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More Divisibility Properties for Binomial Coefficients

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Abstract Let a, b and n be positive integers with $a > b$, we prove the following divisibility property: For all positive integers n, we have

$$(2bn+1)(2bn+3)(2bn+5)\binom{2bn}{bn}\Big|15(a-b)(3a-b)(5a-b)(5a-3b)\binom{2an}{an}\binom{an}{bn},$$

which extends the result of Yang. And for all positive integers n, we show the following divisibility properties:

$$(6n+1)\binom{4n}{n}\Big|\binom{12n}{6n}\binom{2n}{n},\ (12n+1)\binom{5n}{n}\Big|\binom{15n}{3n}\binom{3n-1}{n-1},$$

$$(18n+1)\binom{12n}{9n}\binom{8n}{2n}\Big|\binom{24n}{18n}\binom{4n}{2n}\binom{6n}{3n}.$$

Other more similar divisibility properties are given also.

Keywords binomial coefficients; p-adic order; divisibility property

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1. Introduction

There are many interesting consequences in studying the divisibility property of binomial coefficients. In 2009, by using the work of Beukers and Heckman [1], Bober [2] proved Vasyunin's conjectures concerning the completeness of his classification of some step functions [3, Conjectures 8 and 11]. That is, Bober [2] showed that: Let $r, a_1, \ldots, a_r, b_1, \ldots, b_{r+1}$ be positive integers such that $a_k \neq b_l$ for all integers k and l, $\sum a_k = \sum b_l$ and $\gcd(a_1, \ldots, a_r, b_1, \ldots, b_{r+1}) = 1$. Then

$$u_n(\vec{a}, \vec{b}) = \frac{(a_1 n)!(a_2 n)!\cdots(a_r n)!}{(b_1 n)!(b_2 n)!\cdots(b_{r+1} n)!}$$

is an integer for all $n \in \mathbb{N}$ if and only if either of the following statements holds:

(i) $u_n = u_n(\vec{a}, \vec{b})$ takes one of the following forms:

$$u_n = \frac{[(a+b)n]!}{(an)!(bn)!} \text{ for } \gcd(a,b) = 1. \tag{1.1}$$

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$$u_n=\frac{(2an)!(bn)!}{(an)!(2bn)![(a-b)n]!}\ \text{ for }\ \gcd(a,b)=1\ \text{ and }\ a>b. \tag{1.2}$$

$$u_n=\frac{(2an)!(2bn)!}{(an)!(bn)![(a+b)n]!}\ \text{ for }\ \gcd(a,b)=1. \tag{1.3}$$

(ii) $(\vec{a},\vec{b})$ is one of the 52 sporadic parameter sets listed in the second column of Table 2 in Bober [2].

In [1] Beukers and Heckman proved a powerful result. We can check Table 8.3 in [1] to find all integral factorial ratios. Moreover, we find the integral factorial ratios in Table 8.3 of [1] can be written as fraction ratios of binomial coefficients, such as

$$\frac{(12n)!(3n)!(2n)!}{(6n)!(6n)!(4n)!(n)!}=\frac{\binom{12n}{6n}\binom{2n}{n}}{\binom{4n}{n}}\ \text{ and }\ \frac{(9n)!(2n)!}{(6n)!(4n)!(n)!}=\frac{\binom{9n}{3n}\binom{2n}{n}}{\binom{4n}{n}}.$$

In 2012, Sun [4,5] proved some divisibility properties of binomial coefficients, such as

$$2(2n+1)\binom{2n}{n}\Big|\binom{6n}{3n}\binom{3n}{n}$$

and

$$(10n+1)\binom{3n}{n}\Big|\binom{15n}{5n}\binom{5n-1}{n-1}.$$

In 2014, Guo [6] proved two conjectures of Sun in [4,5]. That is, for all positive integers n, we have

$$(2n+3)\binom{2n}{n}\Big|3\binom{6n}{3n}\binom{3n}{n}$$

and

$$(10n+3)\binom{3n}{n}\Big|21\binom{15n}{5n}\binom{5n}{n}.$$

In this paper, we first show more divisibility properties for other binomial coefficients in Bober [2].

Theorem 1.1 *Let n be a positive integer. Then*

$$(6n+1)\binom{4n}{n}\Big|\binom{12n}{6n}\binom{2n}{n}\ \text{ and }\ (6n+5)\binom{4n}{n}\Big|5\binom{12n}{6n}\binom{2n}{n}. \tag{1.4}$$

Theorem 1.2 *Let n be a positive integer. Then*

$$(12n+1)\binom{5n}{n}\Big|\binom{15n}{3n}\binom{3n-1}{n-1}. \tag{1.5}$$

Theorem 1.3 *Let n be a positive integer. Then*

$$(18n+1)\binom{12n}{9n}\binom{8n}{2n}\Big|\binom{24n}{18n}\binom{4n}{2n}\binom{6n}{3n}. \tag{1.6}$$

We also have more divisibility properties for other binomial coefficients which appeared in Bober [2].

Theorem 1.4 *Let n be a positive integer. Then*

$$(8n+1)\Big|\frac{(12n)!n!}{(8n)!(3n)!(2n)!}=\frac{\binom{12n}{8n}\binom{4n}{n}}{\binom{2n}{n}},\quad (8n+1)\Big|\frac{(12n)!(3n)!}{(8n)!(6n)!n!}=\frac{\binom{12n}{8n}\binom{4n}{n}}{\binom{6n}{3n}},\quad (\mathit{lines}\,3,4),$$

$$(8n+1)\Big|\frac{(9n)!(4n)!}{(8n)!(3n)!(2n)!}=\frac{\binom{9n}{8n}\binom{4n}{n}}{\binom{2n}{n}},\quad (8n+1)\Big|\frac{7(15n)!(4n)!}{(8n)!(6n)!(5n)!}=\frac{7\binom{15n}{7n}\binom{7n}{n}}{\binom{5n}{n}},\quad (\textit{lines}\,9,27),$$

$$(8n+1)\Big|\frac{7(15n)!n!}{(8n)!(5n)!(3n)!}=\frac{7\binom{15n}{7n}\binom{7n}{2n}}{\binom{3n}{n}},\quad (\textit{line}\,37),$$

where the above indicated lines are the corresponding lines of Table 2 in [2].

Theorem 1.5 *Let n be a positive integer. Then*

$$(10n+1)\Big|\frac{(12n)!(5n)!}{(10n)!(4n)!(3n)!}=\frac{\binom{12n}{10n}\binom{5n}{n}}{\binom{3n}{n}},\qquad (6n+1)\Big|\frac{(9n)!(2n)!}{(6n)!(4n)!n!}=\frac{\binom{9n}{6n}\binom{3n}{n}}{\binom{4n}{2n}},\ (\textit{lines}\,6,8),$$

$$(5n+1)\Big|\frac{4(9n)!n!}{(5n)!(3n)!(2n)!}=\frac{4\binom{9n}{5n}\binom{4n}{n}}{\binom{2n}{n}},\qquad (10n+1)\Big|\frac{(18n)!(5n)!(3n)!}{(10n)!(9n)!(6n)!n!}=\frac{\binom{18n}{10n}\binom{3n}{n}\binom{2n}{n}}{\binom{9n}{8n}\binom{6n}{n}},\ (11,12),$$

$$(12n+1)\Big|\frac{(18n)!(4n)!}{(12n)!(9n)!n!}=\frac{\binom{18n}{6n}\binom{4n}{n}}{\binom{9n}{3n}},\qquad (9n+1)\Big|\frac{(12n)!(2n)!}{(9n)!(4n)!n!}=\frac{\binom{12n}{3n}\binom{2n}{n}}{\binom{4n}{n}},\ (13,14),$$

$$(9n+1)\Big|\frac{(10n)!(6n)!}{(9n)!(5n)!(2n)!}=\frac{\binom{10n}{n}\binom{6n}{n}}{\binom{2n}{n}},\qquad (9n+1)\Big|\frac{5(14n)!(3n)!}{(9n)!(7n)!n!}=\frac{5\binom{14n}{5n}\binom{3n}{n}}{\binom{7n}{2n}},\ (16,17),$$

$$(7n+1)\Big|\frac{5(12n)!(2n)!}{(7n)!(4n)!(3n)!}=\frac{5\binom{12n}{5n}\binom{5n}{n}}{\binom{3n}{n}},\qquad (12n+1)\Big|\frac{(14n)!(6n)!(4n)!}{(12n)!(7n)!(3n)!(2n)!}=\frac{\binom{14n}{2n}\binom{4n}{n}}{\binom{7n}{n}},\ (19,20),$$

$$(7n+1)\Big|\frac{3(10n)!(6n)!n!}{(7n)!(5n)!(3n)!(2n)!}=\frac{3\binom{10n}{7n}\binom{6n}{2n}}{\binom{5n}{n}},\qquad (9n+1)\Big|\frac{2(15n)!n!}{(9n)!(5n)!(2n)!}=\frac{2\binom{15n}{9n}\binom{6n}{n}}{\binom{2n}{n}},\ (22,23),$$

$$(18n+1)\Big|\frac{(30n)!(9n)!(5n)!}{(18n)!(15n)!(10n)!n!}=\frac{\binom{30n}{18n}\binom{5n}{3n}\binom{2n}{n}}{\binom{15n}{12n}\binom{10n}{n}},\qquad (10n+1)\Big|\frac{(15n)!(2n)!}{(10n)!(4n)!(3n)!}=\frac{\binom{15n}{10n}\binom{5n}{2n}}{\binom{4n}{2n}},\ (24,29),$$

$$(10n+1)\Big|\frac{(15n)!(2n)!}{(10n)!(6n)!n!}=\frac{\binom{15n}{10n}\binom{2n}{n}}{\binom{6n}{n}},\qquad (14n+1)\Big|\frac{(15n)!(7n)!}{(14n)!(5n)!(3n)!}=\frac{\binom{15n}{n}\binom{7n}{2n}}{\binom{3n}{n}},\ (32,33),$$

$$(12n+1)\Big|\frac{(15n)!(6n)!n!}{(12n)!(5n)!(3n)!(2n)!}=\frac{\binom{15n}{3n}\binom{6n}{n}}{\binom{2n}{n}},\qquad (12n+1)\Big|\frac{(20n)!(3n)!}{(12n)!(10n)!n!}=\frac{\binom{20n}{12n}\binom{3n}{n}}{\binom{10n}{2n}},\ (36,39),$$

$$(12n+1)\Big|\frac{(20n)!(6n)!n!}{(12n)!(10n)!(3n)!(2n)!}=\frac{\binom{20n}{12n}\binom{6n}{3n}\binom{3n}{n}}{\binom{10n}{2n}\binom{2n}{n}},\qquad (14n+1)\Big|\frac{3(20n)!(7n)!(2n)!}{(14n)!(10n)!(4n)!n!}=\frac{3\binom{20n}{14n}\binom{6n}{2n}\binom{2n}{n}}{\binom{10n}{7n}\binom{3n}{n}},\ (40,44),$$

$$(18n+1)\Big|\frac{(20n)!(9n)!(6n)!}{(18n)!(10n)!(4n)!(3n)!}=\frac{\binom{20n}{18n}\binom{6n}{2n}\binom{2n}{n}}{\binom{10n}{n}\binom{3n}{n}},\qquad (14n+1)\Big|\frac{5(24n)!(7n)!(4n)!}{(14n)!(12n)!(8n)!n!}=\frac{5\binom{24n}{14n}\binom{10n}{2n}\binom{2n}{n}}{\binom{12n}{5n}\binom{5n}{n}},\ (46),$$

where the above indicated lines are the corresponding lines of Table 2 in [2].

Remark 1.6 The fraction ratios of binomial coefficients in Theorems 1.1–1.3 correspond to lines ♯ 2, ♯ 25 and ♯ 52 of Table 2 (see [2]), respectively. And the fraction ratios of binomial coefficients in Theorems 1.4 and 1.5 correspond to other related lines in [2] Table 2.

In [7], Yang proved the conjecture by Guo [8]. Let a, b and n be positive integers with $a > b$. Yang showed that

$$(2bn+1)(2bn+3)\binom{2bn}{bn}\Big|3(a-b)(3a-b)\binom{2an}{an}\binom{an}{bn}.$$

Another purpose of this paper is to extend the result of Yang in [7]. We have the following results.

Theorem 1.7 *Let a, b and n be positive integers with $a > b$. Then*

$$(2bn+1)(2bn+3)(2bn+5)\binom{2bn}{bn}\Big|15(a-b)(3a-b)(5a-b)(5a-3b)\binom{2an}{an}\binom{an}{bn}. \tag{1.7}$$

Theorem 1.8 *Let a, b and n be positive integers with $a > b$. Then*

$$(2bn+7)\binom{2bn}{bn}\Big|7(a-b)(7a-5b)(7a-3b)(7a-b)\binom{2an}{an}\binom{an}{bn}. \tag{1.8}$$

Theorem 1.9 *Let a, b and n be positive integers with $a > b$. Then*

$$(2bn+9)\binom{2bn}{bn}\Big|9(a-b)(9a-7b)(9a-5b)(9a-3b)(9a-b)\binom{2an}{an}\binom{an}{bn}. \tag{1.9}$$

The rest of the paper is organized as follows. In Section 2, we prove some preliminary results. Then we use these results to prove Theorems 1.2–1.5, 1.7–1.9 in Section 4.

2. Preliminaries

Let n be a positive integer. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x and $\{x\}$ the fractional part of x. For a prime p, we use $\nu_p(n)$ to symbol the largest nonnegative integer k such that $p^k \| n$. By Legendre's theorem, we know that

$$\nu_p(n!) = \sum_{i=1}^{\infty} \Big\lfloor \frac{n}{p^i} \Big\rfloor. \tag{2.1}$$

We have the following well-known result of Kummer [9].

Lemma 2.1 (Kummer's Theorem) *For any integers $0 \le k \le n$ and any prime p,*

$$\nu_p\left(\binom{n}{k}\right) = \sharp\{\text{carries when adding } k \text{ to } n-k \text{ in base } p\}.$$

Lemma 2.2 *Let n be a positive integer. Then*

$$\frac{(12n)!(3n)!(2n)!}{(6n)!(6n)!(4n)!n!} = \frac{\binom{12n}{6n}\binom{2n}{n}}{\binom{4n}{n}} \in \mathbb{N}, \quad \frac{(15n)!(4n)!}{(12n)!(5n)!(2n)!} = \frac{\binom{15n}{3n}\binom{3n}{n}}{\binom{5n}{n}} \in \mathbb{N},$$

$$\frac{(24n)!(9n)!(6n)!(4n)!}{(18n)!(12n)!(8n)!(3n)!(2n)!} = \frac{\binom{24n}{6n}\binom{9n}{n}\binom{4n}{n}}{\binom{12n}{6n}\binom{2n}{n}} \in \mathbb{N}, \quad \frac{(12n)!n!}{(8n)!(3n)!(2n)!} = \frac{\binom{12n}{8n}\binom{4n}{n}}{\binom{2n}{n}} \in \mathbb{N},$$

$$\frac{(12n)!(3n)!}{(8n)!(6n)!n!} = \frac{\binom{12n}{8n}\binom{4n}{n}}{\binom{6n}{3n}} \in \mathbb{N}, \quad \frac{(9n)!(4n)!}{(8n)!(3n)!(2n)!} = \frac{\binom{9n}{8n}\binom{4n}{n}}{\binom{2n}{n}} \in \mathbb{N},$$

$$\frac{(15n)!(4n)!}{(8n)!(6n)!(5n)!} = \frac{\binom{15n}{7n}\binom{7n}{n}}{\binom{5n}{n}} \in \mathbb{N}, \quad \frac{(15n)!n!}{(8n)!(5n)!(3n)!} = \frac{\binom{15n}{7n}\binom{7n}{2n}}{\binom{3n}{n}} \in \mathbb{N}.$$

Proof These are immediate consequences of Theorem 1.4 in [2] and three of the 52 sporadic step functions given in [2], Table 2, lines 2, 25, 52, 3, 4, 9, 27 and 37. □

Lemma 2.3 *Let $m > 1$ and n be positive integers such that $m|6n+1$. Then*

$$\Big\lfloor \frac{12n}{m} \Big\rfloor + \Big\lfloor \frac{3n}{m} \Big\rfloor + \Big\lfloor \frac{2n}{m} \Big\rfloor = \Big\lfloor \frac{6n}{m} \Big\rfloor + \Big\lfloor \frac{6n}{m} \Big\rfloor + \Big\lfloor \frac{4n}{m} \Big\rfloor + \Big\lfloor \frac{n}{m} \Big\rfloor + 1. \tag{2.2}$$

Proof We know that $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. Hence, the identity (2.2) is equivalent to

$$\left\{\frac{12n}{m}\right\} + \left\{\frac{3n}{m}\right\} + \left\{\frac{2n}{m}\right\} = \left\{\frac{6n}{m}\right\} + \left\{\frac{6n}{m}\right\} + \left\{\frac{4n}{m}\right\} + \left\{\frac{n}{m}\right\} - 1. \tag{2.3}$$

As $m|6n+1$, then we have $6n+1 = mt$, $t \equiv 1, 5 \pmod 6$. We calculate directly and get

$$\left\{\frac{12n}{m}\right\} = \frac{m-2}{m},$$

$$\left\{\frac{3n}{m}\right\} = \frac{m-1}{2m},$$

$$\left\{\frac{2n}{m}\right\} = \begin{cases} \frac{m-1}{3m}, & \text{if } t \equiv 1 \pmod 6, \\ \frac{2m-1}{3m}, & \text{if } t \equiv 5 \pmod 6. \end{cases}$$

$$\left\{\frac{6n}{m}\right\} = \frac{m-1}{m},$$

$$\left(\left\{\frac{4n}{m}\right\}, \left\{\frac{n}{m}\right\}\right) = \begin{cases} \left(\frac{2m-2}{3m}, \frac{m-2}{3m}\right), & \text{if } t \equiv 1 \pmod 6, \\ \left(\frac{m-1}{6m}, \frac{5m-1}{6m}\right), & \text{if } t \equiv 5 \pmod 6. \end{cases}$$

Therefore, we can easily verify that the identity (2.3) holds for any positive integer n. □

Lemma 2.4 *Let $m > 1$ and n be positive integers such that $m|12n+1$. Then*

$$\left\lfloor\frac{15n}{m}\right\rfloor + \left\lfloor\frac{4n}{m}\right\rfloor = \left\lfloor\frac{12n}{m}\right\rfloor + \left\lfloor\frac{5n}{m}\right\rfloor + \left\lfloor\frac{2n}{m}\right\rfloor + 1. \tag{2.4}$$

Proof The proof is similar to that of Lemma 2.3. It suffices to show that

$$\left\{\frac{15n}{m}\right\} + \left\{\frac{4n}{m}\right\} = \left\{\frac{12n}{m}\right\} + \left\{\frac{5n}{m}\right\} + \left\{\frac{2n}{m}\right\} - 1. \tag{2.5}$$

Since $m|12n+1$, then $12n+1 = mt$ with $t \equiv 1, 5, 7, 11 \pmod{12}$. By direct computations, we have

$$\left\{\frac{12n}{m}\right\} = \frac{m-1}{m},$$

$$\left(\left\{\frac{5n}{m}\right\}, \left\{\frac{2n}{m}\right\}\right) = \begin{cases} \left(\frac{5m-5}{12m}, \frac{m-1}{6m}\right), & \text{if } t \equiv 1 \pmod{12}, \\ \left(\frac{m-5}{12m}, \frac{5m-1}{6m}\right), & \text{if } t \equiv 5 \pmod{12}, \\ \left(\frac{11m-5}{12m}, \frac{m-1}{6m}\right), & \text{if } t \equiv 7 \pmod{12}, \\ \left(\frac{7m-5}{12m}, \frac{5m-1}{6m}\right), & \text{if } t \equiv 11 \pmod{12}. \end{cases}$$

$$\left(\left\{\frac{15n}{m}\right\}, \left\{\frac{4n}{m}\right\}\right) = \begin{cases} \left(\frac{m-5}{4m}, \frac{m-1}{3m}\right), & \text{if } t \equiv 1 \pmod{12}, \\ \left(\frac{m-5}{4m}, \frac{2m-1}{3m}\right), & \text{if } t \equiv 5 \pmod{12}, \\ \left(\frac{3m-5}{4m}, \frac{m-1}{3m}\right), & \text{if } t \equiv 7 \pmod{12}, \\ \left(\frac{3m-5}{4m}, \frac{2m-1}{3m}\right), & \text{if } t \equiv 11 \pmod{12}. \end{cases}$$

In the above computations, we have used the fact that $m \equiv t \equiv 1, 5, 7, 11 \pmod{12}$ since $mt = 12n + 1 \equiv 1 \pmod{12}$. Thus, the identity (2.5) holds. □

Lemma 2.5 *Let $m>1$ and n be positive integers such that $m|18n+1$. Then*

$$\Big\lfloor\frac{24n}{m}\Big\rfloor+\Big\lfloor\frac{9n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor+\Big\lfloor\frac{4n}{m}\Big\rfloor=\Big\lfloor\frac{18n}{m}\Big\rfloor+\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{8n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor+1. \tag{2.6}$$

Proof We prove in the same way as in Lemma 2.3, it is sufficient to show that

$$\Big\{\frac{24n}{m}\Big\}+\Big\{\frac{9n}{m}\Big\}+\Big\{\frac{6n}{m}\Big\}+\Big\{\frac{4n}{m}\Big\}=\Big\{\frac{18n}{m}\Big\}+\Big\{\frac{12n}{m}\Big\}+\Big\{\frac{8n}{m}\Big\}+\Big\{\frac{3n}{m}\Big\}+\Big\{\frac{2n}{m}\Big\}-1. \tag{2.7}$$

In this case, we have $m|18n+1$ and $18n+1=mt, t\equiv 1,5,7,11,13,17 \pmod{18}$. By computations, we get the following results.

$$\Big\{\frac{18n}{m}\Big\}=\frac{m-1}{m},$$

$$\Big(\Big\{\frac{12n}{m}\Big\},\ \Big\{\frac{8n}{m}\Big\},\ \Big\{\frac{3n}{m}\Big\},\ \Big\{\frac{2n}{m}\Big\}\Big)=\begin{cases}\Big(\frac{2m-2}{3m},\ \frac{4m-4}{9m},\ \frac{m-1}{6m},\ \frac{m-1}{9m}\Big), & \text{if } t\equiv 1 \pmod{18},\\ \Big(\frac{m-2}{3m},\ \frac{2m-4}{9m},\ \frac{5m-1}{6m},\ \frac{5m-1}{9m}\Big), & \text{if } t\equiv 5 \pmod{18},\\ \Big(\frac{2m-2}{3m},\ \frac{m-4}{9m},\ \frac{m-1}{6m},\ \frac{7m-1}{9m}\Big), & \text{if } t\equiv 7 \pmod{18},\\ \Big(\frac{m-2}{3m},\ \frac{8m-4}{9m},\ \frac{5m-1}{6m},\ \frac{2m-1}{9m}\Big), & \text{if } t\equiv 11 \pmod{18},\\ \Big(\frac{2m-2}{3m},\ \frac{7m-4}{9m},\ \frac{m-1}{6m},\ \frac{4m-1}{9m}\Big), & \text{if } t\equiv 13 \pmod{18},\\ \Big(\frac{m-2}{3m},\ \frac{5m-4}{9m},\ \frac{5m-1}{6m},\ \frac{8m-1}{9m}\Big), & \text{if } t\equiv 17 \pmod{18},\end{cases}$$

$$\Big\{\frac{9n}{m}\Big\}=\frac{m-1}{2m},$$

$$\Big(\Big\{\frac{24n}{m}\Big\},\ \Big\{\frac{6n}{m}\Big\},\ \Big\{\frac{4n}{m}\Big\}\Big)=\begin{cases}\Big(\frac{m-4}{3m},\ \frac{m-1}{3m},\ \frac{2m-2}{9m}\Big), & \text{if } t\equiv 1 \pmod{18},\\ \Big(\frac{2m-4}{3m},\ \frac{2m-1}{3m},\ \frac{m-2}{9m}\Big), & \text{if } t\equiv 5 \pmod{18},\\ \Big(\frac{m-4}{3m},\ \frac{m-1}{3m},\ \frac{5m-2}{9m}\Big), & \text{if } t\equiv 7 \pmod{18},\\ \Big(\frac{2m-4}{3m},\ \frac{2m-1}{3m},\ \frac{4m-2}{9m}\Big), & \text{if } t\equiv 11 \pmod{18},\\ \Big(\frac{m-4}{3m},\ \frac{m-1}{3m},\ \frac{8m-2}{9m}\Big), & \text{if } t\equiv 13 \pmod{18},\\ \Big(\frac{2m-4}{3m},\ \frac{2m-1}{3m},\ \frac{7m-2}{9m}\Big), & \text{if } t\equiv 17 \pmod{18}.\end{cases}$$

Therefore, the identity (2.7) holds for any positive integers by checking the six cases. The proof of Lemma 2.5 is completed. □

Lemma 2.6 *Let $m>1$ and n be positive integers such that $m|8n+1$. Then*

$$\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor=\Big\lfloor\frac{8n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor+1,$$

$$\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor=\Big\lfloor\frac{8n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor+1,$$

and

$$\Big\lfloor\frac{9n}{m}\Big\rfloor+\Big\lfloor\frac{4n}{m}\Big\rfloor=\Big\lfloor\frac{8n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor+1.$$

Let $m>1$ and n be positive integers such that $m|8n+1$ and $m\neq 7$. Then

$$\Big\lfloor\frac{15n}{m}\Big\rfloor+\Big\lfloor\frac{4n}{m}\Big\rfloor=\Big\lfloor\frac{8n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor+\Big\lfloor\frac{5n}{m}\Big\rfloor+1,$$

and

$$\left\lfloor\frac{15n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor=\left\lfloor\frac{8n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor+1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|8n+1$ and $8n+1=mt, t\equiv 1,3,5,7 \pmod 8$. By computations, we have

$$\left\{\frac{8n}{m}\right\}=\frac{m-1}{m},\quad \left\{\frac{4n}{m}\right\}=\frac{m-1}{2m},\quad \left\{\frac{12n}{m}\right\}=\frac{m-3}{2m},$$

$$\left(\left\{\frac{6n}{m}\right\},\ \left\{\frac{2n}{m}\right\},\ \left\{\frac{3n}{m}\right\},\ \left\{\frac{n}{m}\right\}\right)=\begin{cases}\left(\frac{3m-3}{4m},\ \frac{m-1}{4m},\ \frac{3m-3}{8m},\ \frac{m-1}{8m}\right), & \text{if } t\equiv 1 \pmod 8,\\ \left(\frac{m-3}{4m},\ \frac{3m-1}{4m},\ \frac{m-3}{8m},\ \frac{3m-1}{8m}\right), & \text{if } t\equiv 3 \pmod 8,\\ \left(\frac{3m-3}{4m},\ \frac{m-1}{4m},\ \frac{7m-3}{8m},\ \frac{5m-1}{8m}\right), & \text{if } t\equiv 5 \pmod 8,\\ \left(\frac{m-3}{4m},\ \frac{3m-1}{4m},\ \frac{5m-3}{8m},\ \frac{7m-1}{8m}\right), & \text{if } t\equiv 7 \pmod 8,\end{cases}$$

$$\left(\left\{\frac{5n}{m}\right\},\ \left\{\frac{9n}{m}\right\},\ \left\{\frac{15n}{m}\right\}\right)=\begin{cases}\left(\frac{5m-5}{8m},\ \frac{m-9}{8m},\ \frac{7m-15}{8m}\right), & \text{if } t\equiv 1 \pmod 8,\\ \left(\frac{7m-5}{8m},\ \frac{3m-9}{8m},\ \frac{5m-15}{8m}\right), & \text{if } t\equiv 3 \pmod 8,\\ \left(\frac{m-5}{8m},\ \frac{5m-9}{8m},\ \frac{3m-15}{8m}\right), & \text{if } t\equiv 5 \pmod 8,\\ \left(\frac{3m-5}{8m},\ \frac{7m-9}{8m},\ \frac{m-15}{8m}\right), & \text{if } t\equiv 7 \pmod 8, m\neq 7,\\ \left(\frac{2}{7},\ \frac{5}{7}, \frac{6}{7}\right), & \text{if } t\equiv 7 \pmod 8, m=7.\end{cases}$$

In the above computations, we have used the fact that $mt=8n+1\equiv 1 \pmod 8$. Similarly, the identities hold for any positive integers by checking the four cases. The proof of Lemma 2.6 is completed. □

3. Some lemmas for Theorem 1.5

In this section, we prove some more lemmas for Theorem 1.5.

Lemma 3.1 *Let $m>$ and n be positive integers such that $m|10n+1$. Then*

$$\left\lfloor\frac{12n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor=\left\lfloor\frac{10n}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor+1,$$

$$\left\lfloor\frac{18n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor=\left\lfloor\frac{10n}{m}\right\rfloor+\left\lfloor\frac{9n}{m}\right\rfloor+\left\lfloor\frac{6n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor+1$$

and

$$\left\lfloor\frac{15n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor=\left\lfloor\frac{10n}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor+1,$$

$$\left\lfloor\frac{15n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor=\left\lfloor\frac{10n}{m}\right\rfloor+\left\lfloor\frac{6n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor+1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|10n+1$ and $10n+1=mt, t\equiv 1,3,7,9 \pmod{10}$. By computations, we have

$$\left\{\frac{10n}{m}\right\}=\frac{m-1}{m},\quad \left\{\frac{5n}{m}\right\}=\frac{m-1}{2m},\quad \left\{\frac{15n}{m}\right\}=\frac{m-3}{2m},$$

$$\Big(\Big\{\frac{n}{m}\Big\},\ \Big\{\frac{2n}{m}\Big\},\ \Big\{\frac{3n}{m}\Big\},\ \Big\{\frac{4n}{m}\Big\}\Big)=\begin{cases}\big(\frac{m-1}{10m},\ \frac{m-1}{5m},\ \frac{3m-3}{10m},\ \frac{2m-2}{5m}\big), & \text{if } t\equiv 1 \pmod{10},\\ \big(\frac{3m-1}{10m},\ \frac{3m-1}{5m},\ \frac{9m-3}{10m},\ \frac{m-2}{5m}\big), & \text{if } t\equiv 3 \pmod{10},\\ \big(\frac{7m-1}{10m},\ \frac{2m-1}{5m},\ \frac{m-3}{10m},\ \frac{4m-2}{5m}\big), & \text{if } t\equiv 7 \pmod{10},\\ \big(\frac{9m-1}{10m},\ \frac{4m-1}{5m},\ \frac{7m-3}{10m},\ \frac{3m-2}{5m}\big), & \text{if } t\equiv 9 \pmod{10},\end{cases}$$

$$\Big(\Big\{\frac{6n}{m}\Big\},\ \Big\{\frac{9n}{m}\Big\},\ \Big\{\frac{12n}{m}\Big\},\ \Big\{\frac{18n}{m}\Big\}\Big)=\begin{cases}\big(\frac{3m-3}{5m},\ \frac{9m-9}{10m},\ \frac{m-6}{5m},\ \frac{4m-9}{5m}\big), & \text{if } t\equiv 1 \pmod{10},\\ \big(\frac{4m-3}{5m},\ \frac{7m-9}{10m},\ \frac{3m-6}{5m},\ \frac{2m-9}{5m}\big), & \text{if } t\equiv 3 \pmod{10},\\ \big(\frac{m-3}{5m},\ \frac{3m-9}{10m},\ \frac{2m-6}{5m},\ \frac{3m-9}{5m}\big), & \text{if } t\equiv 7 \pmod{10},\\ \big(\frac{2m-3}{5m},\ \frac{m-9}{10m},\ \frac{4m-6}{5m},\ \frac{m-9}{5m}\big), & \text{if } t\equiv 9 \pmod{10}.\end{cases}$$

Here we have used the fact that $mt\equiv 1 \pmod{10}$. For example, we have $m\equiv 9 \pmod{10}$ when $t\equiv 9 \pmod{10}$. Similarly, the identities hold for any positive integers by checking the four cases. The proof of Lemma 3.1 is completed. □

Lemma 3.2 *Let $m>1$ and n be positive integers such that $m|12n+1$. Then*

$$\Big\lfloor\frac{18n}{m}\Big\rfloor+\Big\lfloor\frac{4n}{m}\Big\rfloor=\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{9n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor+1,$$

$$\Big\lfloor\frac{14n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor+\Big\lfloor\frac{4n}{m}\Big\rfloor=\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{7n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor+1,$$

$$\Big\lfloor\frac{15n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor=\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{5n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor+1$$

and

$$\Big\lfloor\frac{20n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor=\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{10n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor+1,$$

$$\Big\lfloor\frac{20n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor=\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{10n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor+1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|12n+1$ and $12n+1=mt, t\equiv 1,5,7,11 \pmod{12}$. By computations, we have

$$\Big\{\frac{12n}{m}\Big\}=\frac{m-1}{m},\quad \Big\{\frac{6n}{m}\Big\}=\frac{m-1}{2m},\quad \Big\{\frac{18n}{m}\Big\}=\frac{m-3}{2m},$$

$$\Big(\Big\{\frac{n}{m}\Big\},\ \Big\{\frac{2n}{m}\Big\},\ \Big\{\frac{3n}{m}\Big\},\ \Big\{\frac{4n}{m}\Big\}\Big)=\begin{cases}\big(\frac{m-1}{12m},\ \frac{m-1}{6m},\ \frac{m-1}{4m},\ \frac{m-1}{3m}\big), & \text{if } t\equiv 1 \pmod{12},\\ \big(\frac{5m-1}{12m},\ \frac{5m-1}{6m},\ \frac{m-1}{4m},\ \frac{2m-1}{3m}\big), & \text{if } t\equiv 5 \pmod{12},\\ \big(\frac{7m-1}{12m},\ \frac{m-1}{6m},\ \frac{3m-1}{4m},\ \frac{m-1}{3m}\big), & \text{if } t\equiv 7 \pmod{12},\\ \big(\frac{11m-1}{12m},\ \frac{5m-1}{6m},\ \frac{3m-1}{4m},\ \frac{2m-1}{3m}\big), & \text{if } t\equiv 11 \pmod{12},\end{cases}$$

$$\Big(\Big\{\frac{5n}{m}\Big\},\ \Big\{\frac{7n}{m}\Big\},\ \Big\{\frac{9n}{m}\Big\}\Big)=\begin{cases}\big(\frac{5m-5}{12m},\ \frac{7m-7}{12m},\ \frac{3m-3}{4m}\big), & \text{if } t\equiv 1 \pmod{12},\\ \big(\frac{m-5}{12m},\ \frac{11m-7}{12m},\ \frac{3m-3}{4m}\big), & \text{if } t\equiv 5 \pmod{12},\\ \big(\frac{11m-5}{12m},\ \frac{m-7}{12m},\ \frac{m-3}{4m}\big), & \text{if } t\equiv 7 \pmod{12},\\ \big(\frac{7m-1}{12m},\ \frac{5m-7}{12m},\ \frac{m-3}{4m}\big), & \text{if } t\equiv 11 \pmod{12},\end{cases}$$

$$\left(\left\{\frac{10n}{m}\right\},\ \left\{\frac{15n}{m}\right\},\ \left\{\frac{20n}{m}\right\}\right)=\begin{cases}\left(\frac{5m-5}{6m},\ \frac{m-5}{4m},\ \frac{2m-5}{3m}\right), & \text{if } t\equiv 1 \pmod{12},\\ \left(\frac{m-5}{6m},\ \frac{m-5}{4m},\ \frac{m-5}{3m}\right), & \text{if } t\equiv 5 \pmod{12},\\ \left(\frac{5m-5}{6m},\ \frac{3m-5}{4m},\ \frac{2m-5}{3m}\right), & \text{if } t\equiv 7 \pmod{12},\\ \left(\frac{m-1}{6m},\ \frac{3m-5}{4m},\ \frac{m-5}{3m}\right), & \text{if } t\equiv 11 \pmod{12}.\end{cases}$$

Here we have used the fact that $m\equiv t \pmod{12}$ since $mt\equiv 1 \pmod{12}$. For example, we have $m\equiv 7 \pmod{12}$ when $t\equiv 7 \pmod{12}$. Similarly, the identities hold for any positive integers by checking the four cases. The proof of Lemma 3.2 is completed. □

Lemma 3.3 *Let $m>1$ and n be positive integers such that $m|14n+1$ and $m\neq 3,5$. Then*

$$\left\lfloor\frac{15n}{m}\right\rfloor+\left\lfloor\frac{7n}{m}\right\rfloor=\left\lfloor\frac{14n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor+1,$$

$$\left\lfloor\frac{20n}{m}\right\rfloor+\left\lfloor\frac{7n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor=\left\lfloor\frac{14n}{m}\right\rfloor+\left\lfloor\frac{10n}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor+1,$$

$$\left\lfloor\frac{24n}{m}\right\rfloor+\left\lfloor\frac{7n}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor=\left\lfloor\frac{14n}{m}\right\rfloor+\left\lfloor\frac{12n}{m}\right\rfloor+\left\lfloor\frac{8n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor+1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|14n+1$ and $14n+1=mt, t\equiv 1,3,5,9,11,13 \pmod{14}$. By computations, we have

$$\left\{\frac{14n}{m}\right\}=\frac{m-1}{m},\quad \left\{\frac{7n}{m}\right\}=\frac{m-1}{2m},$$

$$\left(\left\{\frac{n}{m}\right\},\ \left\{\frac{2n}{m}\right\},\ \left\{\frac{3n}{m}\right\},\ \left\{\frac{4n}{m}\right\}\right)=\begin{cases}\left(\frac{m-1}{14m},\ \frac{m-1}{7m},\ \frac{3m-3}{14m},\ \frac{2m-2}{7m}\right), & \text{if } t\equiv 1 \pmod{14},\\ \left(\frac{3m-1}{14m},\ \frac{3m-1}{7m},\ \frac{9m-3}{14m},\ \frac{6m-2}{7m}\right), & \text{if } t\equiv 3 \pmod{14},\\ \left(\frac{5m-1}{14m},\ \frac{5m-1}{7m},\ \frac{m-3}{14m},\ \frac{3m-2}{7m}\right), & \text{if } t\equiv 5 \pmod{14},\\ \left(\frac{9m-1}{14m},\ \frac{2m-1}{7m},\ \frac{13m-3}{14m},\ \frac{4m-2}{7m}\right), & \text{if } t\equiv 9 \pmod{14},\\ \left(\frac{11m-1}{14m},\ \frac{4m-1}{7m},\ \frac{5m-3}{14m},\ \frac{m-2}{7m}\right), & \text{if } t\equiv 11 \pmod{14},\\ \left(\frac{13m-1}{14m},\ \frac{6m-1}{7m},\ \frac{11m-3}{14m},\ \frac{5m-2}{7m}\right), & \text{if } t\equiv 13 \pmod{14},\end{cases}$$

$$\left(\left\{\frac{8n}{m}\right\},\ \left\{\frac{10n}{m}\right\},\ \left\{\frac{15n}{m}\right\},\ \left\{\frac{20n}{m}\right\}\right)=\begin{cases}\left(\frac{4m-4}{7m},\ \frac{5m-5}{7m},\ \frac{m-15}{14m},\ \frac{3m-10}{7m}\right), & \text{if } t\equiv 1 \pmod{14},\\ \left(\frac{5m-4}{7m},\ \frac{m-5}{7m},\ \frac{3m-15}{14m},\ \frac{2m-10}{7m}\right), & \text{if } t\equiv 3 \pmod{14},\\ \left(\frac{6m-4}{7m},\ \frac{4m-5}{7m},\ \frac{5m-15}{14m},\ \frac{m-10}{7m}\right), & \text{if } t\equiv 5 \pmod{14}, m\neq 3,\\ \left(\frac{2}{3},\ \frac{1}{3},\ 0,\ \frac{2}{3}\right), & \text{if } t\equiv 5 \pmod{14}, m=3,\\ \left(\frac{m-4}{7m},\ \frac{3m-5}{7m},\ \frac{9m-15}{14m},\ \frac{6m-10}{7m}\right), & \text{if } t\equiv 9 \pmod{14},\\ \left(\frac{2m-4}{7m},\ \frac{6m-5}{7m},\ \frac{11m-15}{14m},\ \frac{5m-10}{7m}\right), & \text{if } t\equiv 11 \pmod{14},\\ \left(\frac{3m-4}{7m},\ \frac{2m-5}{7m},\ \frac{13m-15}{14m},\ \frac{4m-10}{7m}\right), & \text{if } t\equiv 13 \pmod{14},\end{cases}$$

$$\Big(\Big\{\frac{5n}{m}\Big\},\Big\{\frac{12n}{m}\Big\},\ \Big\{\frac{24n}{m}\Big\}\Big)=\begin{cases}\Big(\frac{5m-5}{14m},\ \frac{6m-6}{7m},\ \frac{5m-12}{7m}\Big), & \text{if } t\equiv 1 \pmod{14},\\ \Big(\frac{m-5}{14m},\ \frac{4m-6}{7m},\ \frac{m-12}{7m}\Big), & \text{if } t\equiv 3 \pmod{14}, m\neq 5,\\ \Big(0,\ \frac{2}{5},\frac{4}{5}\Big), & \text{if } t\equiv 3 \pmod{14}, m=5,\\ \Big(\frac{11m-5}{14m},\ \frac{2m-6}{7m},\ \frac{4m-12}{7m}\Big), & \text{if } t\equiv 5 \pmod{14},\\ \Big(\frac{3m-5}{14m},\ \frac{5m-6}{7m},\ \frac{3m-12}{7m}\Big), & \text{if } t\equiv 9 \pmod{14},\\ \Big(\frac{13m-5}{14m},\ \frac{3m-6}{7m},\ \frac{6m-12}{7m}\Big), & \text{if } t\equiv 11 \pmod{14},\\ \Big(\frac{9m-5}{14m},\ \frac{m-6}{7m},\ \frac{2m-12}{7m}\Big), & \text{if } t\equiv 13 \pmod{14}.\end{cases}$$

Here we have used the fact that $mt\equiv 1 \pmod{14}$. For example, we have $m\equiv 5 \pmod{14}$ when $t\equiv 3 \pmod{14}$. Similarly, the identities hold for any positive integers by checking the six cases. The proof of Lemma 3.3 is completed. □

Lemma 3.4 *Let $m>1$ and n be positive integers such that $m|18n+1$. Then*

$$\Big\lfloor\frac{30n}{m}\Big\rfloor+\Big\lfloor\frac{9n}{m}\Big\rfloor+\Big\lfloor\frac{5n}{m}\Big\rfloor=\Big\lfloor\frac{18n}{m}\Big\rfloor+\Big\lfloor\frac{15n}{m}\Big\rfloor+\Big\lfloor\frac{10n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor+1,$$

$$\Big\lfloor\frac{20n}{m}\Big\rfloor+\Big\lfloor\frac{9n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor=\Big\lfloor\frac{18n}{m}\Big\rfloor+\Big\lfloor\frac{10n}{m}\Big\rfloor+\Big\lfloor\frac{4n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|18n+1$ and $18n+1=mt, t\equiv 1,5,7,11,13,17 \pmod{18}$. By computations, we have

$$\Big\{\frac{18n}{m}\Big\}=\frac{m-1}{m},\quad \Big\{\frac{9n}{m}\Big\}=\frac{m-1}{2m},$$

$$\Big(\Big\{\frac{n}{m}\Big\},\ \Big\{\frac{5n}{m}\Big\},\ \Big\{\frac{10n}{m}\Big\},\ \Big\{\frac{20n}{m}\Big\}\Big)=\begin{cases}\Big(\frac{m-1}{18m},\ \frac{5m-5}{18m},\ \frac{5m-5}{9m},\ \frac{m-10}{9m}\Big), & \text{if } t\equiv 1 \pmod{18},\\ \Big(\frac{5m-1}{18m},\ \frac{7m-5}{18m},\ \frac{7m-5}{9m},\ \frac{5m-10}{9m}\Big), & \text{if } t\equiv 5 \pmod{18},\\ \Big(\frac{7m-1}{18m},\ \frac{17m-5}{18m},\ \frac{8m-5}{9m},\ \frac{7m-10}{9m}\Big), & \text{if } t\equiv 7 \pmod{18},\\ \Big(\frac{11m-1}{18m},\ \frac{m-5}{18m},\ \frac{m-5}{9m},\ \frac{2m-10}{9m}\Big), & \text{if } t\equiv 11 \pmod{18},\\ \Big(\frac{13m-1}{18m},\ \frac{11m-5}{18m},\ \frac{2m-5}{9m},\ \frac{4m-10}{9m}\Big), & \text{if } t\equiv 13 \pmod{18},\\ \Big(\frac{17m-1}{18m},\ \frac{13m-5}{18m},\ \frac{4m-5}{9m},\ \frac{8m-10}{9m}\Big), & \text{if } t\equiv 17 \pmod{18},\end{cases}$$

$$\Big(\Big\{\frac{15n}{m}\Big\},\ \Big\{\frac{30}{m}\Big\}\Big)=\begin{cases}\Big(\frac{5m-5}{6m},\ \frac{2m-5}{3m}\Big), & \text{if } t\equiv 1 \pmod{18},\\ \Big(\frac{m-5}{6m},\ \frac{m-5}{3m}\Big), & \text{if } t\equiv 5 \pmod{18},\\ \Big(\frac{5m-5}{6m},\ \frac{2m-5}{3m}\Big), & \text{if } t\equiv 7 \pmod{18},\\ \Big(\frac{m-5}{6m},\ \frac{m-5}{3m}\Big), & \text{if } t\equiv 11 \pmod{18},\\ \Big(\frac{5m-5}{6m},\ \frac{2m-5}{3m}\Big), & \text{if } t\equiv 13 \pmod{18},\\ \Big(\frac{m-5}{6m},\ \frac{m-5}{3m}\Big), & \text{if } t\equiv 17 \pmod{18},\end{cases}$$

$$\left(\left\{\frac{3n}{m}\right\},\ \left\{\frac{6n}{m}\right\},\ \left\{\frac{4n}{m}\right\}\right)=\begin{cases}\left(\frac{m-1}{6m},\ \frac{m-1}{3m},\ \frac{2m-2}{9m}\right), & \text{if } t\equiv 1 \pmod{18},\\ \left(\frac{5m-1}{6m},\ \frac{2m-1}{3m},\ \frac{m-2}{9m}\right), & \text{if } t\equiv 5 \pmod{18},\\ \left(\frac{m-1}{6m},\ \frac{m-1}{3m},\ \frac{5m-2}{9m}\right), & \text{if } t\equiv 7 \pmod{18},\\ \left(\frac{5m-1}{6m},\ \frac{2m-1}{3m},\ \frac{4m-2}{9m}\right), & \text{if } t\equiv 11 \pmod{18},\\ \left(\frac{m-1}{6m},\ \frac{m-1}{3m},\ \frac{8m-2}{9m}\right), & \text{if } t\equiv 13 \pmod{18},\\ \left(\frac{5m-1}{6m},\ \frac{2m-1}{3m},\ \frac{7m-2}{9m}\right), & \text{if } t\equiv 17 \pmod{18}.\end{cases}$$

Here we have used the fact that $mt\equiv 1 \pmod{18}$. For example, we have $m\equiv 11 \pmod{18}$ when $t\equiv 5 \pmod{18}$. Similarly, the identities hold for any positive integers by checking the six cases. The proof of Lemma 3.4 is completed. □

Lemma 3.5 *Let $m>1$ and n be positive integers such that $m|9n+1$ and $m\neq 2,5$. Then*

$$\left\lfloor\frac{12n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor=\left\lfloor\frac{9n}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor+1,$$
$$\left\lfloor\frac{10n}{m}\right\rfloor+\left\lfloor\frac{6n}{m}\right\rfloor=\left\lfloor\frac{9n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor+1,$$
$$\left\lfloor\frac{14n}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor=\left\lfloor\frac{9n}{m}\right\rfloor+\left\lfloor\frac{7n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor+1,$$
$$\left\lfloor\frac{15n}{m}\right\rfloor+\left\lfloor\frac{n}{m}\right\rfloor=\left\lfloor\frac{9n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor+1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|9n+1$ and $9n+1=mt$, $t\equiv 1,2,4,5,7,8 \pmod 9$. By computations, we have

$$\left\{\frac{9n}{m}\right\}=\frac{m-1}{m},$$

$$\left(\left\{\frac{3n}{m}\right\},\left\{\frac{6n}{m}\right\},\left\{\frac{12n}{m}\right\},\left\{\frac{15n}{m}\right\}\right)=\begin{cases}\left(\frac{m-1}{3m},\ \frac{2m-2}{3m},\ \frac{m-4}{3m},\ \frac{2m-5}{3m}\right), & \text{if } t\equiv 1 \pmod 9,\\ \left(\frac{2m-1}{3m},\ \frac{m-2}{3m},\ \frac{2m-4}{3m},\ \frac{m-5}{3m}\right), & \text{if } t\equiv 2 \pmod 9,\\ \left(\frac{m-1}{3m},\ \frac{2m-2}{3m},\ \frac{m-4}{3m},\ \frac{2m-5}{3m}\right), & \text{if } t\equiv 4 \pmod 9,\\ \left(\frac{2m-1}{3m},\ \frac{m-2}{3m},\ \frac{2m-4}{3m},\ \frac{m-5}{3m}\right), & \text{if } t\equiv 5 \pmod 9, m\neq 2,\\ \left(\frac{1}{2},\ 0,\ 0,\ \frac{1}{2}\right), & \text{if } t\equiv 5 \pmod 9, m=2,\\ \left(\frac{m-1}{3m},\ \frac{2m-2}{3m},\ \frac{m-4}{3m},\ \frac{2m-5}{3m}\right), & \text{if } t\equiv 7 \pmod 9,\\ \left(\frac{2m-1}{3m},\ \frac{m-2}{3m},\ \frac{2m-4}{3m},\ \frac{m-5}{3m}\right), & \text{if } t\equiv 8 \pmod 9,\end{cases}$$

$$\left(\left\{\frac{n}{m}\right\},\ \left\{\frac{2n}{m}\right\},\ \left\{\frac{4n}{m}\right\},\left\{\frac{10n}{m}\right\}\right)=\begin{cases}\left(\frac{m-1}{9m},\ \frac{2m-2}{9m},\ \frac{4m-4}{9m},\frac{m-10}{9m}\right), & \text{if } t\equiv 1 \pmod 9,\\ \left(\frac{2m-1}{9m},\ \frac{4m-2}{9m},\ \frac{8m-4}{9m},\frac{2m-10}{9m}\right), & \text{if } t\equiv 2 \pmod 9,\\ \left(\frac{4m-1}{9m},\ \frac{8m-2}{9m},\ \frac{7m-4}{9m},\frac{4m-10}{9m}\right), & \text{if } t\equiv 4 \pmod 9,\\ \left(\frac{5m-1}{9m},\ \frac{m-2}{9m},\ \frac{2m-4}{9m},\frac{5m-10}{9m}\right), & \text{if } t\equiv 5 \pmod 9,\\ \left(\frac{7m-1}{9m},\ \frac{5m-2}{9m},\ \frac{m-4}{9m},\frac{7m-10}{9m}\right), & \text{if } t\equiv 7 \pmod 9,\\ \left(\frac{8m-1}{9m},\ \frac{7m-2}{9m},\ \frac{5m-4}{9m},\frac{8m-10}{9m}\right), & \text{if } t\equiv 8 \pmod 9,\end{cases}$$

$$\Big(\Big\{\frac{5n}{m}\Big\},\ \Big\{\frac{7n}{m}\Big\},\ \Big\{\frac{14n}{m}\Big\}\Big)=\begin{cases}\Big(\frac{5m-5}{9m},\ \frac{7m-7}{9m},\ \frac{5m-14}{9m}\Big), & \text{if } t\equiv 1 \pmod 9,\\ \Big(\frac{m-5}{9m},\ \frac{5m-7}{9m},\ \frac{m-14}{9m}\Big), & \text{if } t\equiv 2 \pmod 9, m\neq 5,\\ \Big(0,\ \frac{2}{5},\ \frac{4}{5}\Big), & \text{if } t\equiv 2 \pmod 9, m=5,\\ \Big(\frac{2m-5}{9m},\ \frac{m-7}{9m},\ \frac{2m-14}{9m}\Big), & \text{if } t\equiv 4 \pmod 9,\\ \Big(\frac{7m-5}{9m},\ \frac{8m-7}{9m},\ \frac{7m-14}{9m}\Big), & \text{if } t\equiv 5 \pmod 9,\\ \Big(\frac{8m-5}{9m},\ \frac{4m-7}{9m},\ \frac{8m-14}{9m}\Big), & \text{if } t\equiv 7 \pmod 9,\\ \Big(\frac{4m-5}{9m},\ \frac{2m-7}{9m},\ \frac{4m-14}{9m}\Big), & \text{if } t\equiv 8 \pmod 9.\end{cases}$$

Here we have used the fact that $mt\equiv 1 \pmod 9$. For example, we have $m\equiv 5 \pmod 9$ when $t\equiv 2 \pmod 9$. Similarly, the identities hold for any positive integers by checking the six cases. The proof of Lemma 3.5 is completed. □

Lemma 3.6 *Let $m>1$ and n be positive integers such that $m|7n+1$ and $m\neq 3,5$. Then*

$$\Big\lfloor\frac{12n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor=\Big\lfloor\frac{7n}{m}\Big\rfloor+\Big\lfloor\frac{4n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+1,$$

$$\Big\lfloor\frac{10n}{m}\Big\rfloor+\Big\lfloor\frac{6n}{m}\Big\rfloor+\Big\lfloor\frac{n}{m}\Big\rfloor=\Big\lfloor\frac{7n}{m}\Big\rfloor+\Big\lfloor\frac{5n}{m}\Big\rfloor+\Big\lfloor\frac{3n}{m}\Big\rfloor+\Big\lfloor\frac{2n}{m}\Big\rfloor+1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|7n+1$ and $7n+1=mt$, $t\equiv 1,2,3,4,5,6 \pmod 7$. By computations, we have

$$\Big\{\frac{7n}{m}\Big\}=\frac{m-1}{m},$$

$$\Big(\Big\{\frac{n}{m}\Big\},\ \Big\{\frac{2n}{m}\Big\},\ \Big\{\frac{3n}{m}\Big\},\ \Big\{\frac{4n}{m}\Big\}\Big)=\begin{cases}\Big(\frac{m-1}{7m},\ \frac{2m-2}{7m},\ \frac{3m-3}{7m},\ \frac{4m-4}{7m}\Big), & \text{if } t\equiv 1 \pmod 7,\\ \Big(\frac{2m-1}{7m},\ \frac{4m-2}{7m},\ \frac{6m-3}{7m},\ \frac{m-4}{7m}\Big), & \text{if } t\equiv 2 \pmod 7,\\ \Big(\frac{3m-1}{7m},\ \frac{6m-2}{7m},\ \frac{2m-3}{7m},\ \frac{5m-4}{7m}\Big), & \text{if } t\equiv 3 \pmod 7,\\ \Big(\frac{4m-1}{7m},\ \frac{m-2}{7m},\ \frac{5m-3}{7m},\ \frac{2m-4}{7m}\Big), & \text{if } t\equiv 4 \pmod 7,\\ \Big(\frac{5m-1}{7m},\ \frac{3m-2}{7m},\ \frac{m-3}{7m},\ \frac{6m-4}{7m}\Big), & \text{if } t\equiv 5 \pmod 7,\\ \Big(\frac{6m-1}{7m},\ \frac{5m-2}{7m},\ \frac{4m-3}{7m},\ \frac{3m-4}{7m}\Big), & \text{if } t\equiv 6 \pmod 7,\end{cases}$$

$$\Big(\Big\{\frac{5n}{m}\Big\},\ \Big\{\frac{10n}{m}\Big\},\ \Big\{\frac{6n}{m}\Big\},\ \Big\{\frac{12n}{m}\Big\}\Big)=\begin{cases}\Big(\frac{5m-5}{7m},\ \frac{3m-10}{7m},\ \frac{6m-6}{7m},\ \frac{5m-12}{7m}\Big), & \text{if } t\equiv 1 \pmod 7,\\ \Big(\frac{3m-5}{7m},\ \frac{6m-10}{7m},\ \frac{5m-6}{7m},\ \frac{3m-12}{7m}\Big), & \text{if } t\equiv 2 \pmod 7,\\ \Big(\frac{m-5}{7m},\ \frac{2m-10}{7m},\ \frac{4m-6}{7m},\ \frac{m-12}{7m}\Big), & \text{if } t\equiv 3 \pmod 7, m\neq 5,\\ \Big(0,\ 0,\ \frac{2}{5},\ \frac{4}{5}\Big), & \text{if } t\equiv 3 \pmod 7, m=5,\\ \Big(\frac{6m-5}{7m},\ \frac{5m-10}{7m},\ \frac{3m-6}{7m},\ \frac{6m-12}{7m}\Big), & \text{if } t\equiv 4 \pmod 7,\\ \Big(\frac{4m-5}{7m},\ \frac{m-10}{7m},\ \frac{2m-6}{7m},\ \frac{4m-12}{7m}\Big), & \text{if } t\equiv 5 \pmod 7, m\neq 3,\\ \Big(\frac{1}{3},\ \frac{2}{3},\ 0,0\Big), & \text{if } t\equiv 5 \pmod 7, m=3,\\ \Big(\frac{2m-5}{7m},\ \frac{4m-10}{7m},\ \frac{m-6}{7m},\ \frac{2m-12}{7m}\Big), & \text{if } t\equiv 6 \pmod 7.\end{cases}$$

Here we have used the fact that $mt\equiv 1 \pmod 7$. For example, we have $m\equiv 5 \pmod 7$ when $t\equiv 3 \pmod 7$. Similarly, the identities hold for any positive integers by checking the six cases.

The proof of Lemma 3.6 is completed. □

Lemma 3.7 *Let $m > 1$ and n be positive integers such that $m|5n+1$ and $m \neq 2, 4$. Then*

$$\left\lfloor \frac{9n}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{5n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + 1.$$

Proof We prove in the same way as in Lemma 2.3. In this case, we have $m|5n+1$ and $5n+1 = mt$, $t \equiv 1, 2, 3, 4 \pmod 5$. By computations, we have

$$\left\{ \frac{5n}{m} \right\} = \frac{m-1}{m},$$

$$\left(\left\{ \frac{n}{m} \right\}, \left\{ \frac{2n}{m} \right\}, \left\{ \frac{3n}{m} \right\}, \left\{ \frac{9n}{m} \right\} \right) = \begin{cases} \left(\frac{m-1}{5m}, \frac{2m-2}{5m}, \frac{3m-3}{5m}, \frac{4m-9}{5m} \right), & \text{if } t \equiv 1 \pmod 5, \\ \left(\frac{2m-1}{5m}, \frac{4m-2}{5m}, \frac{m-3}{5m}, \frac{3m-9}{5m} \right), & \text{if } t \equiv 2 \pmod 5, \\ \left(\frac{3m-1}{5m}, \frac{m-2}{5m}, \frac{4m-3}{5m}, \frac{2m-9}{5m} \right), & \text{if } t \equiv 3 \pmod 5, m \neq 2, \\ \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \right), & \text{if } t \equiv 3 \pmod 5, m = 2, \\ \left(\frac{4m-1}{5m}, \frac{3m-2}{5m}, \frac{2m-3}{5m}, \frac{m-9}{5m} \right), & \text{if } t \equiv 4 \pmod 5, m \neq 4, \\ \left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \right), & \text{if } t \equiv 4 \pmod 5, m = 4. \end{cases}$$

Here we have used the fact that $mt \equiv 1 \pmod 5$. For example, we have $m \equiv 2 \pmod 5$ when $t \equiv 3 \pmod 5$. Similarly, the identities hold for any positive integers by checking the four cases. The proof of Lemma 3.7 is completed. □

4. Proofs of Theorems 1.1–1.5 and 1.7–1.9

In this section, we will give the proofs of Theorems 1.1–1.5 and 1.7–1.9.

Proof of Theorem 1.1 Let

$$S_n := \frac{(12n)!(3n)!(2n)!}{(6n)!(6n)!(4n)!(n)!} = \frac{\binom{12n}{6n}\binom{2n}{n}}{\binom{4n}{n}}.$$

By Lemma 2.2, $S_n \in \mathbb{N}$ for all positive integers n, so to prove Theorem 1.1, it suffices to show that for any prime $p|6n+1$, the p-adic order $\nu_p(S_n)$ of S_n is not less than $\nu_p(6n+1)$. Now $\nu_p(S_n)$ is given by

$$\sum_{i=1}^{\infty} \left(\left\lfloor \frac{12n}{p^i} \right\rfloor + \left\lfloor \frac{3n}{p^i} \right\rfloor + \left\lfloor \frac{2n}{p^i} \right\rfloor - \left\lfloor \frac{6n}{p^i} \right\rfloor - \left\lfloor \frac{6n}{p^i} \right\rfloor - \left\lfloor \frac{4n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor \right). \tag{4.1}$$

For $i \leq \nu_p(6n+1)$, by Lemma 2.3, we have

$$\left(\left\lfloor \frac{12n}{p^i} \right\rfloor + \left\lfloor \frac{3n}{p^i} \right\rfloor + \left\lfloor \frac{2n}{p^i} \right\rfloor - \left\lfloor \frac{6n}{p^i} \right\rfloor - \left\lfloor \frac{6n}{p^i} \right\rfloor - \left\lfloor \frac{4n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor \right) = 1.$$

It follows that the summation (4.1) is greater than or equal to $\nu_p(6n+1)$. Therefore, we have proved that

$$(6n+1)\binom{4n}{n} \Big| \binom{12n}{6n}\binom{2n}{n}.$$

We will use another method to show that

$$(6n+5)\binom{4n}{n}\Big|5\binom{12n}{6n}\binom{2n}{n}.$$

Replace n by $n+1$ in S_n, after reduction, we obtain

$$S_{n+1}=\frac{2\cdot 3^2(12n+1)(12n+5)(12n+7)(12n+11)S_n}{(n+1)(2n+1)(6n+1)(6n+5)}\in\mathbb{N}.$$

Hence

$$(6n+5)|2\cdot 3^2(12n+1)(12n+5)(12n+7)(12n+11)S_n.$$

Since

$$\begin{aligned}\gcd(6n+5,2)=&\gcd(6n+5,3)=\gcd(6n+5,12n+1)=\gcd(6n+5,12n+7)\\=&\gcd(6n+5,12n+11)=1\end{aligned}$$

and

$$\gcd(6n+5,12n+5)=5,$$

we get $(6n+5)|5S_n$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 The proof is similar to that of Theorem 1.1. By Lemma 2.2,

$$T_n:=\frac{(15n)!(4n)!}{(12n)!(5n)!(2n)!}=\frac{\binom{15n}{3n}\binom{3n}{n}}{\binom{5n}{n}}\in\mathbb{N}.$$

For any odd prime p with $p|12n+1$, the p-adic order of T_n is given by

$$\nu_p(T_n)=\sum_{i=1}^{\infty}\Big(\Big\lfloor\frac{15n}{p^i}\Big\rfloor+\Big\lfloor\frac{4n}{p^i}\Big\rfloor-\Big\lfloor\frac{12n}{p^i}\Big\rfloor-\Big\lfloor\frac{5n}{p^i}\Big\rfloor-\Big\lfloor\frac{2n}{p^i}\Big\rfloor\Big).\tag{4.2}$$

In view of Lemmas 2.2 and 2.4, the summation (4.2) is greater than or equal to $\nu_p(12n+1)$. That is

$$(12n+1)\binom{5n}{n}\Big|3\binom{15n}{3n}\binom{3n-1}{n-1}.$$

It is easy to see that the carries when adding $(15n-3n)$ to $3n$ in base 3 is the same as the carries when adding $(5n-n)$ to n in base 3. It follows from Lemma 2.1 that

$$\nu_3\left(\binom{15n}{3n}\right)=\nu_3\left(\binom{5n}{n}\right).$$

Since $\gcd(12n+1,3)=1$, we have

$$(12n+1)\binom{5n}{n}\Big|\binom{15n}{3n}\binom{3n-1}{n-1}.$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3 Similarly, by Lemma 2.2

$$W_n:=\frac{(24n)!(9n)!(6n)!(4n)!}{(18n)!(12n)!(8n)!(3n)!(2n)!}=\frac{\binom{24n}{6n}\binom{9n}{n}\binom{4n}{n}}{\binom{12n}{6n}\binom{2n}{n}}\in\mathbb{N}.$$

For any odd prime p with $p|18n+1$,

$$\nu_p(W_n) = \sum_{i=1}^{\infty}\Big(\Big\lfloor\frac{24n}{p^i}\Big\rfloor + \Big\lfloor\frac{9n}{p^i}\Big\rfloor + \Big\lfloor\frac{6n}{p^i}\Big\rfloor + \Big\lfloor\frac{4n}{p^i}\Big\rfloor - \Big\lfloor\frac{18n}{p^i}\Big\rfloor - \Big\lfloor\frac{12n}{p^i}\Big\rfloor - \Big\lfloor\frac{8n}{p^i}\Big\rfloor - \Big\lfloor\frac{3n}{p^i}\Big\rfloor - \Big\lfloor\frac{2n}{p^i}\Big\rfloor\Big). \tag{4.3}$$

By Lemmas 2.2 and 2.5, the summation (4.3) is clearly greater than or equal to $\nu_p(18n+1)$. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4 Since the proofs are similar, we only prove the last statement of Theorem 1.4. Similarly, by Lemma 2.2

$$Y_n := \frac{(15n)!n!}{(8n)!(5n)!(3n)!} = \frac{\binom{15n}{7n}\binom{7n}{2n}}{\binom{3n}{n}} \in \mathbb{N}.$$

For any odd prime p with $p|8n+1$,

$$\nu_p(Y_n) = \sum_{i=1}^{\infty}\Big(\Big\lfloor\frac{15n}{p^i}\Big\rfloor + \Big\lfloor\frac{n}{p^i}\Big\rfloor - \Big\lfloor\frac{8n}{p^i}\Big\rfloor - \Big\lfloor\frac{5n}{p^i}\Big\rfloor - \Big\lfloor\frac{3n}{p^i}\Big\rfloor\Big). \tag{4.4}$$

By Lemmas 2.2 and 2.6, the summation (4.4) is clearly greater than or equal to $\nu_p(8n+1)$ when $p \neq 7$, and $\nu_p(8n+1) - 1$ when $p = 7$. This completes the proof of Theorem 1.4. □

The proof of Theorem 1.5 is similar to that of Theorems 1.1–1.4, so we omit details.

Proof of Theorem 1.7 Let

$$T(a,b,n) := \frac{\binom{2an}{an}\binom{an}{bn}}{\binom{2bn}{bn}} = \frac{(2an)!(bn)!}{(an)!(an-bn)!(2bn)!}.$$

By [2, Theorem 1.4], we know that $T(a,b,n) \in \mathbb{N}$ for all integers n. It is easy to see that $2bn+1, 2bn+3$ and $2bn+5$ are pairwise coprime positive integers. Hence to prove Theorem 1.7, it suffices to prove that

$$2bn+5|15(a-b)(3a-b)(5a-b)(5a-3b)T(a,b,n), \tag{4.5}$$

$$2bn+3|15(a-b)(3a-b)(5a-b)(5a-3b)T(a,b,n), \tag{4.6}$$

$$2bn+1|15(a-b)(3a-b)(5a-b)(5a-3b)T(a,b,n). \tag{4.7}$$

By [7, Theorem 1], we have

$$(2bn+1)(2bn+3)\binom{2bn}{bn}\Big|3(a-b)(3a-b)\binom{2an}{an}\binom{an}{bn}.$$

Hence we only prove the first statement. Suppose that $p^\alpha\|2bn+5$ with $\alpha \geq 1$. We will prove that

$$p^\alpha|15(a-b)(3a-b)(5a-b)(5a-3b)T(a,b,n). \tag{4.8}$$

Let $p^{c_1}\|a-b$, $p^{c_2}\|5a-b$, $p^{c_3}\|5a-3b$, $p^{c_4}\|3a-b$ with non-negative integers c_i, $i = 1,2,3,4$. Write $\beta = \max\{c_1,c_2,c_3,c_4\}$. If $\alpha \leq \beta$, then (4.8) holds clearly. Therefore, we may assume that

$\alpha > \beta$. Assume $p \geq 7$, we will show that

$$\left\lfloor \frac{2an}{p^i} \right\rfloor + \left\lfloor \frac{bn}{p^i} \right\rfloor - \left\lfloor \frac{an}{p^i} \right\rfloor - \left\lfloor \frac{an-bn}{p^i} \right\rfloor - \left\lfloor \frac{2bn}{p^i} \right\rfloor = 1,$$

for $i = \beta+1, \beta+2, \ldots, \alpha$. In fact, since $p^\alpha | 2bn+5$ and $p \geq 7$, we have $\gcd(p,n) = 1$. It follows that $p^\alpha \| 2bn+5$ and $bn \equiv \frac{p^\alpha - 5}{2} \pmod{p^\alpha}$. Take $i \in \beta+1, \beta+2, \ldots, \alpha$, we get $2bn \equiv p^i - 5 \pmod{p^i}$ and $bn \equiv \frac{p^i-5}{2} \pmod{p^i}$. Now we divide the proof into five cases according to the value of $an \pmod{p^i}$.

Case 1. $an \equiv t \pmod{p^i}$ with $0 \leq t < \frac{p^i-5}{2}$. Then $2an \equiv 2t \pmod{p^i}$ and $0 \leq 2t < p^i - 5$. Also

$$an - bn \equiv t - \frac{p^i-5}{2} + p^i \pmod{p^i}, \quad 0 \leq t - \frac{p^i-5}{2} + p^i < p^i.$$

Therefore, we obtain

$$\begin{aligned}
&\left\lfloor \frac{2an}{p^i} \right\rfloor + \left\lfloor \frac{bn}{p^i} \right\rfloor - \left\lfloor \frac{an}{p^i} \right\rfloor - \left\lfloor \frac{an-bn}{p^i} \right\rfloor - \left\lfloor \frac{2bn}{p^i} \right\rfloor \\
&= \frac{2an-2t}{p^i} + \frac{bn - \frac{p^i-5}{2}}{p^i} - \frac{an-t}{p^i} - \frac{an-bn-(t-\frac{p^i-5}{2}+p^i)}{p^i} - \frac{2bn-(p^i-5)}{p^i} \\
&= 1.
\end{aligned}$$

Case 2. $an \equiv \frac{p^i-5}{2} \pmod{p^i}$. Then $an - bn \equiv 0 \pmod{p^i}$. Since $\gcd(p,n) = 1$, we have $p^i \mid a-b$, which contradicts $p^{c_1} \parallel a-b$, $c_1 \leq \beta < i$.

Case 3. $an \equiv \frac{p^i-3}{2} \pmod{p^i}$. We have

$$5an - 3bn \equiv \frac{5(p^i-3)}{2} - \frac{3(p^i-5)}{2} \equiv 0 \pmod{p^i}.$$

Now $\gcd(p,n) = 1$ yields that $p^i \mid 5a-3b$, but $p^{c_3} \parallel 5a-3b$, $c_3 \leq \beta < i$. A contradiction.

Case 4. $an \equiv \frac{p^i-1}{2} \pmod{p^i}$. In this case, we have

$$5an - bn \equiv \frac{5(p^i-1)}{2} - \frac{p^i-5}{2} \equiv 0 \pmod{p^i}.$$

Since $\gcd(p,n) = 1$, then $p^i \mid 5a-b$. This also contradicts $p^{c_2} \parallel 5a-b$ and $c_2 \leq \beta < i$.

Case 5. $an \equiv t \pmod{p^i}$ with $\frac{p^i+1}{2} \leq t < p^i$. Then $2an \equiv 2t - p^i \pmod{p^i}$ and $1 \leq 2t - p^i < p^i$. Also

$$an - bn \equiv t - \frac{p^i-5}{2} \pmod{p^i}, \quad 0 \leq t - \frac{p^i-5}{2} < \frac{p^i+5}{2} < p^i.$$

Hence

$$\begin{aligned}
&\left\lfloor \frac{2an}{p^i} \right\rfloor + \left\lfloor \frac{bn}{p^i} \right\rfloor - \left\lfloor \frac{an}{p^i} \right\rfloor - \left\lfloor \frac{an-bn}{p^i} \right\rfloor - \left\lfloor \frac{2bn}{p^i} \right\rfloor \\
&= \frac{2an-(2t-p^i)}{p^i} + \frac{bn - \frac{p^i-5}{2}}{p^i} - \frac{an-t}{p^i} - \frac{an-bn-(t-\frac{p^i-5}{2})}{p^i} - \frac{2bn-(p^i-5)}{p^i} \\
&= 1.
\end{aligned}$$

Consequently, $\nu_p(T(a,b,n)) \geq \alpha - \beta$, and so

$$v_p(15(a-b)(3a-b)(5a-b)(5a-3b)T(a,b,n)) \geq \alpha.$$

To sum up, we have shown that the identity (4.8) is true for $p \geq 7$.

Now suppose that $p = 5$. If $25 \mid n$, then $5 \mid 2bn+5$, while $25 \nmid 2bn+5$. It follows that $\alpha = 1$. It is easily seen that the identity (4.8) holds. If $25 \nmid n$, in this situation, we apply the proof of the case $p \geq 7$. In Case 2, $an - bn \equiv 0 \pmod{5^i}$ implies $5^{i-1} \mid a - b$. In Case 3, $5an - 3bn \equiv 0 \pmod{5^i}$, which means that $5^{i-1} \mid 5a - 3b$. In Case 4, $5an - bn \equiv 0 \pmod{5^i}$ is equivalent to $5^{i-1} \mid 5a - b$. Hence, if $i \geq \beta + 2$, then $i - 1 \geq \beta + 1$. Contradictions in both cases. Namely,

$$\left\lfloor \frac{2an}{5^i} \right\rfloor + \left\lfloor \frac{bn}{5^i} \right\rfloor - \left\lfloor \frac{an}{5^i} \right\rfloor - \left\lfloor \frac{an-bn}{5^i} \right\rfloor - \left\lfloor \frac{2bn}{5^i} \right\rfloor = 1,$$

for $i = \beta+2, \beta+3, \ldots, \alpha$. Hence $\nu_5(T(a,b,n)) \geq \alpha - \beta - 1$, and then

$$\nu_5(15(a-b)(3a-b)(5a-b)(5a-3b)T(a,b,n)) \geq \alpha,$$

therefore, the identity (4.8) holds. By the similar argument as for $p = 5$, (4.8) holds for $p = 3$. This completes the proof of Theorem 1.7. $\square$

Proofs of Theorems 1.8 and 1.9 The proof is similar to that of Theorem 1 in [7]. To prove Theorem 1.8, it is sufficient to prove that

$$2bn + 7 | 7(a-b)(7a-5b)(7a-3b)(7a-b)T(a,b,n). \tag{4.9}$$

Suppose that $p^\alpha \| 2bn+7$ with $\alpha \geq 1$. We will prove that

$$p^\alpha | 7(a-b)(7a-5b)(7a-3b)(7a-b)T(a,b,n). \tag{4.10}$$

Let $p^{c_1} \| a-b$, $p^{c_2} \| 7a-5b$, $p^{c_3} \| 7a-3b$, $p^{c_4} \| 7a-b$ with nonnegative integers $c_i \geq 0$, $i = 1,2,3,4$. Write $\beta = \max\{c_1, c_2, c_3, c_4\}$. If $\alpha \leq \beta$, it is clear that (4.10) holds. Therefore, we assume that $\alpha > \beta$. Suppose that $p > 7$, we will prove that

$$\left\lfloor \frac{2an}{p^i} \right\rfloor + \left\lfloor \frac{bn}{p^i} \right\rfloor - \left\lfloor \frac{an}{p^i} \right\rfloor - \left\lfloor \frac{an-bn}{p^i} \right\rfloor - \left\lfloor \frac{2bn}{p^i} \right\rfloor = 1,$$

for $i = \beta+1, \beta+2, \ldots, \alpha$. Since $p^\alpha | 2bn+7$ and $p > 7$, we have $\gcd(p,n) = 1$. Note that $p^\alpha \| 2bn+7$, so $bn \equiv \frac{p^\alpha - 7}{2} \pmod{p^\alpha}$. Take $i \in \beta+1, \beta+2, \ldots, \alpha$, we must have

$$bn \equiv \frac{p^i - 7}{2} \pmod{p^i}.$$

Now we divide the proof into six cases according to the value of $an \pmod{p^i}$.

Case 1. $an \equiv t \pmod{p^i}$ with $0 \leq t < \frac{p^i-7}{2}$. Then $2an \equiv 2t \pmod{p^i}$ and $0 \leq 2t < p^i - 7$. Also

$$an - bn \equiv t - \frac{p^i-7}{2} + p^i \pmod{p^i}, \quad 0 \leq t - \frac{p^i-7}{2} + p^i < p^i.$$

Therefore, we have

$$\begin{aligned}
&\left\lfloor \frac{2an}{p^i} \right\rfloor + \left\lfloor \frac{bn}{p^i} \right\rfloor - \left\lfloor \frac{an}{p^i} \right\rfloor - \left\lfloor \frac{an-bn}{p^i} \right\rfloor - \left\lfloor \frac{2bn}{p^i} \right\rfloor \\
&= \frac{2an-2t}{p^i} + \frac{bn - \frac{p^i-7}{2}}{p^i} - \frac{an-t}{p^i} - \frac{an-bn-(t-\frac{p^i-7}{2}+p^i)}{p^i} - \frac{2bn-(p^i-7)}{p^i} \\
&= 1.
\end{aligned}$$

Case 2. $an \equiv \frac{p^i-7}{2} \pmod{p^i}$. Then $an-bn \equiv 0 \pmod{p^i}$. Note that $\gcd(p,n)=1$, we get $p^i \mid a-b$, while $p^{c_1} \parallel a-b$, $c_1 \le \beta < i$. A contradiction.

Case 3. $an \equiv \frac{p^i-5}{2} \pmod{p^i}$. Then

$$7an-5bn \equiv \frac{7(p^i-5)}{2} - \frac{5(p^i-7)}{2} \equiv 0 \pmod{p^i}.$$

Since $\gcd(p,n)=1$, we have $p^i \mid 7a-5b$, however $p^{c_2} \parallel 7a-5b$ with $c_2 \le \beta < i$. A contradiction again.

Case 4. $an \equiv \frac{p^i-3}{2} \pmod{p^i}$. Then

$$7an-3bn \equiv \frac{7(p^i-3)}{2} - \frac{3(p^i-7)}{2} \equiv 0 \pmod{p^i}.$$

The fact that $\gcd(p,n)=1$ implies $p^i \mid 7a-3b$, which contradicts $p^{c_3} \parallel 7a-3b$ and $c_3 \le \beta < i$.

Case 5. $an \equiv \frac{p^i-1}{2} \pmod{p^i}$. Then

$$7an-bn \equiv \frac{7(p^i-1)}{2} - \frac{(p^i-7)}{2} \equiv 0 \pmod{p^i}.$$

As $\gcd(p,n)=1$, we have $p^i \mid 7a-b$. However, $p^{c_4} \parallel 7a-b$, $c_4 \le \beta < i$, again a contradiction.

Case 6. $an \equiv t \pmod{p^i}$ with $\frac{p^i+1}{2} \le t < p^i$. We have $2an \equiv 2t-p^i \pmod{p^i}$ and $1 \le 2t-p^i < p^i$. Further,

$$an-bn \equiv t-\frac{p^i-7}{2} \pmod{p^i}, \quad 0 \le t-\frac{p^i-7}{2} < \frac{p^i+7}{2} < p^i.$$

Hence, we have

$$\begin{aligned}
&\Big\lfloor \frac{2an}{p^i} \Big\rfloor + \Big\lfloor \frac{bn}{p^i} \Big\rfloor - \Big\lfloor \frac{an}{p^i} \Big\rfloor - \Big\lfloor \frac{an-bn}{p^i} \Big\rfloor - \Big\lfloor \frac{2bn}{p^i} \Big\rfloor \\
&= \frac{2an-(2t-p^i)}{p^i} + \frac{bn-\frac{p^i-7}{2}}{p^i} - \frac{an-t}{p^i} - \frac{an-bn-(t-\frac{p^i-7}{2})}{p^i} - \frac{2bn-(p^i-7)}{p^i} \\
&= 1.
\end{aligned}$$

Therefore, $\nu_p(T(a,b,n)) \ge \alpha-\beta$ and $v_p(7(a-b)(7a-5b)(7a-3b)(7a-b)T(a,b,n)) \ge \alpha$. That is, the identity (4.10) holds for $p > 7$.

Now assume $p=7$. If $49 \mid n$, then $7 \mid 2bn+7$, but $49 \nmid 2bn+7$. It implies that $\alpha = 1$. Obviously, the identity (4.10) is true. If $49 \nmid n$, by the same argument as in the proof of the case $p>7$. Here in Case 2, $an-bn \equiv 0 \pmod{7^i}$ implies $7^{i-1} \mid a-b$. In Case 3, $7an-5bn \equiv 0 \pmod{7^i}$, that is $7^{i-1} \mid 7a-5b$. In Case 4, $7an-3bn \equiv 0 \pmod{7^i}$, so $7^{i-1} \mid 7a-3b$. In Case 5, $7an-bn \equiv 0 \pmod{7^i}$ means that $7^{i-1} \mid 7a-b$. Hence, if $i \ge \beta+2$, then $i-1 \ge \beta+1$. Contradictions in both cases. That is,

$$\Big\lfloor \frac{2an}{7^i} \Big\rfloor + \Big\lfloor \frac{bn}{7^i} \Big\rfloor - \Big\lfloor \frac{an}{7^i} \Big\rfloor - \Big\lfloor \frac{an-bn}{7^i} \Big\rfloor - \Big\lfloor \frac{2bn}{7^i} \Big\rfloor = 1,$$

for $i=\beta+2, \beta+3, \ldots, \alpha$. It follows that $\nu_7(T(a,b,n)) \ge \alpha-\beta-1$. Thus, we have

$$\nu_7(7(a-b)(7a-5b)(7a-3b)(7a-b)T(a,b,n)) \ge \alpha,$$

and so we prove (4.10). By the same argument as above, the identity (4.10) also holds for $p=5$ and $p=3$. This completes the proof of Theorem 1.8. $\square$

The proof of Theorem 1.9 is similar to that of Theorem 1.8, so we omit details here.

Remark 5.1 By the similar argument as in the paper, we can get other more divisibility properties for the binomial coefficients in Bober [2].

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