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Set Sharing Results for Derivatives of Meromorphic Functions

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Abstract In this paper, we investigate the uniqueness of the derivatives of meromorphic functions sharing two different sets, and obtain the result that if two transcendental meromorphic functions f and g satisfy $\overline{E}_{f^{(k)}}(S) = \overline{E}_{g^{(k)}}(T)$, then $f^{(k)} = Ag^{(k)}$, where S, T are two finite sets and A is a nonzero constant. In particular, $k = 0$ implies $f = Ag$.

Keywords meromorphic function; shared set; uniqueness theorem

MR(2020) Subject Classification 30D35

1. Introduction

In this paper, the term "meromorphic" means that a function is meromorphic in the complex plane $\mathbb{C}$. It is assumed that the readers are familiar with the notations of Nevanlinna theory that can be found, for instance, in [1]. Let S be the subset of distinct elements in $\mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{z | f(z) - a = 0, \text{ counting multiplicities}\},$$

$$\overline{E}_f(S) = \bigcup_{a \in S} \{z | f(z) - a = 0, \text{ ignoring multiplicities}\}.$$

Let f and g be two non-constant meromorphic functions. If $E_f(S) = E_g(S)$, then we say f and g share the set S CM (counting multiplicities). Similarly, we say f and g share the set S IM (ignoring multiplicities) if $\overline{E}_f(S) = \overline{E}_g(S)$.

During the last few decades, the uniqueness theory of entire or meromorphic functions has been grown up as an important subfield of the value distribution theory. The main intention of the uniqueness theory is to determine an entire or meromorphic function uniquely satisfying some prescribed condition. The remarkable Five-Value Theorem and Four-Value Theorem by Nevanlinna [2] can be considered as the inception of this extensive theory. Later, research became more interesting when Gross [3] transferred the study of uniqueness theory to a more general setup, namely sets of distinct elements instead of values. For instance, he proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical. In the same paper, Gross proposed the

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following question:

Question 1.1 ([3]) Can one find two (or possible even one) finite sets S_j $(j=1,2)$ such that any two entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ $(j=1,2)$ must be identical?

This problem has been investigated by many mathematicians and lots of remarkable works have been obtained in this aspect [4–9]. In 1995, Yi proved the following result.

Theorem 1.2 ([10]) *Let z_j, $j=1,2,\ldots,n$ be n distinct roots of algebraic equation $z^n+az^{n-m}+b=0$, where n and m are relatively prime positive integers with $n\geq 15$ and $n>m\geq 5$, and let a and b be nonzero constants satisfying*

$$\frac{b^{n-m}}{a^n}\neq\frac{(-1)^n m^m(n-m)^{n-m}}{n^n}.$$

Suppose that $S=\{c+dz_1,\ldots,c+dz_n\}$, where $d\neq 0$ and c are constants. If f and g are non-constant entire functions such that $E_f(S)=E_g(S)$, then $f=g$.

At the same time, Yi further posed a question: Is the condition $n\geq 15$ sharp in Theorem 1.2? To answer this question, Li and Yang [11] showed that the set $S=\{z^9-z^8+1=0\}$ with only 9 elements is a unique range set of entire functions. Afterwards, Li and Yang [12] also proved that there exists a set with 15 elements such that any two meromorphic functions f and g satisfying $E_f(S)=E_g(S)$ must be identical. Further, Yi [13] and Mues [14] independently improved the above results in the following manner.

Theorem 1.3 ([13,14]) *Let $S=\{z|z^n+az^{n-m}+b=0\}$, where n and m are two positive integers such that n and m have no common factor, $m\geq 2$ and $n>2m+8$, let a and b be two nonzero constants such that the algebraic equation $z^n+az^{n-m}+b=0$ has no multiple root. If f and g are two non-constant meromorphic functions satisfying $E_f(S)=E_g(S)$, then $f=g$.*

In 1998, Qiu did further study on what happens when the k-th derivatives of two meromorphic functions share values [15]. From then on, many mathematicians have obtained lots of elegant results related to the k-th derivatives [16–21]. In this direction, Yi and the second authors [22] proved the following theorem.

Theorem 1.4 ([22]) *Let $S=\{z|z^n+az^{n-1}+b=0\}$, where n (≥ 7) and k are two positive integers and let a and b be two nonzero constants such that the algebraic equation $z^n+az^{n-1}+b=0$ has no multiple root. If f and g are two non-constant entire functions satisfying $E_{f^{(k)}}(S)=E_{g^{(k)}}(S)$, then $f^{(k)}=g^{(k)}$.*

On the other hand, there is an extensive literature on entire or meromorphic functions concerning sharing sets without counting multiplicity [23–26]. And by Yi's polynomial, he [27] has also proved

Theorem 1.5 ([27]) *Let $S=\{z|z^n+az^{n-m}+b=0\}$, where n and m are two positive integers such that n and m have no common factor, $m\geq 2$ and $n>2m+14$, a and b are two nonzero constants such that the algebraic equation $z^n+az^{n-m}+b=0$ has no multiple root. If f and g are two non-constant meromorphic functions satisfying $\overline{E}_f(S)=\overline{E}_g(S)$, then $f=g$.*

Theorem 1.6 ([27]) *Let $S = \{z|z^n + az^{n-m} + b = 0\}$, where n and m are two positive integers such that n and m have no common factor and $n > 2m + 7$, and let a and b be two nonzero constants such that the algebraic equation $z^n + az^{n-m} + b = 0$ has no multiple root. If f and g are two non-constant entire functions satisfying $\overline{E}_f(S) = \overline{E}_g(S)$, then $f = g$.*

Note that the results mentioned above are based on the functions sharing common sets, but the uniqueness of two meromorphic functions sharing two different sets is seldom studied.

Here, we assume that S and T are zero sets of Yi's polynomials P and Q respectively of the following forms

$$P(z) = az^n + bz^{n-m} + d, \; Q(z) = uz^n + vz^{n-m} + t, \tag{1.1}$$

where n and m are two positive integers, and $a, \; b, \; d, \; u, \; v, \; t$ are nonzero complex numbers such that P and Q have no multiple zero.

For two meromorphic functions f and g, does there exist $\overline{E}_f(S) = \overline{E}_g(T)$? The purpose of this paper is to seek the possible answer of the above question. Indeed, we obtained the following results.

Theorem 1.7 *Let f and g be two meromorphic functions and let k be a non-negative integer such that $f^{(k)}$ is not constant. P and Q are defined as (1.1). If $\overline{E}_{f^{(k)}}(S) = \overline{E}_{g^{(k)}}(T)$ and $n > 2m + 7 + \frac{7}{k+1}$, where either $(n, m) = 1$, $m \geq 2$, or $m \geq 4$, then $f^{(k)} = Ag^{(k)}$ for some constant A such that $A^n = \frac{du}{at}$, $A^{n-m} = \frac{dv}{bt}$.*

As instant consequence of Theorem 1.7, we have the following corollary when $k = 0$.

Corollary 1.8 *Let f and g be two non-constant meromorphic functions and let P and Q be defined as (1.1). If $\overline{E}_f(S) = \overline{E}_g(T)$ and $n > 2m+14$, where either $(n, m) = 1$, $m \geq 2$, or $m \geq 4$, then $f = Ag$ for some constant A such that $A^n = \frac{du}{at}$, $A^{n-m} = \frac{dv}{bt}$.*

Remark 1.9 Under the condition of Corollary 1.8, when $S = T$, that is, $\overline{E}_f(S) = \overline{E}_g(S)$ and $A^n = A^{n-m} = 1$, we can obtain that $f = g$. Therefore, Theorem 1.7 and Corollary 1.8 improve Theorem 1.5.

When $k \geq 1$, there is $2m + 7 + \frac{7}{k+1} \geq 2m + 10.5$. Therefore, $n > 2m + 10$ implies that $n > 2m + 7 + \frac{7}{k+1}$, and hence, it can be inferred from Theorem 1.7 that the following corollary holds.

Corollary 1.10 *Let f and g be two meromorphic functions and let k be a positive integer such that $f^{(k)}$ is not constant. Suppose P and Q are defined as (1.1). If $\overline{E}_{f^{(k)}}(S) = \overline{E}_{g^{(k)}}(T)$ and $n > 2m + 10$, where either $(n, m) = 1$, $m \geq 2$, or $m \geq 4$, then $f^{(k)} = Ag^{(k)}$ for some constant A such that $A^n = \frac{du}{at}$, $A^{n-m} = \frac{dv}{bt}$.*

Furthermore, for the case that f and g are two entire functions, we have

Theorem 1.11 *Let f and g be two entire functions and let k be a non-negative integer such that $f^{(k)}$ is not constant. P and Q are defined as (1.1). If $\overline{E}_{f^{(k)}}(S) = \overline{E}_{g^{(k)}}(T)$ and $n > 2m + 7$ with $(n, m) = 1$, then $f^{(k)} = Ag^{(k)}$ for some constant A such that $A^n = \frac{du}{at}$, $A^{n-m} = \frac{dv}{bt}$.*

Similarly, we have the following corollary when $k=0$.

Corollary 1.12 *Let f and g be two non-constant entire functions and let P and Q be defined as (1.1). If $\overline{E}_f(S)=\overline{E}_g(T)$ and $n>2m+7$ with $(n,m)=1$, then $f=Ag$ for some constant A such that $A^n=\frac{du}{at}$, $A^{n-m}=\frac{dv}{bt}$.*

Remark 1.13 Under the condition of Corollary 1.12, when $S=T$, that is, $\overline{E}_f(S)=\overline{E}_g(S)$ and $A^n=A^{n-m}=1$, then $f=g$. Hence, Theorem 1.11 and Corollary 1.12 improve Theorem 1.6.

Remark 1.14 The conditions of Theorem 1.7 and Theorem 1.11 "$f^{(k)}$ is not constant" cannot be dropped. For example: Let $n=15$, $m=2$, $f=7z^k$, $g=z^k$ and $S=T=\{z|z^{15}+z^{13}+1=0\}$. It is clear that $f^{(k)}(z)=7k!$, $g^{(k)}(z)=k!$, $7k!\notin S$, $k!\notin S$ and $\overline{E}_{f^{(k)}}(S)=\overline{E}_{g^{(k)}}(S)=\emptyset$, but $f^{(k)}(z)\not\equiv g^{(k)}(z)$ for any z.

2. Some lemmas

In this section, we present some important lemmas which will be needed in the sequel. Firstly, we introduce some notations and definitions as follows.

Let a be a finite complex number and let f and g be two meromorphic functions sharing the value a IM. Let z_0 be an a-point of f with multiplicity p, and an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,\frac{1}{f-a})$ the reduced counting function of the zeros of $f-a$ with $p>q\geq 1$. In the same way, we can define $\overline{N}_L(r,\frac{1}{g-a})$. We denote by $N_2(r,\frac{1}{f-a})$ the counting function of the zeros of $f-a$, where an a-point of multiplicity m is counted m times if $m\leq 2$ and is counted twice if $m>2$.

Definition 2.1 ([1]) *We denote by $\Theta(a,f)$ the quantity*

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)},$$

which is called the ramification index.

Lemma 2.2 ([1]) *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n\cdot b_m\neq 0$. Then $T(r,R(f))=dT(r,f)+S(r,f)$, where $d=\max\{n,m\}$.

Lemma 2.3 ([1]) *Let f be a non-constant meromorphic function in $\mathbb{C}$ and let a_j $(j=1,2,\ldots,q)$ be distinct points in $\mathbb{C}\cup\{\infty\}$. Then*

$$(q-2)T(r,f)\leq\sum_{j=1}^{q}\overline{N}(r,\frac{1}{f-a_j})+S(r,f).$$

Lemma 2.4 ([1]) *Let f be a non-constant meromorphic function and let k be a positive integer.*

Then

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.5 ([1]) *Let f be a non-constant meromorphic function and let a be any value in the extended complex plane. Then the set of values a for which $\Theta(a, f) > 0$ is countable and*

$$\sum_{a} \Theta(a, f) \leq 2.$$

Lemma 2.6 ([28]) *Let f and g be two non-constant meromorphic functions sharing the value 1 IM. Let*

$$H = (\frac{f''}{f'} - \frac{2f'}{f-1}) - (\frac{g''}{g'} - \frac{2g'}{g-1}). \tag{2.1}$$

If $H \not\equiv 0$, then

$$\begin{aligned} T(r, f) + T(r, g) \leq & 2[N_2(r, f) + N_2(r, g) + N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g})] + \\ & 3\overline{N}_L(r, \frac{1}{f-1}) + 3\overline{N}_L(r, \frac{1}{g-1}) + S(r, f) + S(r, g). \end{aligned}$$

Lemma 2.7 *Let f and g be two meromorphic functions. P and Q are given by (1.1). If there exist two constants A ($\neq 0$) and B such that*

$$\frac{1}{P(f)} = \frac{A}{Q(g)} + B, \tag{2.2}$$

and $n > 2m + 3$, where $(n, m) = 1$ or $m \geq 3$, then $Q(g) = AP(f)$ for some constant A such that $A = \frac{t}{d}$, where t and d are defined in (1.1).

Proof Set

$$\begin{aligned} & F := P(f) = af^n + bf^{n-m} + d, \quad G := Q(g) = ug^n + vg^{n-m} + t, \\ & F_1 := -\frac{a}{d}f^n - \frac{b}{d}f^{n-m}, \quad G_1 := -\frac{u}{t}g^n - \frac{v}{t}g^{n-m}. \end{aligned}$$

It is easy to see that $F_1 - 1 = -\frac{1}{d}F$, $G_1 - 1 = -\frac{1}{t}G$. Then we can rewrite (2.2) as

$$F_1 = \frac{(t - dtB)G_1 + (dA + dtB - t)}{(dA + dtB) - dtBG_1}. \tag{2.3}$$

By Lemma 2.2, we get

$$T(r, f) = T(r, g) + O(1). \tag{2.4}$$

We claim that $B = 0$, otherwise, we consider the following two cases:

Case 1. Suppose that $B \neq 0, \frac{1}{d}$. If $dA + dtB - t \neq 0$, we have from (2.3) that

$$\overline{N}(r, \frac{1}{(t - dtB)G_1 + (dA + dtB - t)}) = \overline{N}(r, \frac{1}{F_1}).$$

Thus, together with (2.4) and using the second fundamental theorem, we get

$$nT(r, g) = T(r, G_1) + S(r, g) \leq \overline{N}(r, G_1) + \overline{N}(r, \frac{1}{G_1}) +$$

$$\begin{aligned}
&\overline{N}(r,\frac{1}{(t-dtB)G_1+(dA+dtB-t)})+S(r,g)\\
\leq&\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{ug^m+v})+\\
&\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{af^m+b})+S(r,g)\\
\leq&(2m+3)T(r,g)+S(r,g),
\end{aligned}$$

which contradicts the assumption $n>2m+3$. Therefore, we obtain $dA+dtB-t=0$, and hence, we can rewrite (2.3) as $F_1=\frac{(1-dB)G_1}{1-dBG_1}$. It follows that $\overline{N}(r,\frac{1}{1-dBG_1})=\overline{N}(r,F_1)$. Again from the second fundamental theorem and (2.4), we obtain

$$\begin{aligned}
nT(r,g)=T(r,G_1)+S(r,g)&\leq\overline{N}(r,G_1)+\overline{N}(r,\frac{1}{G_1})+\overline{N}(r,\frac{1}{1-dBG_1})+S(r,g)\\
&\leq\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{ug^m+v})+\overline{N}(r,f)+S(r,g)\\
&\leq(m+3)T(r,g)+S(r,g),
\end{aligned}$$

which is also a contradiction.

Case 2. Suppose that $B=\frac{1}{d}$. Then (2.3) becomes $F_1=\frac{dA}{(dA+t)-tG_1}$. If $dA+t\neq0$, we get

$$\overline{N}(r,\frac{1}{(dA+t)-tG_1})=\overline{N}(r,F_1).$$

In the same manner as Case 1, we have a contradiction. Therefore, we have $dA+t=0$, and hence, $F_1G_1=1$, that is

$$f^{n-m}\{af^m+b\}g^{n-m}\{ug^m+v\}=dt. \tag{2.5}$$

Write

$$\begin{aligned}
&af^m+b=a(f-s_1)(f-s_2)\cdots(f-s_m),\quad s_i\neq s_j,\ i\neq j,\\
&ug^m+v=u(g-t_1)(g-t_2)\cdots(g-t_m),\quad t_i\neq t_j,\ i\neq j.
\end{aligned}$$

We consider the following two subcases:

Subcase 2.1. $(n,m)=1$.

Let z_0 be a zero of f with multiplicity p_0. Then by (2.5), z_0 is a pole of g with multiplicity q_0 such that $(n-m)p_0=nq_0$. Noting that $(n,m)=1$, we have $p_0\geq n$. Therefore

$$\overline{N}(r,\frac{1}{f})\leq\frac{1}{n}N(r,\frac{1}{f}). \tag{2.6}$$

Let z_{i1} be a zero of $f-s_i$ with multiplicity p_{i1} for $i=1,2,\dots,m$. By (2.5) again, z_{i1} is a pole of g with multiplicity q_{i1} such that $p_{i1}=nq_{i1}$. Then $p_{i1}\geq n$ and thus

$$\overline{N}(r,\frac{1}{f-s_i})\leq\frac{1}{n}N(r,\frac{1}{f-s_i}). \tag{2.7}$$

Similarly, we have results for the zeros of $g^{n-m}\{ug^m+v\}$. On the other hand, suppose z_2 is a pole of f, from (2.5), we get z_2 is a zero of $g^{n-m}\{ug^m+v\}$. Then

$$\overline{N}(r,f)\leq\overline{N}(r,\frac{1}{g})+\sum_{i=1}^{m}\overline{N}(r,\frac{1}{g-t_i})$$

$$\leq \frac{1}{n}N(r,\frac{1}{g})+\frac{1}{n}\sum_{i=1}^{m}N(r,\frac{1}{g-t_i})$$
$$\leq \frac{m+1}{n}T(r,g)+S(r,g). \tag{2.8}$$

From (2.4) and (2.6)–(2.8), by the second fundamental theorem, we have

$$\begin{aligned} mT(r,f) \leq &\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\sum_{i=1}^{m}\overline{N}(r,\frac{1}{f-s_i})+S(r,f)\\ \leq &\frac{m+1}{n}T(r,g)+\frac{1}{n}N(r,\frac{1}{f})+\frac{1}{n}\sum_{i=1}^{m}N(r,\frac{1}{f-s_i})+S(r,f)+S(r,g)\\ \leq &\frac{2m+2}{n}T(r,f)+S(r,f)+S(r,g), \end{aligned}$$

which contradicts $n > 2m+3$.

Subcase 2.2. $m \geq 3$.

Let z_3 be a zero of f with multiplicity p_3. Then by (2.5), z_3 is a pole of g with multiplicity q_3 such that $(n-m)p_3 = nq_3$, then $p_3 \geq \frac{n}{n-m}$. Let z_{i4} be a zero of $f-s_i$ with multiplicity p_{i4} for $i = 1, 2, \ldots, m$. By (2.5) again, z_{i4} is a pole of g with multiplicity q_{i4} such that $p_{i4} = nq_{i4}$, then $p_{i4} \geq n$.

In the same manner as Subcase 2.1, we have

$$\begin{aligned} mT(r,f) &\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\sum_{i=1}^{m}\overline{N}(r,\frac{1}{f-s_i})+S(r,f)\\ &\leq T(r,g)+\frac{n-m}{n}N(r,\frac{1}{f})+\frac{1}{n}\sum_{i=1}^{m}N(r,\frac{1}{f-s_i})+S(r,,f)+S(r,g)\\ &\leq 2T(r,f)+S(r,f)+S(r,g), \end{aligned}$$

which contradicts $m \geqslant 3$.

Therefore, we have $B = 0$, and hence, we can rewrite (2.3) as

$$d(F_1-1)=\frac{1}{A}\cdot t(G_1-1),$$

that is

$$af^n+bf^{n-m}+d=\frac{u}{A}g^n+\frac{v}{A}g^{n-m}+\frac{t}{A}. \tag{2.9}$$

Eq. (2.9) can be rewritten as

$$f_1+f_2=\frac{t}{A}-d,$$

where

$$f_1=f^{n-m}\{af^m+b\},\ f_2=-\frac{1}{A}g^{n-m}\{ug^m+v\}.$$

If $A \neq \frac{t}{d}$, together with (2.4) and applying the second fundamental theorem, we have

$$\begin{aligned} nT(r,f)=T(r,f_1)+S(r,f) \leq &\overline{N}(r,f_1)+\overline{N}(r,\frac{1}{f_1})+\overline{N}(r,\frac{1}{f_1-(\frac{t}{A}-d)})+S(r,f)\\ \leq &\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{af^m+b})+ \end{aligned}$$

$$\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{ug^m+v})+S(r,f)$$
$$\leq (2m+3)T(r,f)+S(r,f)+S(r,g),$$

which contradicts $n>2m+3$. Therefore, $A=\frac{t}{d}$.

This completes the proof of Lemma 2.7. □

Lemma 2.8 *Let f and g be two non-constant meromorphic functions. P and Q are given by (1.1). If $P(f)=Q(g)$ and $n>2m+3$, where either $(n,m)=1$, $m\geq 2$, or $m\geq 4$, then $f=Ag$ for some constant A such that $A^n=\frac{u}{a}$, $A^{n-m}=\frac{v}{b}$.*

Proof From the condition, we have

$$af^n+bf^{n-m}+d=ug^n+vg^{n-m}+t. \tag{2.10}$$

Using the same manner as (2.9), we have $t=d$. Thus (2.10) becomes

$$af^n+bf^{n-m}=ug^n+vg^{n-m}. \tag{2.11}$$

For simplicity, let $A(z):=\frac{f(z)}{g(z)}$, $\alpha:=\frac{u}{a}\neq 0$ and $\beta:=\frac{v}{b}\neq 0$. Then (2.11) becomes

$$g^m(A^n-\alpha)=-\frac{b}{a}(A^{n-m}-\beta),\ \ g^m=-\frac{b(A^{n-m}-\beta)}{a(A^n-\alpha)}. \tag{2.12}$$

Assume that A is not a constant, we discuss the following two cases.

Case 1. $(n,m)=1$, $m\geq 2$.

If $A^{n-m}-\beta$ and $A^n-\alpha$ have no common zeros, write

$$A^n-\alpha=(A-\alpha_1)(A-\alpha_2)\cdots(A-\alpha_n),\quad \alpha_i\neq\alpha_j,\ i\neq j.$$

It follows from (2.12) that each zero of $A-\alpha_i$, $i=1,\dots,n$ is also a pole of g. Suppose that z_{i1} is a zero of $A(z)-\alpha_i$ with multiplicity p_{i1}, then z_{i1} is a pole of g with multiplicity q_{i1}. Again by (2.12), we obtain $p_{i1}=mq_{i1}$, then $p_{i1}\geq m$, which implies

$$\overline{N}(r,\frac{1}{A-\alpha_i})\leq\frac{1}{m}N(r,\frac{1}{A-\alpha_i}).$$

Therefore,

$$\Theta(\alpha_i,A)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{A-\alpha_i})}{T(r,A)}\geq 1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{A-\alpha_i})}{N(r,\frac{1}{A-\alpha_i})}\geq 1-\frac{1}{m}>0.$$

By Lemma 2.5, $n(1-\frac{1}{m})\leq 2$, that is, $n\leq\frac{2m}{m-1}$, which contradicts $n>2m+3$.

If $z^{n-m}-\beta$ and $z^n-\alpha$ have common zeros, then there exists z_2 such that $A^n(z_2)=\alpha,\ A^{n-m}(z_2)=\beta$. We can rewrite (2.12) as

$$g^m=-\frac{b\beta}{a\alpha}\cdot\frac{(\frac{A}{A(z_2)})^{n-m}-1}{(\frac{A}{A(z_2)})^n-1}. \tag{2.13}$$

Since $(n,m)=1$, z^n-1 and $z^{n-m}-1$ have different zeros except for $z=1$. Let r_i, $i=1,2,\dots,n-1$ be all the roots of the equation $z^n-1=0$ except for the value 1. Suppose that z_{i3} is a zero of $\frac{A(z)}{A(z_2)}-r_i$ with multiplicity p_{i2}, then from (2.13), z_{i3} is a pole of g with multiplicity

q_{i2}. By (2.13) again, $p_{i2} = mq_{i2}$, then $p_{i2} \geq m$, so that

$$\overline{N}(r, \frac{1}{\frac{A}{A(z_2)} - r_i}) \leq \frac{1}{m} N(r, \frac{1}{\frac{A}{A(z_2)} - r_i}).$$

Therefore,

$$\Theta(r_i, \frac{A}{A(z_2)}) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{\frac{A}{A(z_2)} - r_i})}{T(r, \frac{A}{A(z_2)})} \geq 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{\frac{A}{A(z_2)} - r_i})}{N(r, \frac{1}{\frac{A}{A(z_2)} - r_i})} \geq 1 - \frac{1}{m} > 0.$$

By Lemma 2.5, we obtain $(n-1)(1-\frac{1}{m}) \leq 2$, that is, $n \leq \frac{3m-1}{m-1}$, which contradicts $n > 2m+3$.

Case 2. $m \geq 4$.

Noting that $z^n - \alpha$ and $z^{n-m} - \beta$ have at most $n-m$ common simple zeros. Therefore, there are at least m distinct roots of the equation $z^n - \alpha = 0$, say $\gamma_1, \gamma_2, \ldots, \gamma_m$, that are not roots of $z^{n-m} - \beta = 0$. Suppose that z_{i4} is a zero of $A(z) - \gamma_i$, $i = 1, 2, \ldots, m$ with multiplicity p_{i3}. Then from (2.12), z_{i4} is a pole of g with multiplicity q_{i3}. By (2.12) again, we obtain $p_{i3} = mq_{i3}$, then $p_{i3} \geq m$, so that

$$\overline{N}(r, \frac{1}{A - \gamma_i}) \leq \frac{1}{m} N(r, \frac{1}{A - \gamma_i}).$$

Therefore,

$$\Theta(\gamma_i, A) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{A-\gamma_i})}{T(r, A)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{A-\gamma_i})}{N(r, \frac{1}{A-\gamma_i})} \geq 1 - \frac{1}{m} > 0.$$

By Lemma 2.5, we have $m(1-\frac{1}{m}) \leq 2$, that is, $m \leq 3$, which contradicts $m \geq 4$.

In conclusion, A is a constant, we have $f = Ag$. From (2.12), and since f and g are non-constant functions, we obtain $A^n = \frac{u}{a}$ and $A^{n-m} = \frac{v}{t}$.

This completes the proof of Lemma 2.8. □

3. Proofs of theorems

Let f and g be two meromorphic functions in $\mathbb{C}$. For convenience, we assume that

$$\begin{aligned}
F &:= P(f^{(k)}) = a[f^{(k)}]^n + b[f^{(k)}]^{n-m} + d, \\
G &:= Q(g^{(k)}) = u[g^{(k)}]^n + v[g^{(k)}]^{n-m} + t, \\
F_1 &:= -\frac{1}{d}[f^{(k)}]^{n-m}\{a[f^{(k)}]^m + b\}, \\
G_1 &:= -\frac{1}{t}[g^{(k)}]^{n-m}\{u[g^{(k)}]^m + v\},
\end{aligned} \tag{3.1}$$

where $a,\ b,\ d,\ u,\ v,\ t$ are nonzero complex numbers.

Proof of Theorem 1.7 Let $F,\ G,\ F_1,\ G_1$ and H be defined as (3.1) and (2.1), respectively. One can verify that F and G share the value 0 IM, thus F_1 and G_1 share the value 1 IM. By Lemma 2.6, if $H \not\equiv 0$, we have

$$T(r, F_1) + T(r, G_1) \leq 2[N_2(r, F_1) + N_2(r, G_1) + N_2(r, \frac{1}{F_1}) + N_2(r, \frac{1}{G_1})] +$$

$$3\overline{N}_L(r,\frac{1}{F_1-1})+3\overline{N}_L(r,\frac{1}{G_1-1})+S(r,f^{(k)})+S(r,g^{(k)}). \tag{3.2}$$

Note that

$$N_2(r,F_1)+N_2(r,\frac{1}{F_1})\le 2\overline{N}(r,f^{(k)})+2\overline{N}(r,\frac{1}{f^{(k)}})+N(r,\frac{1}{a[f^{(k)}]^m+b}),$$
$$N_2(r,G_1)+N_2(r,\frac{1}{G_1})\le 2\overline{N}(r,g^{(k)})+2\overline{N}(r,\frac{1}{g^{(k)}})+N(r,\frac{1}{u[g^{(k)}]^m+v}),$$
$$\overline{N}_L(r,\frac{1}{F_1-1})\le \overline{N}_{(2}(r,\frac{1}{F})\le N(r,\frac{1}{f^{(k+1)}})\le N(r,\frac{1}{f^{(k)}})+\overline{N}(r,f^{(k)}),$$
$$\overline{N}_L(r,\frac{1}{G_1-1})\le \overline{N}_{(2}(r,\frac{1}{G})\le N(r,\frac{1}{g^{(k+1)}})\le N(r,\frac{1}{g^{(k)}})+\overline{N}(r,g^{(k)}).$$

Combining the above inequalities and (3.2), we have

$$\begin{aligned} n\{T(r,f^{(k)})+T(r,g^{(k)})\} =& T(r,F_1)+T(r,G_1)+S(r,f^{(k)})+S(r,g^{(k)})\\ \le& 7\overline{N}(r,f^{(k)})+7\overline{N}(r,g^{(k)})+7N(r,\frac{1}{f^{(k)}})+S(r,f^{(k)})+S(r,g^{(k)})+\\ & 7N(r,\frac{1}{g^{(k)}})+2N(r,\frac{1}{a[f^{(k)}]^m+b})+2N(r,\frac{1}{u[g^{(k)}]^m+v})\\ \le& (2m+7+\frac{7}{k+1})\{T(r,f^{(k)})+T(r,g^{(k)})\}+S(r,f^{(k)})+S(r,g^{(k)}), \end{aligned}$$

which contradicts the assumption $n>2m+7+\frac{7}{k+1}$. Thus $H\equiv 0$, that is

$$\frac{F_1''}{F_1'}-\frac{2F_1'}{F_1-1}=\frac{G_1''}{G_1'}-\frac{2G_1'}{G_1-1}.$$

Integrating both sides of the above equality twice, we get

$$\frac{1}{F_1-1}=\frac{A^*}{G_1-1}+B^*, \tag{3.3}$$

where $A^*(\neq 0)$ and B^* are constants. Note that

$$F_1-1=-\frac{1}{d}F,\ G_1-1=-\frac{1}{t}G. \tag{3.4}$$

Then substituting (3.4) into (3.3) yields

$$\frac{1}{F}=\frac{tA^*}{d}\cdot\frac{1}{G}-\frac{B^*}{d}.$$

Applying Lemma 2.7, we get $tF=dG$. Further, applying Lemma 2.8, we can obtain $f^{(k)}=Ag^{(k)}$, where $A^n=\frac{du}{at}$, $A^{n-m}=\frac{dv}{bt}$.

This completes the proof of Theorem 1.7. □

Proof of Theorem 1.11 Let $F,\ G,\ F_1,\ G_1$ and H be defined as (3.1) and (2.1), respectively. It can be seen that F_1 and G_1 share the value 1 IM. Note that $N(r,f^{(k)})=N(r,g^{(k)})=0$. In the same way as done in the proof of Theorem 1.7, we can show that

$$\frac{1}{F}=\frac{tA^*}{d}\cdot\frac{1}{G}-\frac{B^*}{d},$$

where A^* $(\neq 0)$ and B^* are constants.

Thus by Lemma 2.7, we get $tF = dG$, which implies

$$at[f^{(k)}]^n + bt[f^{(k)}]^{n-m} = du[g^{(k)}]^n + dv[g^{(k)}]^{n-m}. \tag{3.5}$$

Let $A(z) := \frac{f^{(k)}(z)}{g^{(k)}(z)}$, $\alpha := \frac{du}{at} \neq 0$ and $\beta := \frac{dv}{bt} \neq 0$ and substituting into (3.5), we obtain

$$[g^{(k)}]^m(A^n - \alpha) = -\frac{b}{a}(A^{n-m} - \beta),\ [g^{(k)}]^m = -\frac{b(A^{n-m} - \beta)}{a(A^n - \alpha)}. \tag{3.6}$$

If A is not a constant. Since $(n, m) = 1$, $z^n - \alpha$ and $z^{n-m} - \beta$ have at most one common simple zero z_0. Let z_i, $i = 1, 2, \ldots, n-1$ be roots of the equation $z^n - \alpha = 0$ except for the value z_0. Noting that each zero of $A(z) - z_i$ is also a pole of $g^{(k)}$, and g is an entire function, then z_i are Picard exceptional values of A, which is impossible.

Thus, A is a constant, we have $f^{(k)} = Ag^{(k)}$. From (3.6), if $A^n \neq \frac{du}{at}$, we will deduce that $f^{(k)}$ is a constant, which contradicts the assumption. Therefore, we have $A^n = \frac{du}{at}$ and $A^{n-m} = \frac{dv}{bt}$.

This completes the proof of Theorem 1.11. □

4. Some open questions

Recently, we have studied on what will happen if the condition "$\overline{E}_{f^{(k)}}(S) = \overline{E}_{g^{(k)}}(T)$" in Theorems 1.7 and 1.11 is replaced by "$E_{f^{(k)}}(S) = E_{g^{(k)}}(T)$", and we obtained the following results.

Theorem 4.1 *Let f and g be two non-constant meromorphic functions and k be a non-negative integer, and let P, Q, S and T be defined as (1.1). If $E_{f^{(k)}}(S) = E_{g^{(k)}}(T)$ and $n > 2m + 4 + \frac{4}{k+1}$, where either $(n, m) = 1$, $m \geq 2$, or $m \geq 4$, then $f^{(k)} = Ag^{(k)}$ for some constant A.*

Theorem 4.2 *Let f and g be two non-constant entire functions and k be a non-negative integer, and let P, Q, S and T be defined as (1.1). If $E_{f^{(k)}}(S) = E_{g^{(k)}}(T)$ and $n > 2m + 4$, then $f^{(k)} = Ag^{(k)}$ for some constant A.*

There are still some open questions for further study.

Question 4.3 Is it possible to additionally weaken the relationship conditions n and m in Theorems 1.7 and 4.1?

Question 4.4 What happens when Yi's polynomials P and Q are replaced by the other style of polynomials?

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References

[1] Hongxun YI, Chongjun YANG. *Uniqueness Theory of Meromorphic Functions.* Science Press, Beijing, 2003.

[2] R. NEVANLINNA. *Le Théorème de Picard-Borel et la Théorie des Functions Méromorphes.* Gauthier-Villars, Paris, 1929.

[3] F. GROSS. *Factorization of Meromorphic Functions and Some Open Problems.* Lecture Notes in Math., Vol. 599, Springer, Berlin, 1977.

[4] Mingliang FANG, Hui GUO. *On meromorphic functions sharing two values*. Analysis, 1997, **17**(4): 355–366.

[5] G. FRANK, M. REINDERS. *A unique range set for meromorphic functions with 11 elements*. Complex Variables Theory Appl., 1998, **37**(1): 185–193.

[6] Weichuan LIN, Hongxun YI. *Some further results on meromorphic functions that share two sets*. Kyungpook Math. J., 2003, **43**(1): 73–85.

[7] Hongxun YI. *Unicity theorems for meromorphic or entire functions II*. Bull. Austral. Math. Soc., 1995, **52**(2): 215–224.

[8] I. LAHIRI. *On a question of Hong Xun Yi*. Arch. Math., 2002, **38**(2): 119–128.

[9] I. LAHIRI, A. BANERJEE. *Weighted sharing of two sets*. Kyungpook Math. J., 2006, **46**(1): 79–87.

[10] Hongxun YI. *On a question of gross*. Sci. China Ser. A, 1995, **38**(1): 8–16.

[11] Ping LI, Chongjun YANG. *On the unique range set of meromorphic functions*. Proc. Amer. Math. Soc., 1996, **124**(1): 177–185.

[12] Ping LI, Chongjun YANG. *Some further results on the unique range sets of meromorphic functions*. Kodai Math. J., 1995, **18**(3): 437–450.

[13] Hongxun YI. *Unicity theorems for meromorphic or entire functions III*. Bull. Austral. Math. Soc., 1996, **53**(1): 71–82.

[14] E. MUES, M. REINDERS. *Meromorphic functions sharing one value and unique range sets*. Kodai Math. J., 1995, **18**(3): 515–522.

[15] Gandi QIU. *Four values theorem of derivatives of meromorphic functions*. Systems Sci. Math. Sci., 1998, **11**(3): 245–248.

[16] Mingliang FANG, I. LAHIRI. *Unique range set for certain meromorphic functions*. Indian J. Math., 2003, **45**(2): 141–150.

[17] Junfan CHEN, Weichuan LIN. *Derivatives of meromorphic functions sharing four small functions*. Complex Var. Elliptic Equ., 2008, **53**(3): 265–274.

[18] A. BANERJEE, P. BHATTACHARJEE. *Uniqueness of derivatives of meromorphic functions sharing two or three sets*. Turkish J. Math., 2010, **34**(1): 21–34.

[19] A. BANERJEE, P. BHATTACHARJEE. *Uniqueness and set sharing of derivatives of meromorphic functions*. Math. Slovaca, 2011, **61**(2): 197–214.

[20] A. BANERJEE, B. CHAKRABORTY. *Further results on the uniqueness of meromorphic functions and their derivative counterpart sharing one or two sets*. Jordan J. Math. Stat., 2016, **9**(2): 117–139.

[21] A. SARKAR. *Derivatives of meromorphic functions sharing two sets with least cardinalities*. Adv. Pure Appl. Math., 2019, **10**(3): 251–262.

[22] Hongxun YI, Weichuan LIN. *Uniqueness of meromorphic functions and a question of Gross*. Kyungpook Math. J., 2006, **46**(3): 437–444.

[23] Hongxun YI. *On the reduced unique range sets of meromorphic functions*. J. Shandong Univ. Nat. Sci., 1998, **33**(4): 3–10. (in Chinese)

[24] S. BARTELS. *Meromorphic functions sharing a set with 17 elements ignoring multiplicities*. Complex Variables Theory Appl., 1999, **39**(1): 85–92.

[25] Hongxun YI, Weiran Lü. *Meromorphic functions that share two sets II*. Acta Math. Sci. Ser. B, 2004, **24**(1): 83–90.

[26] Jindong LI, Qibin ZHANG. *On uniqueness of meromorphic functions sharing two IM sets*. J. Math. Res. Exposition, 2005, **25**(2): 299–306. (in Chinese)

[27] Hongxun YI. *Uniqueness theorems for meromorphic functions*. Chinese Ann. Math. Ser. A, 1996, **17**(4): 397–404. (in Chinese)

[28] Mingliang FANG, Hui GUO. *On unique range sets for meromorphic or entire functions*. Acta Math. Sinica, 1998, **14**(4): 569–576.