcal{H}} : g'(0) = f_{\overline{z}}(0) = 0\}$. The family $\mathcal{S}_{\mathcal{H}_0}$ is known to be compact.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f(z) \prec g(z)$, provided there exists an analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

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In [2], Ruscheweyh introduced an operator $\mathcal{D}_{\mathcal{S}}^m : \mathcal{S} \to \mathcal{S}$ as

$$\mathcal{D}_{\mathcal{S}}^m(f)(z) = \frac{z(z^{m-1}f(z))^{(m)}}{m!}, \quad m \in \mathbb{N} = \{0, 1, 2, \ldots\}, \ z \in \mathbb{U}.$$

Comparing with the Ruscheweyh operator in analytic functions case, Dziok-Darus-Sokół-Bulboaca [3] constructed a new linear operator $\mathcal{D}_{\mathcal{H}_0}^m : \mathcal{H}_0 \to \mathcal{H}_0$ by [4]

$$\mathcal{D}_{\mathcal{H}_0}^m(f)(z) = \mathcal{D}_{\mathcal{H}_0}^m(h)(z) + (-1)^m \overline{\mathcal{D}_{\mathcal{H}_0}^m(g)(z)}. \tag{1.2}$$

Using (1.1) and (1.2), it is easy to see

$$\mathcal{D}_{\mathcal{H}_0}^m(f)(z) = z + \sum_{k=2}^{\infty} \mathcal{U}(k,m) a_k z^k + (-1)^m \sum_{k=2}^{\infty} \overline{\mathcal{U}(k,m) b_k z^k}, \quad z \in \mathbb{U}, \tag{1.3}$$

where

$$\mathcal{U}(k,0) = 1, \ \mathcal{U}(k,m) = \frac{k(k+1)\cdots(k+m-1)}{m!}, \quad m \in \mathbb{N} - \{0\}. \tag{1.4}$$

Recently, Dziok [4] extended some results with respect to harmonic starlike functions and convex functions by the operator $\mathcal{D}_{\mathcal{H}_0}^m$.

Motivated by Dziok [4] and Yaguchi-Sekine-Saitoh-Owa et al. [5], we introduce the following subclass of $\mathcal{H}$.

Definition 1.1 *If* $-1 \le A < B \le 1$, $0 \le \alpha \le 1$, $m_1, m_2 \in \mathbb{N} - \{0\}$ *and* $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}_0}$, *then*

$$f \in \mathcal{G}_{\mathcal{H}_0}^{m_1,m_2}(A,B) \Leftrightarrow \frac{\alpha \mathcal{D}_{\mathcal{H}_0}^{m_1}(f)(z) + (1-\alpha)\mathcal{D}_{\mathcal{H}_0}^{m_2}(f)(z)}{z} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}.$$

Remark 1.2 (i) We note that the m_1 and m_2 can be adjusted arbitrarily in Definition 1.1, e.g., choosing $m_1 = 1$, $m_2 = 2$, we have $f = h + \overline{g} \in \mathcal{G}_{\mathcal{H}_0}^{1,2}(A,B)$

$$\Leftrightarrow \frac{\alpha(zh'(z) - \overline{zg'(z)}) + (1-\alpha)(zh'(z) + z^2h''(z) + \overline{zg'(z) + z^2g''(z)}}{z} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}.$$

(ii) Let $\varphi = \mathcal{U} + i\mathcal{V}$ be the subclass of $\mathcal{H}$, where

$$\mathcal{U}(z) = \sum_{k=0}^{\infty} u_k z^k, \ \mathcal{V}(z) = \sum_{k=1}^{\infty} v_k z^k, \quad z \in \mathbb{U}.$$

A function $f \in \mathcal{H}$ of the form (1.1) has coefficients correlated with the function φ, if

$$a_k u_k = -|a_k||u_k|, \ b_k v_k = -|b_k||v_k|, \quad k \in \mathbb{N}. \tag{1.5}$$

Let $\mathcal{T}^m$ be the class of functions $f \in \mathcal{H}$ with coefficients correlated with respect to the function

$$\varphi(z) = \frac{z}{(1-z)^{m+1}} + (-1)^m \frac{\overline{z}}{(1-\overline{z})^{m+1}}, \quad m \in \{0, 1, 2, \ldots\}, \ z \in \mathbb{U}. \tag{1.6}$$

We define the class $\mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B) = \mathcal{G}_{\mathcal{H}_0}^{m_1,m_2}(A,B) \bigcap \mathcal{T}^m$.

(iii) In particular, if taking $\alpha = 1$ and $m_1 = m$, then the class $\mathcal{G}_{\mathcal{H}_0}^{m,m_2}(A,B) \equiv \mathcal{R}_{\mathcal{H}}^m(A,B)$ was defined by Dziok [4].

Let the topology on $\mathcal{H}$ be given by a metric ρ which is equivalent to the topological of uniform convergence on compact subsets, where the ρ is determined as

$$\rho(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f-g\|_k}{1 - \|f-g\|_k}$$

whenever f and g belong to $\mathcal{H}$, and $\|f\|_k = \max\{|f(z)| : |z| = r_k, 0 < r_n < 1, \lim_{k\to\infty} r_k = 1\}$. It follows from theorems of Weierstrass and Montel that this topology space is complete [6]. If $\mathcal{F} \in \mathcal{H}$, then $\mathcal{F}$ is called locally uniformly bounded if there is a positive constant M such that $|f(z)| \leq M$ whenever $f \in \mathcal{F}$ (see [7]). Further, $\mathcal{F} \subset \mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded. Denote by $H\mathcal{F}$ the closed convex hull of $\mathcal{F}$, where

$$H\mathcal{F} = \Big\{ \sum_{k=1}^{\infty} t_k f_k, f_k \in \mathcal{F}, t_k \geq 0, \sum_{k=1}^{\infty} t_k = 1 \Big\}.$$

Let $\mathcal{F}$ be a nonempty subset of $\mathcal{S}_{\mathcal{H}_0}$. A point $f \in \mathcal{F}$ is called an extreme point of $\mathcal{F}$ provided that $f = tg + (1-t)h$, where $t \in (0,1), g, h \in \mathcal{F}$, implies $f = g = h$. A point $g \in \mathcal{F}$ is called a support point of $\mathcal{F}$ if there exists a continuous linear functional $L : \mathcal{S}_{\mathcal{H}_0} \to \mathbb{C}$ such that $\Re L$ is nonconstant on $\mathcal{F}$ and

$$\Re L(g) = \max\{\Re L(h) : h \in \mathcal{F}\}.$$

We denote by E$\mathcal{F}$ and supp$\mathcal{F}$ the subsets of $\mathcal{F}$ consisting of extreme points of $\mathcal{F}$ and support points of $\mathcal{F}$, respectively [6, 8].

Establishing various properties of classes of harmonic functions has attracted the attention of many mathematicians [9–11], and interest in the study of univalent harmonic functions prompted the publication of the articles [12–20]. In this paper, we organize the contents as follows. In Section 2, two important lemmas are given, which help to investigate our main results. In Section 3.1, we get the extreme points set of the class $\mathcal{G}^{m_1,m_2}_{\mathcal{H}_\mathcal{T}}(A,B)$, and apply the extremal theory to discuss some geometric properties. In Section 3.2, we obtain the radius equation from the class $\mathcal{R}^1_{\mathcal{H}}(A,B)$ to the class $\mathcal{G}^{1,2}_{\mathcal{H}_\mathcal{T}}(A,B)$. In Section 3.3, we shall give the result of modified-Hadamard product associated with $\mathcal{G}^{m_1,m_2}_{\mathcal{H}_\mathcal{T}}(A,B)$.

2. Preliminaries

The following lemmas are needed.

Lemma 2.1([6]) *Let $\mathbb{X}$ be a locally convex linear topological space and let $\mathcal{F}$ be a compact subset of $\mathbb{X}$.*

(1) If $\mathcal{F}$ is non-empty, then $E\mathcal{F}$ is non-empty.

(2) $HE\mathcal{F}$=$H\mathcal{F}$.

(3) If $H\mathcal{F}$ is compact, then $EH\mathcal{F} \subset \mathcal{F}$.

(4) If J is a real-valued, continuous, convex functional on $H\mathcal{F}$, then

$$\max\{J(f) : f \in H\mathcal{F}\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in EH\mathcal{F}\}.$$

Lemma 2.2 *Let $f \in \mathcal{T}^m$. Then $f \in \mathcal{G}^{m_1,m_2}_{\mathcal{H}_\mathcal{T}}(A,B)$ if and only if the condition*

$$\sum_{k=2}^{\infty} [\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)](|a_k| + |b_k|) \leq \frac{B-A}{1+B} \tag{2.1}$$

holds true, where $\mathcal{U}(k,m_1)$ and $\mathcal{U}(k,m_2)$ are defined by (1.4).

Proof With the similar approach as in [4, Theorem 7], we can obtain the necessary condition. Here, we only need to prove the sufficient condition. Since $f \in \mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(A,B)$, by (1.3) and Definition 1.1, then it is easy to see

$$|\frac{G_1(z)}{G_2(z)}| < 1, \tag{2.2}$$

where

$$\begin{aligned} G_1(z) = &-\sum_{k=2}^{\infty}[\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)]a_k z^k - \\ &\sum_{k=2}^{\infty}[(-1)^{m_1}\alpha\mathcal{U}(k,m_1)+(-1)^{m_2}(1-\alpha)\mathcal{U}(k,m_2)]\overline{b_k}\overline{z}^k, \\ G_2(z) = &(B-A)z + B\sum_{k=2}^{\infty}[\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)]a_k z^k + \\ &B\sum_{k=2}^{\infty}[(-1)^{m_1}\alpha\mathcal{U}(k,m_1)+(-1)^{m_2}(1-\alpha)\mathcal{U}(k,m_2)]\overline{b_k}\overline{z}^k. \end{aligned}$$

Now, taking $0 < z = r < 1$ in (2.2) and using the (1.5), (1.6), then we have

$$\frac{\sum\limits_{k=2}^{\infty}[\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)](|a_k|+|b_k|)r^{k-1}}{(B-A)-B\sum\limits_{k=2}^{\infty}[\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)](|a_k|+|b_k|)r^{k-1}} < 1. \tag{2.3}$$

For $0 < r < 1$, setting

$$G_3(r) = (B-A) - B\sum_{k=2}^{\infty}[\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)](|a_k|+|b_k|)r^{k-1},$$

then we can know that the function $G_3(r)$ is increasing in r, thus,

$$G_3(r) > G_3(0) = B - A > 0.$$

By (2.3), we have

$$\sum_{k=2}^{\infty}(1+B)[\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)](|a_k|+|b_k|)r^{k-1} < B - A. \tag{2.4}$$

Taking $r \to 1$ in (2.4), we complete the proof. □

3. Main results

In these sections from 3.1 to 3.3, we give the main results for the class $\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(A,B)$.

3.1. Applications to extremal theory

In this section, using Lemmas 2.1 and 2.2, we discuss the sharp coefficients bounds, sharp growth theorems and distortion theorems for the subclass $\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(A,B)$ by using the extremal theory.

Theorem 3.1 $E\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(A,B) = \mathbb{W} = \mathbb{S}_1 \cup \mathbb{S}_2$, *where* $\mathbb{S}_1 = \{h_1(z), h_2(z), \ldots, h_k(z), \ldots\}$, $\mathbb{S}_2 =$

$\{g_2(z), g_3(z), \ldots, g_k(z), \ldots\}$, $z \in \mathbb{U}$ *and*

$$\begin{cases} h_k(z) = z, & k = 1, \\ h_k(z) = z - \dfrac{B-A}{(1+B)\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)} z^k, & k \in \{2,3,\ldots\}, \\ g_k(z) = z - \dfrac{B-A}{(1+B)\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)} \overline{z}^k, & k \in \{2,3,\ldots\}. \end{cases}$$

Proof To complete the proof, we divide the process into three steps:

Step 1. First, we show that the class $\mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B)$ is convex subset of $\mathcal{H}_0$. Suppose that $f_i \in \mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B)$, $0 \le \lambda \le 1$ and

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k,i} z^k + \sum_{k=2}^{\infty} \overline{b_{k,i}} \overline{z}^k, \quad i = 1,2, \ z \in \mathbb{U}.$$

Then we can know

$$\lambda f_1(z) + (1-\lambda) f_2(z) = z + \sum_{k=2}^{\infty} [\lambda a_{k,1} + (1-\lambda) a_{k,2}] z^k + \sum_{k=2}^{\infty} [\lambda \overline{b_{k,1}} + (1-\lambda)\overline{b_{k,2}}] \overline{z}^k, \quad z \in \mathbb{U}. \tag{3.1}$$

In the view of Lemma 2.2 and (3.1), we have

$$\begin{aligned} &\sum_{k=2}^{\infty} [\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)][|\lambda a_{k,1} + (1-\lambda) a_{k,2}| + |\lambda \overline{b_{k,1}} + (1-\lambda)\overline{b_{k,2}}|] \\ &\quad \le \lambda \sum_{k=2}^{\infty} [\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)](|a_{k,1}| + |b_{k,1}|) + \\ &\qquad (1-\lambda) \sum_{k=2}^{\infty} [\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)](|a_{k,2}| + |b_{k,2}|) \\ &\quad \le \lambda \frac{B-A}{1+B} + (1-\lambda)\frac{B-A}{1+B} = \frac{B-A}{1+B}, \end{aligned} \tag{3.2}$$

Eq. (3.2) makes sure that the function $\lambda f_1 + (1-\lambda) f_2 \in \mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B)$. Hence, the class $\mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B)$ is convex.

Step 2. Next, we show that the class $\mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B)$ is compact subset of $\mathcal{H}_0$. Suppose that $f \in \mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B), 0 < |z| = r < 1$, and

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k} \overline{z}^k, \quad z \in \mathbb{U}.$$

Then we conclude that

$$\begin{aligned} |f(z)| &= \left| z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k} \overline{z}^k \right| \le r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \\ &\le r + \sum_{k=2}^{\infty} [\alpha k + (1-\alpha)k)](|a_k| + |b_k|) r^k \\ &\le r + \sum_{k=2}^{\infty} [\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)](|a_k| + |b_k|) \le r + \frac{B-A}{1+B}. \end{aligned} \tag{3.3}$$

Thus, (3.3) implies that the class $\mathcal{G}_{\mathcal{H}_\mathcal{T}}^{m_1,m_2}(A,B)$ is locally uniformly bounded. Thus, we need to prove it is closed.

If $\{f_j\}_{j=1}^{\infty} \subset \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$, $f_j(z) \to f(z)$, $j \to \infty$, where

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k + \sum_{k=2}^{\infty} \overline{b_{k,j}} \overline{z}^k, \quad z \in \mathbb{U} \tag{3.4}$$

and

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k} \overline{z}^k, \quad z \in \mathbb{U}. \tag{3.5}$$

Hence using (3.4) and (3.5), we have $a_{k,j} \to a_k$, $b_{k,j} \to b_k, j \to \infty$. Since $f_j \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$, following Lemma 2.2, we obtain

$$\sum_{k=2}^{\infty} [\alpha \mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)](|a_{k,j}| + |b_{k,j}|) \le \frac{B-A}{1+B}. \tag{3.6}$$

Taking $j \to \infty$ in (3.6), it gives that

$$\sum_{k=2}^{\infty} [\alpha \mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)](|a_k| + |b_k|) \le \frac{B-A}{1+B}. \tag{3.7}$$

Eq. (3.7) implies $f \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$. Therefore, the class $\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$ is closed.

Step 3. Finally, we shall give the extreme points for $\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$. Let any $g_k(z) \in \mathbb{W}$, $k \in \{2,3,\ldots\}$ and define the following equality

$$g_k = tG_1(z) + (1-t)G_2(z), \quad 0 < t < 1, \tag{3.8}$$

where

$$g_k(z) = z - \frac{B-A}{(1+B)\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)} \overline{z}^k \in \mathbb{W} \tag{3.9}$$

and

$$\begin{aligned} G_j(z) =& z + \sum_{k=2}^{\infty} a_{k,i} z^k + \sum_{k=2}^{\infty} \overline{b_{k,i}} \overline{z}^k \\ =& z - \sum_{k=2}^{\infty} |a_{k,i}| z^k - \sum_{k=2}^{\infty} |b_{k,i}| \overline{z}^k \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B), \quad i = 1,2. \end{aligned} \tag{3.10}$$

By (3.8)–(3.10), it is easy to know that

$$\frac{B-A}{(1+B)\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)} = t|b_{k,1}| + (1-t)|b_{k,2}|. \tag{3.11}$$

Therefore, (3.11) and Lemma 2.2 imply that

$$b_{k,1} = b_{k,2} = \frac{B-A}{(1+B)\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)}, \quad k \in \{2,3,\ldots\}.$$

Moreover, we can note that $a_{k,1} = a_{k,2} = 0$, this gives us $g_k \in E\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$. Similarly, we can verify that the functions $h_k(z) \in \mathrm{E}\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$. Thus, $\mathbb{W} \subset E\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$.

On the other hand, it is easy to see $\mathbb{W}$ is compact and $\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B) = H\mathbb{W}$. Thus, by Lemma 2.1, we have $\mathrm{E}\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B) = EH\mathbb{W} \subset \mathbb{W}$. So $\mathrm{E}\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B) = \mathbb{W}$. $\square$

Corollary 3.2 *Let the function $f(z) \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$ be given by (ii) in Remark 1.2. Then*

$$|a_k| \le \frac{B-A}{(1+B)\alpha(m_1+1)+(1-\alpha)(m_2+1)}, \quad k \in \{2,3,\ldots\},$$

$$|b_k| \le \frac{B-A}{(1+B)\alpha(m_1+1)+(1-\alpha)(m_2+1)}, \quad k \in \{2,3,\ldots\}.$$

The result are sharp, and the extremal functions are defined by h_2 and g_2 as Theorem 3.1.

Proof For each fixed $k \in \{2,3,\ldots\}$, define the following real-valued continuous functionals

$$J(f) = |a_k|, \ \widetilde{J}(f) = |b_k|, \ f \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B). \tag{3.12}$$

It is easy to know that J and $\widetilde{J}$ are convex on $\mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$. Thus, in view of Lemma 2.1 and Theorem 3.1, we have

$$\max\{J(f) : f \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)\} = \max\{|a_k| : f \in H\mathbb{W}\} = \max\{|a_k| : f \in \mathbb{W}\} \tag{3.13}$$

and

$$\max\{\widetilde{J}(f) : f \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)\} = \max\{|b_k| : f \in H\mathbb{W}\} = \max\{|b_k| : f \in \mathbb{W}\}. \tag{3.14}$$

By (3.13) and (3.14), then

$$J(f)_{f\in\mathbb{W}} = \widetilde{J}(f)_{f\in\mathbb{W}} = |a_k| = |b_k| = \frac{B-A}{(1+B)\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)}$$
$$\le \frac{B-A}{(1+B)\alpha\mathcal{U}(2,m_1)+(1-\alpha)\mathcal{U}(2,m_2)} = \frac{B-A}{(1+B)\alpha(m_1+1)+(1-\alpha)(m_2+1)}.$$

This completes the proof of Corollary 3.2. □

Corollary 3.3 *If the function $f(z) \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$, $|z| = r < 1$, $z \in \mathbb{U}$, then*

$$\mathbb{Y}_1 \le |f(z)| \le \mathbb{Y}_2,$$

where

$$\mathbb{Y}_1 = r - \frac{B-A}{(1+B)\alpha(m_1+1)+(1-\alpha)(m_2+1)} r^2,$$

$$\mathbb{Y}_2 = r + \frac{B-A}{(1+B)\alpha(m_1+1)+(1-\alpha)(m_2+1)} r^2.$$

The results are sharp, and the extremal functions are defined by h_2 and g_2 as Theorem 3.1.

Proof The proof is similar to the case in Corollary 3.2, we omit it. □

Corollary 3.4 *If the function $f(z) \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1,m_2}(A,B)$, $|z| = r < 1$, $z \in \mathbb{U}$, and $\mathbb{N} \ni m \le \max\{m_1, m_2\}$, then*

$$\mathbb{Y}_3 \le |\mathcal{D}_{\mathcal{H}}^m(f)(z)| \le \mathbb{Y}_4,$$

where

$$\mathbb{Y}_3 = r - \frac{(B-A)\mathcal{U}(\widehat{k},m)}{(1+B)\alpha\mathcal{U}(\widehat{k},m_1)+(1-\alpha)\mathcal{U}(\widehat{k},m_2)} r^{\widehat{k}},$$

$$\mathbb{Y}_4 = r + \frac{(B-A)\mathcal{U}(\widehat{k},m)}{(1+B)\alpha\mathcal{U}(\widehat{k},m_1)+(1-\alpha)\mathcal{U}(\widehat{k},m_2)}r^{\widehat{k}}.$$

The results are sharp, and the extremal functions are defined by $h_{\widehat{k}}$ *and* $g_{\widehat{k}}$ *as Theorem 3.1. The* $\widehat{k}$ *is given by the following (3.16).*

Proof Define the following real-valued continuous functionals

$$J(f) = |\mathcal{D}^m_{\mathcal{H}_0}(f)(z)|,\ f\in \mathcal{G}^{m_1,m_2}_{\mathcal{H}\mathcal{T}}(A,B),\quad z\in\mathbb{U}.$$

It is easy to know that J is convex on $\mathcal{G}^{m_1,m_2}_{\mathcal{H}\mathcal{T}}(A,B)$. Thus, in view of Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned}\max\{J(f): f\in \mathcal{G}^{m_1,m_2}_{\mathcal{H}\mathcal{T}}(A,B)\} &= \max\{|\mathcal{D}^m_{\mathcal{H}_0}(f)(z)| : f\in H\mathbb{W}\}\\ &= \max\{|\mathcal{D}^m_{\mathcal{H}_0}(f)(z)| : f\in \mathbb{W}\}.\end{aligned} \tag{3.15}$$

By (3.15), then

$$\mathfrak{M}_1 \le J(f)_{f\in\mathbb{W}} = |\mathcal{D}^m_{\mathcal{H}_0}(h_k)(z)| = |\mathcal{D}^m_{\mathcal{H}_0}(g_k)(z)| \le \mathfrak{M}_2,$$

where

$$\mathfrak{M}_1 = r - \frac{(B-A)\mathcal{U}(k,m)}{(1+B)\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)}r^k,$$

$$\mathfrak{M}_2 = r + \frac{(B-A)\mathcal{U}(k,m)}{(1+B)\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)}r^k.$$

We define the sequence $\{\zeta^m_k\}$ as follows:

$$\zeta^m_k = \frac{(B-A)\mathcal{U}(k,m)}{(1+B)\alpha\mathcal{U}(k,m_1)+(1-\alpha)\mathcal{U}(k,m_2)}r^k,\quad k\in\{2,3,\dots\}.$$

Since $m\le \max\{m_1,m_2\}$ and $0\le r<1$, it is easy to prove that $\zeta^m_k\to 0$ as $k\to\infty$, which implies that there is a $\widehat{k}\in\{2,3,\dots\}$ such that

$$\zeta^m_{\widehat{k}} = \max\{\zeta^m_k : k=2,3,\dots\}. \tag{3.16}$$

This completes the proof of Corollary 3.4. □

3.2. Radius equation

In this section, we discuss the radius equation between $\mathcal{R}^1_{\mathcal{H}}(A_1,B_1)$ and $\mathcal{G}^{1,2}_{\mathcal{H}\mathcal{T}}(A_2,B_2)$.

Theorem 3.5 *Let* $0\le\alpha\le 1$, $-1\le A_i<B_i\le 1$ $(i=1,2)$. *If* $f\in\mathcal{R}^1_{\mathcal{H}}(A_1,B_1)$ *and* x *are some complex numbers with* $0<|x|<1$, *then* $\frac{1}{x}f(xz)\in\mathcal{G}^{1,2}_{\mathcal{H}\mathcal{T}}(A_2,B_2)$ *for* $0<|x|\le|x_0|$, *where* $|x_0|$ *is the smallest positive root of the equation*

$$\begin{aligned}F(|x|) = &\sqrt{\frac{B_1-A_1}{1+B_1}}\frac{\alpha|x|\sqrt{2-|x|^2}}{1-|x|^2} - \frac{B_2-A_2}{1+B_2} + \\ &\sqrt{\frac{B_1-A_1}{1+B_1}}\frac{(1-\alpha)|x|\sqrt{18-24|x|^2+16|x|^4-4|x|^6}}{2(1-|x|^2)^2} = 0.\end{aligned}$$

Proof Suppose that $f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=2}^{\infty}\overline{b_k}\overline{z}^k\in\mathcal{R}_{\mathcal{H}}^{1}(A_1,B_1)$. Then for complex numbers x $(0<|x|<1)$, we may write

$$\frac{1}{x}f(xz)=z+\sum_{k=2}^{\infty}a_kx^{k-1}z^k+\sum_{k=2}^{\infty}\overline{b_k}x^{k-1}\overline{z}^k,\quad z\in\mathbb{U}.$$

In order to prove that $\frac{1}{x}f(xz)\in\mathcal{G}_{\mathcal{H}\mathcal{T}}^{1,2}(A_2,B_2)$, by applying the Lemma 2.2, it needs to show that

$$\mathcal{W}=\sum_{k=2}^{\infty}\big[\alpha k+\frac{1-\alpha}{2}k(k+1)\big]|x|^{k-1}(|a_k|+|b_k|)\le\frac{B_2-A_2}{1+B_2}.\tag{3.17}$$

With the aid of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathcal{W}=&\sum_{k=2}^{\infty}\big[\alpha k+\frac{1-\alpha}{2}k(k+1)\big]|x|^{k-1}(|a_k|+|b_k|)\\ =&\alpha\sum_{k=2}^{\infty}k|x|^{k-1}(|a_k|+|b_k|)+\frac{1-\alpha}{2}\sum_{k=2}^{\infty}k(k+1)|x|^{k-1}(|a_k|+|b_k|)\\ \le&\frac{\alpha}{|x|}\Big(\sum_{k=2}^{\infty}k|x|^{2k}\Big)^{\frac{1}{2}}\Big(\sum_{k=2}^{\infty}k(|a_k|+|b_k|)^2\Big)^{\frac{1}{2}}+\\ &\frac{1-\alpha}{2|x|}\Big(\sum_{k=2}^{\infty}k(k+1)^2|x|^{2k}\Big)^{\frac{1}{2}}\Big(\sum_{k=2}^{\infty}k(|a_k|+|b_k|)^2\Big)^{\frac{1}{2}}\end{aligned}\tag{3.18}$$

Putting $y=|x|^2$, then we have the following important computations:

(i) Let

$$\sum_{k=2}^{\infty}k|x|^{2k}=\sum_{k=2}^{\infty}ky^k\equiv S_1(y).$$

Then

$$S_1(y)=y\frac{d}{dy}\Big(\sum_{k=2}^{\infty}y^k\Big)=y\frac{d}{dy}(\frac{y^2}{1-y})=\frac{y^2(2-y)}{(1-y)^2}.\tag{3.19}$$

(ii) Let

$$\sum_{k=2}^{\infty}k(k+1)^2|x|^{2k}=\sum_{k=2}^{\infty}k(k+1)^2y^k\equiv S_2(y).$$

Because

$$\sum_{k=2}^{\infty}(k+1)y^k=\sum_{k=2}^{\infty}ky^k+\sum_{k=2}^{\infty}y^k=\frac{y^2(2-y)}{(1-y)^2}+\frac{y^2}{1-y}=\frac{y^2(3-2y)}{(1-y)^2},$$

so, we can know that

$$\begin{aligned}S_2(y)=&\frac{d}{dy}\Big[y^2\frac{d}{dy}\Big(\sum_{k=2}^{\infty}(k+1)y^k\Big)\Big]=\frac{d}{dy}(\frac{6y^3-6y^4+2y^5}{(1-y)^3})\\ =&\frac{y^2(18-24y+16y^2-4y^3)}{(1-y)^4}.\end{aligned}\tag{3.20}$$

Furthermore, since $f\in\mathcal{R}_{\mathcal{H}}^{1}(A_1,B_1)$, by [4, Theorem 9], we have

$$\sum_{k=2}^{\infty}k(|a_k|+|b_k|)\le\frac{B_1-A_1}{1+B_1}.\tag{3.21}$$

Thus, following (3.18)–(3.21), we can obtain that

$$\mathcal{W}=\sum_{k=2}^{\infty}\Big[\alpha k+\frac{1-\alpha}{2}k(k+1)\Big]|x|^{k-1}(|a_k|+|b_k|)\le\sqrt{\frac{B_1-A_1}{1+B_1}}\alpha\frac{|x|\sqrt{2-|x|^2}}{1-|x|^2}+\sqrt{\frac{B_1-A_1}{1+B_1}}\frac{1-\alpha}{2}\frac{|x|\sqrt{18-24|x|^2+16|x|^4-4|x|^6}}{(1-|x|^2)^2}. \tag{3.22}$$

In order to prove (3.17), then one needs to consider the complex number x $(0<|x|<1)$ such that

$$\sqrt{\frac{B_1-A_1}{1+B_1}}\alpha\frac{|x|\sqrt{2-|x|^2}}{1-|x|^2}+\sqrt{\frac{B_1-A_1}{1+B_1}}\frac{1-\alpha}{2}\frac{|x|\sqrt{18-24|x|^2+16|x|^4-4|x|^6}}{(1-|x|^2)^2}=\frac{B_2-A_2}{1+B_2}.$$

Hence, we define the following function with $|x|$ by

$$F(|x|)=\sqrt{\frac{B_1-A_1}{1+B_1}}\alpha\frac{|x|\sqrt{2-|x|^2}}{1-|x|^2}-\frac{B_2-A_2}{1+B_2}+\sqrt{\frac{B_1-A_1}{1+B_1}}\frac{1-\alpha}{2}\frac{|x|\sqrt{18-24|x|^2+16|x|^4-4|x|^6}}{(1-|x|^2)^2}.$$

It is easy to see that $F(0)=-\frac{B_2-A_2}{1+B_2}<0$ and $F(1)\to+\infty$ as $|x|\to1^-$, which implies that there is some x_0 such that $F(|x_0|)=0$ $(0<|x_0|<1)$. The proof of the theorem is completed. □

3.3. Modified-Hadamard product

Suppose that the functions $f_j(z)=h_j(z)+\overline{g_j(z)}$, $j=1,2,\ldots,\kappa$, where

$$h_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k,\ \ g_j(z)=\sum_{k=2}^{\infty}\overline{b_{k,j}}z^k,\quad z\in\mathbb{U}.$$

For $\kappa\in\mathbb{N}$, $\mathcal{J}>1$, we introduce the modified-Hadamard product of f_j by

$$\mathcal{F}_{\kappa}^{\mathcal{J}}(z)=z-\sum_{k=2}^{\infty}\Big(\sum_{j=1}^{\kappa}|a_{k,j}|^{\mathcal{J}}\Big)z^k-\overline{\sum_{k=2}^{\infty}\Big(\sum_{j=1}^{\kappa}|b_{k,j}|^{\mathcal{J}}\Big)z^k}. \tag{3.23}$$

In particular, if the parameters $A=1-2\gamma$ $(0\le\gamma\le1)$, $B=1$, then the class

$$\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(1-2\gamma,1)\equiv\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(\gamma).$$

We shall discuss the modified-Hadamard product with the subclass $\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(\gamma)$ in the following Theorem 3.6.

Theorem 3.6 *If the functions $f_j(z)\in\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(\gamma_j)$, $0\le\gamma_j<1$, $j=1,2,\ldots,\kappa$, $z\in\mathbb{U}$ and $\gamma=\min_{1\le j\le\kappa}\{\gamma_j\}$, then when*

$$\kappa(1-\gamma)^{\mathcal{J}}<[\alpha(1+m_1)+(1-\alpha)(1+m_2)]^{\mathcal{J}-1},\quad\mathcal{J}>1, \tag{3.24}$$

we have $\mathcal{F}_{\kappa}^{\mathcal{J}}(z)\in\mathcal{G}_{\mathcal{H}_{\mathcal{T}}}^{m_1,m_2}(\epsilon_\kappa)$, where

$$\epsilon_\kappa=1-\frac{\kappa(1-\gamma)^{\mathcal{J}}}{[\alpha(m_1+1)+(1-\alpha)(m_2+1)]^{\mathcal{J}-1}}.$$

The results are sharp for the functions $f_j(z)$ given by

$$f_j(z) = z - \sum_{k=2}^{\infty} \mathfrak{N}(\gamma_j) z^k - \sum_{k=2}^{\infty} \mathfrak{N}(\gamma_j) \overline{z}^k, \quad j = 1, 2, \ldots, \kappa,$$

where

$$\mathfrak{N}(\gamma_j) = \frac{1 - \gamma_j}{\alpha(m_1 + 1) + (1 - \alpha)(m_2 + 1)}.$$

Proof Since the functions $f_j(z) \in \mathcal{G}_{\mathcal{H}\mathcal{T}}^{m_1, m_2}(\gamma_j)$, $j = 1, 2, \ldots, \kappa$, then from Lemma 2.2, we have

$$\sum_{k=2}^{\infty} \frac{\alpha \mathcal{U}(k, m_1) + (1 - \alpha)\mathcal{U}(k, m_2)}{1 - \gamma_j} (|a_{k,j}| + |b_{k,j}|) \leq 1, \quad j = 1, 2, \ldots, \kappa. \tag{3.25}$$

Using Cauchy-Schwarz inequality with (3.25), we can get

$$\begin{aligned} \sum_{k=2}^{\infty} \{\psi(\gamma_j)\}^{\mathcal{J}} (|a_{k,j}|^{\mathcal{J}} + |b_{k,j}|^{\mathcal{J}}) &\leq \sum_{k=2}^{\infty} \{\psi(\gamma_j)\}^{\mathcal{J}} (|a_{k,j}| + |b_{k,j}|)^{\mathcal{J}} \\ &\leq \Big\{ \sum_{k=2}^{\infty} \psi(\gamma_j)(|a_{k,j}| + |b_{k,j}|) \Big\}^{\mathcal{J}} \leq 1, \quad j = 1, 2, \ldots, \kappa, \end{aligned} \tag{3.26}$$

where

$$\psi(\chi) = \frac{\alpha \mathcal{U}(k, m_1) + (1 - \alpha)\mathcal{U}(k, m_2)}{1 - \chi}, \quad 0 \leq \chi \leq 1.$$

Therefore, (3.26) gives that

$$\sum_{k=2}^{\infty} \Big\{ \frac{1}{\kappa} \sum_{j=1}^{\kappa} [\psi(\gamma_j)]^{\mathcal{J}} (|a_{k,j}|^{\mathcal{J}} + |b_{k,j}|^{\mathcal{J}}) \Big\} \leq 1. \tag{3.27}$$

In order to complete the proof, from (3.23) and Lemma 2.2, we must find the largest ϵ_κ such that

$$\sum_{k=2}^{\infty} \psi(\epsilon_\kappa) \Big[\sum_{j=1}^{\kappa} (|a_{k,j}|^r + |b_{k,j}|^r) \Big] \leq 1. \tag{3.28}$$

By (3.27), for any $k \geq 2$, we can see that (3.28) holds true if

$$\psi(\epsilon_\kappa) \Big[\sum_{j=1}^{\kappa} (|a_{k,j}|^{\mathcal{J}} + |b_{k,j}|^{\mathcal{J}}) \Big] \leq \frac{1}{\kappa} \sum_{j=1}^{\kappa} [\psi(\gamma_j)]^{\mathcal{J}} (|a_{k,j}|^{\mathcal{J}} + |b_{k,j}|^{\mathcal{J}}). \tag{3.29}$$

Let $\gamma = \min_{1 \leq j \leq \kappa} \{\gamma_j\}$. Then we have

$$\frac{1}{\kappa} \sum_{j=1}^{\kappa} [\psi(\gamma_j)]^{\mathcal{J}} (|a_{k,j}|^{\mathcal{J}} + |b_{k,j}|^{\mathcal{J}}) \geq \frac{1}{\kappa} [\psi(\gamma)]^{\mathcal{J}} \sum_{j=1}^{\kappa} (|a_{k,j}|^{\mathcal{J}} + |b_{k,j}|^{\mathcal{J}}). \tag{3.30}$$

From (3.30), the (3.29) is equivalent to

$$\psi(\epsilon_\kappa) \leq \frac{1}{\kappa} [\psi(\gamma)]^{\mathcal{J}}, \quad k \geq 2. \tag{3.31}$$

We can know that (3.31) is true if

$$\epsilon_\kappa \leq 1 - \frac{\kappa(1 - \gamma)^{\mathcal{J}}}{[\alpha \mathcal{U}(k, m_1) + (1 - \alpha)\mathcal{U}(k, m_2)]^{\mathcal{J} - 1}}, \quad k \geq 2.$$

Now let

$$g(k) = 1 - \frac{\kappa(1-\gamma)^{\mathcal{J}}}{[\alpha\mathcal{U}(k,m_1) + (1-\alpha)\mathcal{U}(k,m_2)]^{\mathcal{J}-1}}.$$

It is clear that $g(k)$ is increasing in k, which gives that

$$g(k) \geq g(2) = 1 - \frac{\kappa(1-\gamma)^{\mathcal{J}}}{[\alpha(m_1+1) + (1-\alpha)(m_2+1)]^{\mathcal{J}-1}}.$$

Furthermore, from (3.24), we can see that $0 \leq \epsilon_\kappa \leq 1$, which completes the proof of Theorem 3.6. □

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