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On the Error Estimation of the Ishikawa Iteration Process and Data Dependence for Strongly Demicontractive Mappings in Hilbert Spaces

Chao WANG*, **Honglei FAN**
School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Jiangsu 210044*, P. R. China*

Abstract In this paper, we consider the error estimation of the Ishikawa iteration process for strongly demicontractive (SDC) mappings in real Hilbert spaces (without the Lipschitz condition), some convergence theorems of the Ishikawa iteration process are also obtained. Moreover, we provide data dependence results for SDC mappings in three cases. Some numerical examples are given to verify our results.

Keywords Ishikawa iteration; error estimation; strongly demicontractive mappings; data dependence

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1. Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. C is a nonempty, closed and convex subset of H and $T: C\to C$ is a nonlinear mapping. We denote the set of fixed points of T by $\mathrm{Fix}(T)=\{x\in C: Tx=x\}$. For arbitrary $x_0\in C$, $\{x_n\}$ is called the Ishikawa iteration process [1] of T if

$$\begin{cases} x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n,\\ y_n=(1-\beta_n)x_n+\beta_n Tx_n \end{cases} \tag{1.1}$$

for all $n\ge 0$, where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$. If $\beta_n=0$ for all $n\ge 0$ in (1.1), then we obtain the Mann iteration process [2]. According to the definition of demicontractive mapping, Maruster et al. [3] defined the concept of strongly demicontractive (SDC) mapping as follows:

$$\|Tx-s_*\|^2\le a\|x-s_*\|^2+K\|Tx-x\|^2,$$

where $a\in(0,1)$, $K\ge 0$ and $s_*\in\mathrm{Fix}(T)$ (Notice that if T is an SDC mapping and $\mathrm{Fix}(T)\ne\emptyset$, then T has a unique fixed point). Moreover, they studied the error estimation and $T-$stability of the Mann iteration process for SDC mappings. Wang [4] obtained a new formula of error estimation of the Mann iteration process for SDC mappings which is better than the

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* Corresponding author
E-mail address: wangchaosx@126.com (Chao WANG); fanhonglei95@qq.com (Honglei FAN)

corresponding results in [3]. Later on, Wang et al. [5] presented an error estimation of the Ishikawa iteration process (1.1) for SDC mappings as follows (including the Lipschitz condition):

Theorem 1.1 ([5]) *Let T be L-Lipschitzian (that is, there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for any $x, y \in C$) and strongly demicontractive with $0 < a < 1$ and $K \geq 0$, $s_* \in \mathrm{Fix}(T)$. Assume that there exist positive numbers θ_1, θ_2, $0 < \theta_1 < \theta_2 < \min\{1,\ 1 - (1-a)(1-K)\}$ such that*

$$\frac{\|TT_{\alpha_n,\beta_n}x - T_{\alpha_n,\beta_n}x\|^2}{\|Tx - x\|^2} \leq \frac{\theta_2}{2}, \tag{1.2}$$

where $x \in C$ and $T_{\alpha_n,\beta_n} := (1-\alpha_n)I + \alpha_n T_{\beta_n}$, $T_{\beta_n} := (1-\beta_n)I + \beta_n T$. For $x_0 \in C$, let $\{x_n\}$ be the sequence generated by the Ishikawa iteration process (1.1) with the control sequence $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying

$$\frac{1-\theta_2}{1-a} \leq \alpha_n \leq \frac{1-\theta_1}{1-a}, \quad 0 \leq \beta_n \leq q < 1, \tag{1.3}$$

$$L^2q^2 + q + a < 1 \tag{1.4}$$

for some $0 \leq q < 1$. Then the following error estimation for the sequence $\{x_n\}$ holds

$$\|x_{n+1} - s_*\|^2 \leq \|x_0 - s_*\|^2\theta_2^{n+1} + (1 + Lq)^2 M(2\theta_2\epsilon_{n-1} + \epsilon_n),$$

where $M = (\frac{1-\theta_1}{1-a})^2 - (1-K)\frac{1-\theta_1}{1-a}$ and $\epsilon_n = \|Tx_n - x_n\|^2$.

Remark 1.2 If $x = s_*$ in (1.2), then we need to change the inequality (1.2) into

$$\|TT_{\alpha_n,\beta_n}x - T_{\alpha_n,\beta_n}x\|^2 \leq \frac{\theta_2}{2}\|Tx - x\|^2.$$

Data dependence results of various contractive type mappings have been studied by many authors [6–9]. Recently, Gürsoy et al. [10] proved the following data dependence theorem for SDC mappings in Hilbert spaces:

Theorem 1.3 ([10]) *Let T be strongly demicontractive with $a+2K \in (0,1)$ and $s_* \in \mathrm{Fix}(T) \subset C$ and $\widetilde{T} : C \to C$ be a mapping with $\widetilde{s}_* \in \mathrm{Fix}(\widetilde{T}) \subset C$. Assume that*

$$\sup_{s \in C} \|Ts - \widetilde{T}s\| \leq \epsilon$$

for a fixed number $\epsilon > 0$. Then we have

$$\|s_* - \widetilde{s}_*\| \leq \sqrt{\frac{1-K}{1-a-2K}}\,\epsilon,$$

provided that

$$\langle T\widetilde{s}_* - s_*, T\widetilde{s}_* - \widetilde{s}_*\rangle \geq 0 \tag{1.5}$$

and

$$\langle s_* - \widetilde{s}_*, s_* - T\widetilde{s}_*\rangle \geq 0. \tag{1.6}$$

In this paper, without the Lipschitz condition or the condition (1.4), we first consider the error estimation of the Ishikawa iteration process (1.1) for SDC mappings. Secondly, a simple convergence theorem of the Ishikawa iteration process (1.1) is obtained. Finally, data dependence theorems of SDC mappings (without the condition (1.5) or (1.6)) are discussed. Our results improve and extend Theorems 1.1, 1.3 and some well-known results in [3, 4, 6–8].

Next, the following two lemmas will be needed in the sequel.

Lemma 1.4 ([6]) *Let X be a real Hilbert space and $a \in [0,1]$. Then*

$$\|ax + (1-a)y\|^2 = a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)\|x-y\|^2$$

for any $x, y \in X$. In particular, if $a = \frac{1}{2}$, then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for any $x, y \in X$.

Lemma 1.5 ([4, 5]) *Suppose $\{a_n\}$ and $\{b_n\}$ are two nonnegative real sequences satisfying*

$$a_{n+1} \le \alpha a_n + \beta b_n$$

and

$$\frac{b_{n+1}}{b_n} \le \frac{\alpha}{2}, \tag{1.7}$$

where $n = 0, 1, 2, \ldots$, $0 < \alpha < 1$ and $\beta > 0$. Then

$$\lim_{n\to\infty} a_n = 0$$

and

$$a_{n+1} \le a_0 \alpha^{n+1} + \beta(\alpha^n b_0 + 2\alpha^{n-1} b_1).$$

Remark 1.6 If $b_n = 0$, then we need to change the inequality (1.7) into $b_{n+1} \le \frac{\alpha}{2} b_n$. In this case, the result is obviously true.

2. Error estimation and convergence of the Ishikawa iteration process

In this section, we give the error estimation of the Ishikawa iteration process (1.1) for SDC mappings, and some convergence theorems are also considered.

Theorem 2.1 *Let T be an SDC mapping with $a \in (0,1)$, $K \in [0,1)$ and $s_* \in \mathrm{Fix}(T)$. Assume that there exist positive numbers θ_1, θ_2, $0 < \theta_1 < \theta_2 < \min\{1, 1-(1-a-q)(1-K)\}$ such that*

$$\|TT_{\alpha_n,\beta_n}x - T_{\alpha_n,\beta_n}x\|^2 \le \frac{\theta_2}{2}\|Tx - x\|^2$$

for some $0 \le q < 1$, where $x \in C$, $T_{\alpha_n,\beta_n} := (1-\alpha_n)I + \alpha_n T_{\beta_n}$ and $T_{\beta_n} := (1-\beta_n)I + \beta_n T$. Let $\{x_n\}$ be the sequence generated by the Ishikawa iteration process (1.1) satisfying

$$\frac{1-\theta_2}{1-a-q} \le \alpha_n \le \frac{1-\theta_1}{1-a-q}, \quad 0 \le \beta_n \le q < 1,$$

$$a + q < 1, \quad M = a + \frac{4K}{(1-K)^2} < 1.$$

Then $\{x_n\}$ converges strongly to the fixed point s_* and the following error estimation holds:

$$\|x_{n+1} - s_*\|^2 \le \|x_0 - s_*\|^2 \theta_2^{n+1} + (\frac{1+\sqrt{M}}{1-\sqrt{M}})^2 Q(\theta_2^n \epsilon_0 + 2\theta_2^{n-1}\epsilon_1), \tag{2.1}$$

where $Q=(\frac{1-\theta_1}{1-a-q})^2-(1-K)\frac{1-\theta_1}{1-a-q}$ *and* $\epsilon_n=\|Tx_n-x_n\|^2$.

Proof From the proof in [3, Theorem 2], we know that

$$\|Tx-s_*\|\le\sqrt{M}\|x-s_*\|.$$

Since $M=a+\frac{4K}{(1-K)^2}<1$, we can get

$$\begin{aligned}\|y_n-s_*\|&=\|(1-\beta_n)x_n+\beta_nTx_n-s_*\|\le(1-\beta_n)\|x_n-s_*\|+\beta_n\|Tx_n-s_*\|\\&\le(1-\beta_n)\|x_n-s_*\|+\beta_n\sqrt{M}\|x_n-s_*\|=[1-\beta_n(1-\sqrt{M})]\|x_n-s_*\|\\&\le\|x_n-s_*\|\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}\|y_n-Ty_n\|^2&=\|(1-\beta_n)(x_n-Ty_n)+\beta_n(Tx_n-Ty_n)\|^2\\&=(1-\beta_n)\|Ty_n-x_n\|^2+\beta_n\|Tx_n-Ty_n\|^2-\beta_n(1-\beta_n)\|Tx_n-x_n\|^2\\&\le(1-\beta_n)\|Ty_n-x_n\|^2+\beta_n\|Tx_n-Ty_n\|^2\\&\le(1-\beta_n)\|Ty_n-x_n\|^2+\beta_n[\|Tx_n-s_*\|+\|Ty_n-s_*\|]^2\\&\le(1-\beta_n)\|Ty_n-x_n\|^2+4\beta_nM\|x_n-s_*\|^2.\end{aligned}\tag{2.3}$$

From (2.2) and (2.3), we have

$$\begin{aligned}\|Ty_n-s_*\|^2&\le a\|y_n-s_*\|^2+K\|Ty_n-y_n\|^2\\&\le a\|x_n-s_*\|^2+(1-\beta_n)K\|Ty_n-x_n\|^2+4\beta_nMK\|x_n-s_*\|^2.\end{aligned}\tag{2.4}$$

Since

$$\|x_n-s_*\|\le\|x_n-Tx_n\|+\|Tx_n-s_*\|\le\|x_n-Tx_n\|+\sqrt{M}\|x_n-s_*\|,$$

i.e., $\|x_n-s_*\|\le\frac{1}{1-\sqrt{M}}\|x_n-Tx_n\|$, we have

$$\begin{aligned}\|Ty_n-x_n\|&\le\|Ty_n-s_*\|+\|x_n-s_*\|\le\sqrt{M}\|y_n-s_*\|+\|x_n-s_*\|\\&\le(\sqrt{M}+1)\|x_n-s_*\|\le\frac{1+\sqrt{M}}{1-\sqrt{M}}\|x_n-Tx_n\|.\end{aligned}\tag{2.5}$$

Notice that $K\in(0,3-\sqrt{8})$ (see [5]), using Lemma 1.4, (2.4) and (2.5), we obtain

$$\begin{aligned}\|x_{n+1}-s_*\|^2=&\|(1-\alpha_n)x_n+\alpha_nTy_n-s_*\|^2\\=&(1-\alpha_n)\|x_n-s_*\|^2+\alpha_n\|Ty_n-s_*\|^2-\alpha_n(1-\alpha_n)\|Ty_n-x_n\|^2\\\le&(1-\alpha_n)\|x_n-s_*\|^2+\alpha_n\{a\|x_n-s_*\|^2+K[(1-\beta_n)\|Ty_n-x_n\|^2+\\&4\beta_nM\|x_n-s_*\|^2]\}-\alpha_n(1-\alpha_n)\|Ty_n-x_n\|^2\\\le&[1-\alpha_n(1-a-4\beta_nKM)]\|x_n-s_*\|^2+\\&[\alpha_n^2-\alpha_n(1-K)](\frac{1+\sqrt{M}}{1-\sqrt{M}})^2\|Tx_n-x_n\|^2\\\le&[1-\alpha_n(1-a-q)]\|x_n-s_*\|^2+(\frac{1+\sqrt{M}}{1-\sqrt{M}})^2[\alpha_n^2-\alpha_n(1-K)]\|Tx_n-x_n\|^2.\end{aligned}$$

Similar to the proof in [3, Theorem 1], we have

$$\|x_{n+1}-s_*\|^2 \le \theta_2\|x_n-s_*\|^2+(\frac{1+\sqrt{M}}{1-\sqrt{M}})^2Q\epsilon_n,$$

where $Q=(\frac{1-\theta_1}{1-a-q})^2-(1-K)\frac{1-\theta_1}{1-a-q}$ and $\epsilon_n=\|Tx_n-x_n\|^2$. From Lemma 1.5, it follows that $\{x_n\}$ converges strongly to the fixed point s_* and

$$\|x_{n+1}-s_*\|^2 \le \|x_0-s_*\|^2\theta_2^{n+1}+(\frac{1+\sqrt{M}}{1-\sqrt{M}})^2Q(\theta_2^n\epsilon_0+2\theta_2^{n-1}\epsilon_1). \quad \square$$

Remark 2.2 In Theorem 2.1, T does not need to be L-Lipschitzian. So, Theorem 2.1 improves Theorem 1.1.

Remark 2.3 Suppose T is a differentiable mapping and it is an SDC mapping with $a\in(0,1)$ and $K\in[0,1)$, the sequence $\{x_n\}$ is generated by (1.1) satisfying:

$$\alpha_n=t_1,\ \beta_n=t_2,\ \frac{1-\theta_2}{1-a-q}\le t_1\le\frac{1-\theta_1}{1-a-q},$$
$$0\le t_2\le q<1,\ a+q<1,\ a+\frac{4K}{(1-K)^2}<1.$$

In this case,

$$TT_{t_1t_2}x-T_{t_1t_2}x=[1-t_1t_2(1-T'\xi)](Tx-x),$$

where $\xi=x+\eta t_1t_2(Tx-x)$, $0<\eta<1$. If the derivative T' of T satisfies

$$|1-t_1t_2(1-T'\xi)|^2\le\frac{\theta_2}{2},$$

then

$$\|TT_{t_1t_2}x-T_{t_1t_2}x\|^2\le\frac{\theta_2}{2}\|Tx-x\|^2,\quad \forall x\in C.$$

Therefore, from Theorem 2.1, we know that $\{x_n\}$ converges strongly to the fixed point of T and the error estimation formula (2.1) holds.

Example 2.4 Let $C=[-1,1]$, define a mapping $T:C\to C$ by

$$Tx=\begin{cases}-\frac{1}{10}x^2, & x\in[-1,0.6],\\ -x+0.564, & x\in(0.6,1],\end{cases}$$

$s_*=0$ is the unique fixed point and T is strongly demicontractive with $a=0.03$ and $K=0.1$ (it is not Lipschizian). Let $q=0.55$, we have

$$M=a+\frac{4K}{(1-K)^2}<1 \text{ and } a+q<1.$$

Since $0<\theta_1<\theta_2<\min\{1,1-(1-a-q)(1-K)\}=0.622$, set $\theta_1=0.596$, $\theta_2=0.6215$. From

$$\frac{1-\theta_2}{1-a-q}\le t_1\le\frac{1-\theta_1}{1-a-q} \text{ and } 0\le t_2\le q,$$

we obtain $t_1\in[0.9012,0.9619]$ and $t_2\in[0,0.55]$. Thus $t_1t_2\in[0,0.529]$. We can choose some t_1 and t_2 satisfying $t_1t_2\in[0.5225,0.529]$. Then

$$\frac{||TT_{t_1t_2}x-T_{t_1t_2}x||^2}{||Tx-x||^2}\le 0.31073<0.31075=\frac{\theta_2}{2},\quad \forall x\in C \text{ and } x\ne s_*.$$

Obviously, if $x = s_*$, then

$$\|TT_{t_1t_2}x - T_{t_1t_2}x\|^2 = \frac{\theta_2}{2}\|Tx - x\|^2.$$

So, all the conditions of Theorem 2.1 are satisfied. We give the following error estimation of the Ishikawa iteration process based on (2.1):

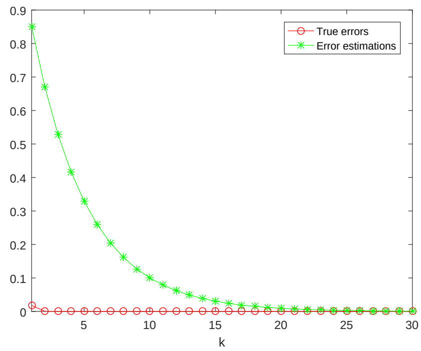

Figure 1 Illustration of true errors $\|x_k - s_*\|$ $(1 \le k \le 30)$ and error estimations with $x_0 = 0.6$

Theorem 2.5 *Let T be L-Lipschitzian and strongly demicontractive with $0 < a < 1$, $K \ge 0$ and $s_* \in \mathrm{Fix}(T)$. Assume that there exist positive numbers θ_1, θ_2, $0 < \theta_1 < \theta_2 < \min\{1, 1-(1-a)(1-K)\}$ such that*

$$\|TT_{\alpha_n,\beta_n}x - T_{\alpha_n,\beta_n}x\|^2 \le \frac{\theta_2}{2}\|Tx - x\|^2,$$

where $x \in C$, $T_{\alpha_n,\beta_n} := (1-\alpha_n)I + \alpha_n T_{\beta_n}$ and $T_{\beta_n} := (1-\beta_n)I + \beta_n T$. Let $\{x_n\}$ be the sequence generated by the Ishikawa iteration process (1.1) satisfying

$$\frac{1-\theta_2}{1-a} \le \alpha_n \le \frac{1-\theta_1}{1-a}, \quad 0 \le \beta_n \le q < 1$$

for some $0 \le q < 1$. Then $\{x_n\}$ converges strongly to the fixed point s_* and the following error estimation holds:

$$\|x_{n+1} - s_*\|^2 \le \|x_0 - s_*\|^2\theta_2^{n+1} + (1+Lq)^2 Q(\theta_2^n \epsilon_0 + 2\theta_2^{n-1}\epsilon_1), \tag{2.6}$$

where $Q = (\frac{1-\theta_1}{1-a})^2 - (1-K)\frac{1-\theta_1}{1-a}$ and $\epsilon_n = \|Tx_n - x_n\|^2$.

Proof From the proof in [5, Theorem 2.1], we have

$$\|x_{n+1} - s_*\|^2 \le [1 - \alpha_n(1-a)]\|x_n - s_*\|^2 + A\|Tx_n - x_n\|^2,$$

where

$$\begin{aligned} A =& \alpha_n\beta_n[aK - (a+K)(1-\beta_n) + KL^2\beta_n^2] + \alpha_n[K(1-\beta_n) - 1 + \alpha_n](1+L\beta_n)^2 \\ =& \alpha_n\beta_n[(a+\beta_n-1)K - a(1-\beta_n)] + \alpha_n[\alpha_n - (1-K)](1+L\beta_n)^2 + KL^2\alpha_n\beta_n^3 - \\ & K\alpha_n\beta_n(1+L\beta_n)^2 \\ \le& (1+L\beta_n)^2[\alpha_n^2 - (1-K)\alpha_n] + K\alpha_n\beta_n(a+\beta_n-1) - K\alpha_n\beta_n(1+2L\beta_n) \end{aligned}$$

$$\leq (1+L\beta_n)^2[\alpha_n^2-(1-K)\alpha_n]+K\alpha_n\beta_n(a+\beta_n-2)$$
$$\leq (1+L\beta_n)^2[\alpha_n^2-(1-K)\alpha_n].$$

Then, we have

$$\|x_{n+1}-s_*\|^2 \leq [1-\alpha_n(1-a)]\|x_n-s_*\|^2+(1+L\beta_n)^2[\alpha_n^2-(1-K)\alpha_n]\|Tx_n-x_n\|^2.$$

From Lemma 1.5, it follows that $\{x_n\}$ converges strongly to the fixed point s_* and

$$\|x_{n+1}-s_*\|^2 \leq \|x_0-s_*\|^2\theta_2^{n+1}+(1+Lq)^2Q(\theta_2^n\epsilon_0+2\theta_2^{n-1}\epsilon_1),$$

where $Q=(\frac{1-\theta_1}{1-a})^2-(1-K)\frac{1-\theta_1}{1-a}$ and $\epsilon_n=\|Tx_n-x_n\|^2$. □

Remark 2.6 In Theorem 2.5, we do not need the condition "$L^2q^2+q+a<1$". So, Theorem 2.5 improves Theorem 1.1.

Now, we give a simple convergence theorem of the Ishikawa iteration process (1.1) for SDC mappings.

Theorem 2.7 *Let $\{x_n\}_{n=0}^\infty$ be the sequence generated by the Ishikawa iteration process (1.1), T be an SDC mapping with $a\in(0,1)$, $K\in[0,1)$ and $s_*\in \mathrm{Fix}(T)$. Assume that*

$$a+\frac{4K}{(1-K)^2}<1 \text{ and } \sum_{n=0}^\infty \alpha_n=\infty,$$

then the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to the fixed point s_* of T.

Proof From the proof in [3, Theorem 2], we have

$$\|Tx-s_*\|\leq \sqrt{M}\|x-s_*\|,$$

where $M=a+\frac{4K}{(1-K)^2}$. Since $M<1$, we have

$$\begin{aligned}
\|x_{n+1}-s_*\| &= \|(1-\alpha_n)x_n+\alpha_nTy_n-s_*\|\\
&\leq (1-\alpha_n)\|x_n-s_*\|+\alpha_n\|Ty_n-s_*\|\\
&\leq (1-\alpha_n)\|x_n-s_*\|+\alpha_n\sqrt{M}\|y_n-s_*\|\\
&\leq (1-\alpha_n)\|x_n-s_*\|+\alpha_n\sqrt{M}[(1-\beta_n)\|x_n-s_*\|+\beta_n\|Tx_n-s_*\|]\\
&\leq (1-\alpha_n)\|x_n-s_*\|+\alpha_n\sqrt{M}(1-\beta_n+\beta_n\sqrt{M})\|x_n-s_*\|\\
&\leq [1-\alpha_n(1-\sqrt{M})]\|x_n-s_*\|\\
&\leq \prod_{k=0}^n[1-(1-\sqrt{M})\alpha_k]\|x_0-s_*\|\\
&\leq e^{-[(1-\sqrt{M})\sum_{k=0}^n\alpha_k]}\|x_0-s_*\|.
\end{aligned}$$

Note that $\sum_{n=0}^\infty\alpha_n=\infty$. Then $\lim_{n\to\infty}x_n=s_*$, i.e., $\{x_n\}$ converges strongly to s_*. □

Remark 2.8 In Theorem 2.7, we change the condition "there exists $\lambda>0$, such that $\lambda\leq\alpha_n<1$" (see [5]) into a weak condition "$\sum_{n=0}^\infty\alpha_n=\infty$". So, Theorem 2.7 generalizes the corresponding result in [5, Theorem 4.1].

Remark 2.9 By [11, Section 4], the condition "$a+\frac{4K}{(1-k)^2}<1$" can be replaced by the condition "$\sqrt{a}+2\sqrt{K}<1$". In this case, the value range of K can be extended from $(0,3-\sqrt{8})$ to $(0,\frac{1}{4})$.

Example 2.10 Let $C=[-1,1]$ and define a mapping $T:C\to C$ by

$$Tx=\begin{cases}\frac{1}{3}x, & x\in[-1,0.6],\\ -x+0.5, & x\in(0.6,1].\end{cases}$$

$s_*=0$ is the unique fixed point of T and T is strongly demicontractive with $a=0.067$ and $K=0.1$. In this case, $a+\frac{4K}{(1-K)^2}<1$.

(1) Set $\alpha_n=1-\frac{1}{n+2}$, $\beta_n=\frac{1}{2}$. Then the conditions of Theorem 2.7 are satisfied (the conditions in [5, Theorem 4.1] are also satisfied). Let $x_0\in[-1,0.6]$. By (1.1), we have

$$y_n=\frac{1}{2}x_n+\frac{1}{2}\cdot\frac{1}{3}x_n=\frac{2}{3}x_n$$

and

$$\begin{aligned}x_{n+1}=&\frac{1}{n+2}x_n+\frac{2}{9}(1-\frac{1}{n+2})x_n\\ =&\frac{1}{9}(2+\frac{7}{n+2})x_n\\ =&\frac{1}{9^{n+1}}\prod_{i=0}^{n}(2+\frac{7}{n+2-i})x_0.\end{aligned}$$

So, $\lim_{n\to\infty}x_n=s_*$.

(2) Set $\alpha_n=\frac{1}{n+2}$, $\beta_n=\frac{1}{2}$. Then $\sum_{n=0}^{\infty}\alpha_n=\infty$. The conditions of Theorem 2.7 are satisfied (but the conditions in [5, Theorem 4.1] are not satisfied). Let $x_0\in[-1,0.6]$. By (1.1), we have

$$y_n=\frac{2}{3}x_n$$

and

$$\begin{aligned}x_{n+1}&=(1-\frac{1}{n+2})x_n+\frac{2}{9}\frac{1}{n+2}x_n\\ &=\frac{1}{9}(9-\frac{7}{n+2})x_n\\ &=\frac{1}{9^{n+1}}\prod_{i=0}^{n}(9-\frac{7}{n+2-i})x_0.\end{aligned}$$

So, we can also get $\lim_{n\to\infty}x_n=s_*$.

3. Data dependence for SDC mappings

Let $\tilde{s}_*$ be a fixed point of $\tilde{T}$. Due to various reasons, it is difficult to calculate $\tilde{s}_*$ by using existing methods or iterative algorithms. But we can calculate the fixed point s_* of another mapping T and use s_* to approximate $\tilde{s}_*$. This method is called "Data Dependence", by using a mapping which is easy to calculate the fixed point to replace the fixed point of the mapping that we need.

Theorem 3.1 *Let $T:C\to C$ be an SDC mapping with $s_*\in\mathrm{Fix}(T)$ and $\tilde{T}:C\to C$ be a* *mapping with* $\tilde{s}_* \in \mathrm{Fix}(\tilde{T})$. *Assume that*

$$\sup_{s\in C}\|Ts-\widetilde{T}s\|\le\epsilon$$

for a fixed number $\epsilon>0$. *Then we have*

$$\|s_*-\tilde{s}_*\|\le\sqrt{\frac{1-2K}{1-a-4K}}\ \epsilon,$$

provided that

$$\langle T\tilde{s}_*-s_*, T\tilde{s}_*-\tilde{s}_*\rangle\ge 0, \tag{3.1}$$

$$0<a+4K<1. \tag{3.2}$$

Proof According to (3.1) and the proof in [10, Theorem 5], we can get

$$\|s_*-\tilde{s}_*\|^2\le a\|\tilde{s}_*-s_*\|^2+K\|\tilde{s}_*-T\tilde{s}_*\|^2+\sup_{s\in C}\|Ts-\tilde{T}s\|^2. \tag{3.3}$$

By Lemma 1.4, we have

$$\begin{aligned}\|T\tilde{s}_*-\tilde{s}_*\|^2&=\|(\tilde{s}_*-s_*)+(s_*-T\tilde{s}_*)\|^2\\&\le\|(\tilde{s}_*-s_*)+(s_*-T\tilde{s}_*)\|^2+\|(\tilde{s}_*-s_*)-(s_*-T\tilde{s}_*)\|^2\\&=2\|\tilde{s}_*-s_*\|^2+2\|s_*-T\tilde{s}_*\|^2\\&\le 2\|s_*-\tilde{s}_*\|^2+2a\|\tilde{s}_*-s_*\|^2+2K\|T\tilde{s}_*-\tilde{s}_*\|^2.\end{aligned}$$

Thus,

$$\|T\tilde{s}_*-\tilde{s}_*\|^2\le\frac{2+2a}{1-2K}\|\tilde{s}_*-s_*\|^2. \tag{3.4}$$

From (3.2)–(3.4), we obtain

$$\|s_*-\tilde{s}_*\|\le\sqrt{\frac{1-2K}{1-a-4K}}\ \epsilon. \quad \square$$

Example 3.2 Let $C=[-1,0.7]$, define mappings $T,\tilde{T}:C\to C$ by

$$Tx=\begin{cases}-\frac{1}{10}x^2, & x\in[-1,0.6],\\ -x+0.8, & x\in(0.6,0.7]\end{cases}$$

and

$$\tilde{T}s=0.05\sin x+0.054.$$

Obviously, $s_*=0$ is the unique fixed point of T and T is strongly demicontractive with $a=0.112$ and $K=10^{-7}$. In this case, $0<a+4K<1$. On the other hand, $\tilde{s}_*=0.056840$ is the unique fixed point of $\tilde{T}$. So, we have $\langle T\tilde{s}_*-s_*, T\tilde{s}_*-\tilde{s}_*\rangle\ge 0$. Note that $\sup_{s\in C}\|Ts-\tilde{T}s\|=0.118232$, we can get the following estimation by using the conclusion of Theorem 3.1:

$$0.056840=\|s_*-\tilde{s}_*\|\le\sqrt{\frac{1-2\times10^{-7}}{1-0.112-4\times10^{-7}}}\times0.118232=0.125467.$$

Remark 3.3 In Example 3.2, since $\langle s_*-\tilde{s}_*, s_*-T\tilde{s}_*\rangle<0$, the fixed point $\tilde{s}_*$ of $\tilde{T}$ cannot be estimated by Theorem 1.3. So, to some extent, Theorem 3.1 generalizes Theorem 1.3.

Theorem 3.4 *Let $T : C \to C$ be an SDC mapping with $s_* \in \mathrm{Fix}(T)$ and $\tilde{T} : C \to C$ be a mapping with $\tilde{s}_* \in \mathrm{Fix}(\tilde{T})$. Assume that*

$$\sup_{s\in C} \|Ts - \widetilde{T}s\| \le \epsilon$$

for a fixed number $\epsilon > 0$. Then we have

$$\|s_* - \tilde{s}_*\| \le \sqrt{\frac{2(1-K)}{1-2a-3K}}\ \epsilon,$$

provided that

$$\langle s_* - \tilde{s}_*, s_* - T\tilde{s}_*\rangle \ge 0, \tag{3.5}$$

$$0 < 2a + 3K < 1. \tag{3.6}$$

Proof By Lemma 1.4 and (3.6), we have

$$\begin{aligned}\|s_* - \tilde{s}_*\|^2 &= \|(s_* - T\tilde{s}_*) + (T\tilde{s}_* - \tilde{s}_*)\|^2 \\ &\le \|(s_* - T\tilde{s}_*) + (T\tilde{s}_* - \tilde{s}_*)\|^2 + \|(s_* - T\tilde{s}_*) - (T\tilde{s}_* - \tilde{s}_*)\|^2 \\ &= 2\|s_* - T\tilde{s}_*\|^2 + 2\|T\tilde{s}_* - \tilde{s}_*\|^2 \\ &\le 2a\|\tilde{s}_* - s_*\|^2 + 2K\|T\tilde{s}_* - \tilde{s}_*\|^2 + 2\|T\tilde{s}_* - \tilde{T}\tilde{s}_*\|^2 \\ &\le 2a\|\tilde{s}_* - s_*\|^2 + 2K\|T\tilde{s}_* - \tilde{s}_*\|^2 + 2\sup_{s\in C}\|Ts - \tilde{T}s\|^2. \end{aligned} \tag{3.7}$$

According to (3.5) and the proof in [10, Theorem 5], we obtain

$$\|\tilde{s}_* - T\tilde{s}_*\| \le \frac{1+a}{1-K}\|\tilde{s}_* - s_*\|^2. \tag{3.8}$$

By (3.7) and (3.8), we can get that

$$\|s_* - \tilde{s}_*\|^2 \le 2a\|\tilde{s}_* - s_*\|^2 + 2K\frac{1+a}{1-K}\|\tilde{s}_* - s_*\|^2 + 2\sup_{s\in C}\|Ts - \tilde{T}s\|^2.$$

Then,

$$\|s_* - \tilde{s}_*\| \le \sqrt{\frac{2(1-K)}{1-2a-3K}}\ \epsilon. \quad \square$$

Example 3.5 Let $C = [-1, 0.7]$, define mappings $T, \tilde{T} : C \to C$ by

$$Tx = \begin{cases} \frac{1}{10}x^2, & x \in [-1, 0.6], \\ -x + 0.8, & x \in (0.6, 0.7] \end{cases}$$

and

$$\tilde{T}s = 0.05\sin x + 0.082.$$

Obviously, $s_* = 0$ is the unique fixed point of T and T is strongly demicontractive with $a = 0.108$ and $K = 0.0071$. In this case, $0 < 2a + 3K < 1$. On the other hand, $\tilde{s}_* = 0.086310$ is the unique fixed point of $\tilde{T}$. So, $\langle s_* - \tilde{s}_*, s_* - T\tilde{s}_*\rangle \ge 0$. Since $\sup_{s\in C}\|Ts - \tilde{T}s\| = 0.089768$, we can get the following estimation by using the conclusion of Theorem 3.4:

$$0.086310 = \|s_* - \tilde{s}_*\| \le \sqrt{\frac{2\times(1-0.0071)}{1-2\times 0.108 - 3\times 0.0071}} \times 0.089768 = 0.144848.$$

Remark 3.6 In Example 3.5, since $\langle T\tilde{s}_* - s_*, T\tilde{s}_* - \tilde{s}_*\rangle < 0$, the fixed point $\tilde{s}_*$ of $\tilde{T}$ cannot be estimated by Theorem 3.1 and Theorem 1.3. So, to some extent, Theorem 3.4 generalizes Theorem 1.3.

Remark 3.7 Notice that

$$\min\{\frac{1-2K}{1-a-4K}, \frac{2(1-K)}{1-2a-3K}\} > \frac{1-K}{1-a-2K}.$$

If (1.5) and (1.6) are all satisfied, we often use Theorem 1.3 to estimate the fixed point $\tilde{s}_*$ of $\tilde{T}$.

Theorem 3.8 *Let $T : C \to C$ be an SDC mapping with $s_* \in \mathrm{Fix}(T)$ and $\tilde{T} : C \to C$ be a mapping with $\tilde{s}_* \in \mathrm{Fix}(\tilde{T})$. Assume that*

$$\sup_{s\in C} \|Ts - \widetilde{T}s\| \leq \epsilon$$

for a fixed number $\epsilon > 0$, and

$$a + 3K < \frac{1}{2}. \tag{3.9}$$

Then we have

$$\|s_* - \tilde{s}_*\| \leq \sqrt{\frac{2(1-2K)}{1-2a-6K}}\ \epsilon.$$

Proof By (3.4) and (3.7), we can get that

$$\|s_* - \tilde{s}_*\|^2 \leq 2a\|s_* - \tilde{s}_*\|^2 + 2K\frac{2+2a}{1-2K}\|s_* - \tilde{s}_*\|^2 + 2\sup_{s\in C}\|Ts - \tilde{T}s\|^2.$$

Therefore,

$$\|s_* - \tilde{s}_*\| \leq \sqrt{\frac{2(1-2K)}{1-2a-6K}}\ \epsilon. \quad \square$$

Remark 3.9 (1) For Example 3.2, if $a = 0.112$, $K = 10^{-7}$, then we have $a + 3K < \frac{1}{2}$. We can get the following estimation by using the conclusion of Theorem 3.8:

$$0.056840 = \|s_* - \tilde{s}_*\| \leq \sqrt{\frac{2\times(1-2\times 10^{-7})}{1-2\times 0.112 - 6\times 10^{-7}}} \times 0.118232 = 0.189810.$$

For Example 3.5, if $a = 0.108$, $K = 0.0071$, then we have $a + 3K < \frac{1}{2}$. We can get the following estimation by using the conclusion of Theorem 3.8:

$$0.086310 = \|s_* - \tilde{s}_*\| \leq \sqrt{\frac{2\times(1-2\times 0.0071)}{1-2\times 0.108 - 6\times 0.0071}} \times 0.089768 = 0.146388.$$

(2) Notice that

$$\max\{\frac{1-2K}{1-a-4K}, \frac{2(1-K)}{1-2a-3K}\} < \frac{2(1-2K)}{1-2a-6K}.$$

If (1.5) or (1.6) is satisfied, then the estimation result of Theorem 3.8 is not better than that of Theorems 3.1 or 3.4. But we know that $\tilde{s}_*$ is often unknown in calculation. In this case, (3.1) and (3.5) cannot be verified. Thus, Theorem 3.8 has more application value than Theorems 3.1, 3.4 and 1.3.

4. Conclusions

This paper studies two problems. Firstly, the error estimation and strong convergence of the Ishikawa process for fixed points of SDC mappings under different conditions are considered. Some examples are given to illustrate the applicability of our results. Secondly, the data dependence of SDC mappings is discussed in three cases, and their respective advantages are illustrated with examples.

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