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Zero Distribution of Solutions of Higher-Order Linear Differential Equations and Zygmund Type Space

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Abstract The aim of this paper is to consider the following two problems:

(1) Establish interrelationships between the growth of coefficients and the geometric distribution of zeros of solutions of non-homogeneous linear differential equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = A_3(z),$$

where $A_0(z), \ldots, A_3(z)$ are analytic functions in the unit disc $\mathbb{D}$;

(2) Find some sufficient conditions on the analytic coefficients of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

for all solutions to belong to the Zygmund type space.

The results we obtain are a generalization of some earlier results by Heittokangas, Gröhn, Korhoneon and Rättyä.

Keywords linear differential equation; uniformly separated sequence; Zygmund type space

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1. Introduction and main results

Let $\mathbb{D}_R$ denote the Euclidian disc of radius R centered at the origin in the complex $\mathbb{C}$, so $\mathbb{D}_1 = \mathbb{D}$. Denote by $\mathcal{H}(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$.

The sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is called uniformly separated if

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} |\frac{z_n - z_k}{1-\overline{z_n}z_k}| > 0,$$

while $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ is said to be separated in the hyperbolic metric if there exists a constant $\delta > 0$ such that $|z_n - z_k|/|1-\overline{z_n}z_k| > \delta$ for any $n \neq k$.

A fundamental objective in the study of complex linear differential equations with analytic coefficients in a complex domain is to relate the growth of coefficients to the growth of solutions and the distribution of their zeros.

We restrict ourselves to the case of the unit disc $\mathbb{D}$. The early results on oscillation theory in the case of unit disc go back to the work of Nehari and his students Beesack and Schwarz in

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the 1940s and 1950s. Nehari proved in [1] that if $A \in \mathcal{H}(\mathbb{D})$ and

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 \tag{1.1}$$

is at most one, then each non-trivial solution of

$$f'' + A(z)f = 0 \tag{1.2}$$

has at most one zero in $\mathbb{D}$.

In 1955, Schwarz [2] showed that if $A \in \mathcal{H}(\mathbb{D})$, then zero-sequences of all nontrivial solutions of (1.2) are separated in the hyperbolic metric if and only if (1.1) is finite.

For recent developments based on localization of the classical results [3]. In the case of higher order linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \quad k \in \mathbb{N}, \tag{1.3}$$

with coefficients $A_j \in \mathcal{H}(\mathbb{D})$, $j = 0, \ldots, k-1$, there are few results.

Let $\varphi_a(z) = (a-z)/(1-\overline{a}z)$, for $a, z \in \mathbb{D}$, denote a conformal automorphism of $\mathbb{D}$ which coincides with its own inverse. Moreover, let $d\sigma_z$ denote the element of the Lebesgue area measure on $\mathbb{D}$.

Very recently, Gröhn etc [4] studied the zero distribution of nontrivial solutions of the linear differential equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0, \tag{1.4}$$

where $A_0, A_1, A_2 \in \mathbb{D}$. They obtained the following result.

Theorem 1.1 ([4]) *Let f be a nontrivial solution of (1.4), where $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$.*

(i) If

$$\sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|^2)^{3-j} < \infty, \quad j = 0, 1, 2,$$

then the sequence of two-fold zeros of f is a finite union of separated sequences.

(ii) If

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1-|z|^2)^{1-j}(1-|\varphi_a(z)|^2)\mathrm{d}\sigma_z < \infty, \quad j = 0, 1, 2,$$

then the sequence of two-fold zeros of f is a finite union of uniformly separated sequences.

Theorem 1.1 is a generalization of the second order case [2]. The proof of Theorem 1.1 is based on a conformal transformation of (1.4), Jensen's formula, and on a sharp growth estimate for solutions of (1.4). A natural question is: what can we say about the nonhomogeneous equation associated to (1.4)

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = A_3(z). \tag{1.5}$$

One purpose of this study is to establish interrelationships between the growth of coefficients and the geometric distribution of zeros of solutions of (1.5). For $0 < p < \infty$, the Ber-type space, denoted by H_p^∞, consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^p < \infty.$$

We obtain the following result.

Theorem 1.2 *Let f be a nontrivial solution of (1.5), where $A_0, A_1, A_2, A_3 \in \mathcal{H}(\mathbb{D})$.*

(i) If

$$\|A_j\|_{H^\infty_{3-j}} = \sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|^2)^{3-j} < \infty, \quad j = 0, 1, 2, \tag{1.6}$$

and

$$\|A_3\|_{H^\infty_3} = \sup_{z\in\mathbb{D}} |A_3(z)|(1-|z|^2)^3 < \infty, \tag{1.7}$$

then the sequence of two-fold zeros of f is a finite union of separated sequences.

(ii) If

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1-|z|^2)^{1-j}(1-|\varphi_a(z)|^2)\mathrm{d}\sigma_z < \infty, \quad j = 0, 1, 2, \tag{1.8}$$

and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_3(z)|(1-|z|^2)(1-|\varphi_a(z)|^2)\mathrm{d}\sigma_z < \infty, \tag{1.9}$$

then the sequence of two-fold zeros of f is a finite union of uniformly separated sequences.

Nevanlinna theory has been applied for fast-growing analytic solutions [5–11], but the analysis on slowly growing solutions seems to require a different approach. An important breakthrough in this regard was [12], where Pommerenke obtained a sharp sufficient condition for the coefficient A which places all solutions f of (1.2) to the classical Hardy space H^2.

Recently, Heittokangas et al. [13] studied equation (1.3) and found sufficient conditions for the analytic coefficients such that all solutions belong to H^∞_p.

Theorem 1.3 ([13]) *Let $0 \le \delta < 1$. For every $p > 0$ there exists a positive constant α, depending only on p and k, such that if the coefficients $A_j(z)$ of (1.1) are analytic in $\mathbb{D}$ and satisfy*

$$\sup_{|z|\ge\delta} |A_j(z)|(1-|z|^2)^{k-j} \le \alpha, \quad j = 0, \dots, k-1,$$

then all solutions of (1.3) belong to H^∞_p.

Sufficient conditions for the coefficients such that all solutions belong to $\mathcal{D}^p$ were found in [14]. For $0 < p < \infty$, the Dirichlet-type space $\mathcal{D}^p$ consists of those analytic functions f in $\mathbb{D}$ for which the integral

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-1}\mathrm{d}\sigma_z$$

converges.

Theorem 1.4 ([14]) *Let $0 \le \delta < 1$. For every $0 < p \le 2$, there is a positive constant α, depending only on p and k, such that if the coefficients $A_j(z)$ of (1.3) are analytic in $\mathbb{D}$ and satisfy*

$$\sup_{|a|\ge\delta} \int_{\mathbb{D}} |A_0(z)|^p (1-|z|^2)^{pk-1} \frac{1-|a|^2}{|1-\overline{a}z|^2}\mathrm{d}\sigma_z \le \alpha$$

and

$$\sup_{|z|\ge\delta} |A_j(z)|(1-|z|^2)^{k-j} \le \alpha, \quad j = 1, \dots, k-1,$$

then all solutions of (1.3) belong to $\mathcal{D}^p \cap H_p^\infty$.

As for the solutions in other function spaces, see, for example [15, 16].

Another purpose of this study is to give some sufficient conditions on the analytic coefficients of (1.3) for all solutions to belong to the Zygmund type space. For $0 < \alpha < \infty$, we denote by $\mathcal{Z}^\alpha$ the Zygmund type space of those functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)| < \infty$$

equipped with the norm $\|f\|_{\mathcal{Z}^\alpha} := |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)|$. The little Zygmund type space, denoted by $\mathcal{Z}_o^\alpha$, is the closed subspace of $\mathcal{Z}^\alpha$ consisting of those functions $f \in \mathcal{Z}^\alpha$ with

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f''(z)| = 0.$$

When $\alpha = 1$, we get the classical Zygmund spaces $\mathcal{Z}$ and $\mathcal{Z}_o$. To the best of our knowledge, solutions of differential equations in the Zygmund type space are considered in the present paper for the first time. We obtain the following results.

Theorem 1.5 *Let $A_0(z), \dots, A_{k-1}(z)$ be the analytic coefficients of (1.3) in $\mathbb{D}$. Let $n_c \in \{1, \dots, k\}$ be the number of nonzero coefficients $A_j(z)$, $j = 0, \dots, k-1$. Fix $l > 0$ and denote $R = (e^{1/l} - 1)/e^{1/l}$. Suppose that*

$$\int_R^1 (\log(1-t)^{-l})^{-1} \sum_{j=0}^{k-1} |A_j(te^{i\theta})|^{\frac{1}{k-j}} \,\mathrm{d}t \le \frac{1}{n_c}. \tag{1.10}$$

Then all solutions of (1.3) belong to the space $\mathcal{Z}^{l+2}$.

Theorem 1.6 *Let $A_0(z), \dots, A_{k-1}(z)$ be the analytic coefficients of (1.3) in $\mathbb{D}$. Assume that*

$$|A_j(z)| \le \frac{\alpha_j}{(1-|z|)^{k-j-1}}, \tag{1.11}$$

where $\alpha_j > 0$. Then all solutions of (1.3) belong to the space $\mathcal{Z}^{2-\frac{2}{k}}$.

Theorem 1.7 *Let $A_0(z), \dots, A_{k-1}(z)$ be the analytic coefficients of (1.3) in $\mathbb{D}$. Assume that*

$$|A_j(z)| \le \alpha_j \log\frac{e}{1-|z|}, \tag{1.12}$$

where $\alpha_j > 0$. Then all solutions of (1.3) belong to the space $\bigcap_{0<\alpha<\infty} \mathcal{Z}_o^\alpha$.

Corollary 1.8 *Let all the coefficients $A_j(z)$ $(j = 0, \dots, k-1)$ of (1.3) belong to the Zygmund type space $\mathcal{Z}^2$. Then all solutions of (1.3) belong to the space $\bigcap_{0<\alpha<\infty} \mathcal{Z}_o^\alpha$.*

2. Auxiliary lemmas

The following lemma gives an estimate for the number of sequences in the finite union appearing in the statement of Theorem 1.1.

Lemma 2.1 ([4]) *Let $\mathcal{L} = \{z_k\}$ be a sequence of points in $\mathbb{D}$ such that the multiplicity of each point is at most $p \in \mathbb{N}$, and let $M \in \mathbb{N}$.*

(i) If

$$\sup_{a\in\mathcal{L}} \sum_{z_k\in\mathcal{L}\setminus\{a\}} (1-|\varphi_a(z_k)|^2)^2 \le M < \infty,$$

then $\{z_k\}$ *can be expressed as a finite union of at most* $M+p$ *separated sequences.*

(ii) If

$$\sup_{a\in\mathcal{L}} \sum_{z_k\in\mathcal{L}\setminus\{a\}} (1-|\varphi_a(z_k)|^2) \le M < \infty,$$

then $\{z_k\}$ *can be expressed as a finite union of at most* $M+p$ *uniformly separated sequences.*

For $0<p<\infty$ and $-1<\alpha<\infty$, the weighted Bergman space A^p_α consists of those functions f, analytic in $\mathbb{D}$, for which

$$\|f\|_{A^p_\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha \mathrm{d}\sigma_z\Big)^{\frac{1}{p}} < \infty.$$

The classical Bergman space A^p is A^p_0. See [17] and [18] for the theory of Bergman spaces. It is well known that an analytic function f belongs to the Bergman space A^p_α if and only if $f^{(n)}$ belongs to $A^p_{np+\alpha}$. This fact follows by Lemma 2.2, which can be found, for example, in [19].

Lemma 2.2 ([19]) *Let* f *be an analytic function in* $\mathbb{D}$*, and let* $0<p<\infty$, $-1<\alpha<\infty$ *and* $n\in\mathbb{N}$. *Then there exist two constants* $C_1>0$ *and* $C_2>0$, *depending only on* p,α *and* n, *such that*

$$C_1\|f\|_{A^p_\alpha} \le \|f^{(n)}\|_{A^p_{np+\alpha}} + \sum_{j=0}^{n-1} |f^{(j)}(0)| \le C_2\|f\|_{A^p_\alpha}.$$

Lemma 2.3 ([20]) *Suppose* $z\in\mathbb{D}, c$ *is real,* $t>-1$, *and*

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\overline{w}|^{2+t+c}} \mathrm{d}\sigma_w.$$

(a) If $c<0$, *then as a function of* z, $I_{c,t}(z)$ *is bounded from above and bounded from below on* $\mathbb{D}$.

(b) If $c>0$, *then* $I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c}$, $|z|\to 1^-$.

(c) If $c=0$, *then* $I_{0,t}(z) \sim \log\frac{1}{1-|z|^2}$, $|z|\to 1^-$.

Lemma 2.4 ([21]) *If* $\alpha>1$ *and* $R=\frac{1+r}{2}$, *then*

$$\int_0^{2\pi} \frac{\mathrm{d}\varphi}{|Re^{i\varphi}-r|^\alpha} = O(\frac{1}{(1-r)^{\alpha-1}}).$$

Lemma 2.5 ([22]) *Let* f *be a solution of (1.3) in* $\mathbb{D}_R$, *where* $0<R\le\infty$, *let* $n_c\in\{1,\ldots,k\}$ *be the number of nonzero coefficients* $A_j(z)$, $j=0,\ldots,k-1$, *and let* $\theta\in[0,2\pi)$ *and* $\varepsilon>0$. *If* $z_\theta=\nu e^{i\theta}\in\mathbb{D}_R$ *is such that* $A_j(z_\theta)\neq 0$ *for some* $j=0,\ldots,k-1$, *then, for all* $\nu<r<R$,

$$|f(re^{i\theta})| \le C\exp\Big(n_c\int_\nu^r \max_{j=0,\ldots,k-1} |A_j(te^{i\theta})|^{1/(k-j)}\mathrm{d}t\Big), \tag{2.1}$$

where $C>0$ *is a constant satisfying*

$$C \le (1+\varepsilon)\max_{j=0,\ldots,k-1}\Big(\frac{|f^{(j)}(z_\theta)|}{(n_c)^j \max_{n=0,\ldots,k-1}|A_n(z_\theta)|^{j/(k-n)}}\Big).$$

An application of Herold's comparison Theorem in its full generality at the end of proof of Lemma 2.5 yields the following pointwise growth estimate for the derivatives of solutions of (1.3). It is also a special case of a more general result in [23].

Lemma 2.6 ([23]) *Suppose the assumptions in Lemma 2.5 hold. Then, for all $r \in (\nu, R)$ and $j = 0, \ldots, k-1$,*

$$\begin{aligned}|f^{(j)}(re^{i\theta})| \le & C\Big(\sup_{\nu\le x\le \frac{1+r}{2}}(n_c \max_{j=0,\ldots,k-1}|A_j(xe^{i\theta})|^{\frac{1}{k-j}})\Big)^j \times \\ & \exp\Big(n_c\int_\nu^r \max_{j=0,\ldots,k-1}|A_j(te^{i\theta})|^{\frac{1}{k-j}}\mathrm{d}t\Big).\end{aligned}$$

3. Proofs of Theorems

Theorem 1.2 can be verified by the following proof of Theorem 1 in [4] with suitable modifications.

Proof of Theorem 1.2 (i) If f is a nontrivial solution of (1.5), then $g = f\circ\varphi_a$ solves

$$g''' + B_2 g'' + B_1 g' + B_0 g = B_3, \tag{3.1}$$

where

$$\begin{aligned}&B_0 = (A_0\circ\varphi_a)(\varphi_a')^3,\\ &B_1 = (A_1\circ\varphi_a)(\varphi_a')^2 - (A_2\circ\varphi_a)\varphi_a'' + 3(\frac{\varphi_a''}{\varphi_a'})^2 - \frac{\varphi_a'''}{\varphi_a'},\\ &B_2 = (A_2\circ\varphi_a)\varphi_a' - 3\frac{\varphi_a''}{\varphi_a'},\quad B_3 = (A_3\circ\varphi_a)(\varphi_a')^3.\end{aligned}\tag{3.2}$$

Thus by the Schwarz-Pick lemma [24, Lemma 1.2], $1-|\varphi_a(z)|^2 = (1-|z|^2)|\varphi_a'(z)|$, we can obtain

$$\begin{aligned}\|B_0\|_{H_3^\infty} &= \sup_{z\in\mathbb{D}}|B_0(z)|(1-|z|^2)^3 = \sup_{z\in\mathbb{D}}|A_0\circ\varphi_a(z)||\varphi'(z)|^3(1-|z|^2)^3\\ &= \sup_{z\in\mathbb{D}}|A_0\circ\varphi_a(z)|(1-|\varphi_a(z)|^2)^3 = \sup_{w\in\mathbb{D}}|A_0(w)|(1-|w|^2)^3 = \|A_0\|_{H_3^\infty}.\end{aligned}$$

Similarly, we have $\|B_3\|_{H_3^\infty} = \|A_3\|_{H_3^\infty}$. Straightforward calculations show that

$$\frac{\varphi_a''(z)}{\varphi_a'(z)} = \frac{2\overline{a}}{1-\overline{a}z},\quad \frac{\varphi_a'''(z)}{\varphi_a'(z)} = \frac{6\overline{a}^2}{(1-\overline{a}z)^2}.$$

Therefore, we have

$$|\frac{\varphi_a''(z)}{\varphi_a'(z)}(1-|z|^2)| \le 4,\quad |\frac{\varphi_a'''(z)}{\varphi_a'(z)}(1-|z|^2)^2| \le 24.$$

An application of above estimates, yields,

$$\begin{aligned}\|B_1\|_{H_2^\infty} &\le \sup_{w\in\mathbb{D}}|A_1(w)|(1-|w|^2)^2 + 4\sup_{w\in\mathbb{D}}|A_2(w)|(1-|w|^2) + 48 + 24\\ &\le \|A_1\|_{H_2^\infty} + 4\|A_2\|_{H_1^\infty} + 72\end{aligned}$$

and

$$\|B_2\|_{H_1^\infty} \le \sup_{w\in\mathbb{D}}|A_2(w)|(1-|w|^2) + 12 = \|A_2\|_{H_1^\infty} + 12.$$

Let $\mathcal{L} = \mathcal{L}(f)$ be the sequence of two-fold zeros of f, and let $a \in \mathcal{L}$; we may assume that $\mathcal{L}$ is not empty, for otherwise there is nothing to prove. Then 0 is a two-fold zero of $g = f \circ \varphi_a$. By applying Jensen's formula to $g(z)/z^2$, we obtain

$$\sum_{\substack{z_k \in \mathcal{L} \\ 0<|\varphi_a(z_k)|<r}} \log \frac{r}{|\varphi_a(z_k)|} \le \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| \mathrm{d}\theta - \log |g''(0)| + \log \frac{2}{r^2}, \tag{3.3}$$

where $0 < r < 1$, $\log^+ x = \max\{0, \log x\}$ for $0 \le x < \infty$. Since

$$\begin{aligned}
&\int_0^1 \Big(\sum_{\substack{z_k \in \mathcal{L} \\ 0<|\varphi_a(z_k)|<r}} \log \frac{r}{|\varphi_a(z_k)|} \Big) r \mathrm{d}r = \sum_{z_k \in \mathcal{L}\setminus\{a\}} \int_{|\varphi_a(z_k)|}^1 r \log \frac{r}{|\varphi_a(z_k)|} \mathrm{d}r \\
&= \frac{1}{4} \sum_{z_k \in \mathcal{L}\setminus\{a\}} [2 \log \frac{1}{|\varphi_a(z_k)|} - (1 - |\varphi_a(z_k)|^2)] \\
&\ge \frac{1}{4} \sum_{z_k \in \mathcal{L}\setminus\{a\}} [2(1 - |\varphi_a(z_k)|) - (1 - |\varphi_a(z_k)|^2)] \\
&= \frac{1}{4} \sum_{z_k \in \mathcal{L}\setminus\{a\}} (1 - |\varphi_a(z_k)|)^2 \ge \frac{1}{16} \sum_{z_k \in \mathcal{L}\setminus\{a\}} (1 - |\varphi_a(z_k)|^2)^2,
\end{aligned}$$

the estimate (3.3) implies

$$\sum_{z_k \in \mathcal{L}\setminus\{a\}} (1 - |\varphi_a(z_k)|^2)^2 \le \frac{8}{\pi} \int_{\mathbb{D}} \log^+ |g(z)| \mathrm{d}\sigma_z - 8 \log |g''(0)| + 8 \log 2 + 8. \tag{3.4}$$

Recall that $g(z)$ is a solution of (3.1). By the proof of the growth estimates [25, Corollary 3], there exists an absolute constant $C_1 > 0$ such that

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| \mathrm{d}\theta \le & C_1 \Big(\int_0^{2\pi} \log^+ \int_0^r |B_3(se^{i\theta})|(1-s)^2 \mathrm{d}s\mathrm{d}\theta + \\
& \sum_{j=0}^{2} \sum_{n=0}^{j} \int_0^{2\pi} \int_0^r |B_j^{(n)}(se^{i\theta})|(1-s)^{2-j+n} \mathrm{d}s\mathrm{d}\theta \Big).
\end{aligned} \tag{3.5}$$

Since $B_j \in H^\infty_{3-j}$ for $j = 0, 1, 2$, we have $B_j^{(n)} \in H^\infty_{3-j+n}$ for $j = 0, 1, 2, n = 0, \ldots, j$ by Cauchy's integral formula. Hence there exists a positive constant $C_2 = C_2(\|A_0\|_{H_3^\infty}, \|A_1\|_{H_2^\infty}, \|A_2\|_{H_1^\infty})$, independent of $a \in \mathbb{D}$, such that

$$|B_j^{(n)}(se^{i\theta})|(1-s)^{3-j+n} \le C_2, \quad j = 0, 1, 2, \; n = 0, \ldots, j, \tag{3.6}$$

for $se^{i\theta} \in \mathbb{D}$.

Now we estimate $\int_0^{2\pi} \log^+ \int_0^r |B_3(se^{i\theta})|(1-s)^2 \mathrm{d}s\mathrm{d}\theta$. Since $B_3 \in H_3^\infty$, there exists a constant $C_3 = C_3(\|A_3\|_{H_3^\infty}) > 1$, independent of $a \in \mathbb{D}$, such that $|B_3(se^{i\theta})|(1-s)^2 \le \frac{C_3}{1-s}$ for $se^{i\theta} \in \mathbb{D}$, which implies

$$\begin{aligned}
&\int_0^{2\pi} \log^+ \int_0^r |B_3(se^{i\theta})|(1-s)^2 \mathrm{d}s\mathrm{d}\theta \le \int_0^{2\pi} \log^+ \int_0^r \frac{C_3}{1-s} \mathrm{d}s\mathrm{d}\theta \\
&\le 2\pi \log^+ \frac{C_3}{1-r} = 2\pi \log \frac{C_3}{1-r}.
\end{aligned} \tag{3.7}$$

Substituting (3.5)-(3.7) into (3.4), we obtain

$$
\begin{aligned}
&\sup_{a\in\mathcal{L}}\sum_{z_k\in\mathcal{L}\setminus\{a\}}(1-|\varphi_a(z_k)|^2)^2\\
&\quad\le 16C_1\Big(2\pi\int_0^1 r\log\frac{C_3}{1-r}\mathrm{d}r+2\pi\sum_{j=0}^{2}\sum_{n=0}^{j}\int_0^1\int_0^r r\frac{C_2}{1-s}\mathrm{d}s\mathrm{d}r\Big)-8\log|g''(0)|+8\log 2+8\\
&\quad\le 32\pi C_1\Big(\int_0^1\log\frac{C_3}{1-r}\mathrm{d}r+6C_2\int_0^1\log\frac{1}{1-r}\mathrm{d}r\Big)-8\log|g''(0)|+8\log 2+8\\
&\quad=32\pi C_1(\log C_3+1+6C_2)-8\log|g''(0)|+8\log 2+8<\infty.
\end{aligned}
$$

This implies the assertion of Theorem 1.2 (i) by Lemma 2.1 (i).

(ii) As in the proof of (i), we conclude that $g=f\circ\varphi_a$ is a solution of (3.1), where the coefficients B_0,B_1,B_2 and B_3 are defined by formula (3.2). Since

$$
\begin{aligned}
\int_{\mathbb{D}}|B_3(z)(1-|z|^2)^2|\mathrm{d}\sigma_z&=\int_{\mathbb{D}}|A_3(\varphi_a(z))||\varphi_a'(z)|^3(1-|z|^2)^2\mathrm{d}\sigma_z\\
&=\int_{\mathbb{D}}|A_3(w)|(1-|w|^2)(1-|\varphi_a(w)|^2)\mathrm{d}\sigma_w,
\end{aligned}
$$

it follows from (1.9) that

$$
\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|B_3(z)|(1-|z|^2)^2\mathrm{d}\sigma_z<\infty. \tag{3.8}
$$

Next we will show that

$$
\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|B_j^{(n)}(z)|(1-|z|^2)^{2-j+n}\mathrm{d}\sigma_z<\infty,\quad j=0,1,2,n=0,\dots,j. \tag{3.9}
$$

According to Lemma 2.2, we need only to show that

$$
\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|B_j(z)|(1-|z|^2)^{2-j}\mathrm{d}\sigma_z<\infty,\quad j=0,1,2.
$$

As a matter of fact, first, since

$$
\begin{aligned}
\int_{\mathbb{D}}|B_0(z)|(1-|z|^2)^2\mathrm{d}\sigma_z&=\int_{\mathbb{D}}|A_0(\varphi_a(z))||\varphi_a'(z)|^3(1-|z|^2)^2\mathrm{d}\sigma_z\\
&=\int_{\mathbb{D}}|A_0(w)|(1-|w|^2)(1-|\varphi_a(w)|^2)\mathrm{d}\sigma_w,
\end{aligned}
$$

we can get

$$
\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|B_0(z)|(1-|z|)^2\mathrm{d}\sigma_z<\infty
$$

from (1.8).

Next, we can obtain from (3.2) and Lemma 2.3 that

$$
\begin{aligned}
&\int_{\mathbb{D}}|B_1(z)|(1-|z|^2)\mathrm{d}\sigma_z\\
&\quad\le\int_{\mathbb{D}}|A_1(\varphi_a(z))||\varphi_a'(z)|^2(1-|z|^2)\mathrm{d}\sigma_z+\int_{\mathbb{D}}|A_2(\varphi_a(z))||\varphi_a''(z)|(1-|z|^2)\mathrm{d}\sigma_z+\\
&\qquad 3\int_{\mathbb{D}}|\frac{\varphi_a''(z)}{\varphi_a'(z)}|^2(1-|z|^2)\mathrm{d}\sigma_z+\int_{\mathbb{D}}|\frac{\varphi_a'''(z)}{\varphi_a'(z)}|(1-|z|^2)\mathrm{d}\sigma_z
\end{aligned}
$$

$$\leq \int_{\mathbb{D}} |A_1(w)|(1-|\varphi_a(w)|^2)\mathrm{d}\sigma_w + 4\int_{\mathbb{D}} |A_2(\varphi_a(z))|\frac{1}{|\varphi_a'(z)|}|\varphi_a'(z)|^2\mathrm{d}\sigma_z +$$
$$3\int_{\mathbb{D}} \frac{4|a|^2}{|1-\overline{a}z|^2}(1-|z|^2)\mathrm{d}\sigma_z + \int_{\mathbb{D}} \frac{6|a|^2}{|1-\overline{a}z|^2}(1-|z|^2)\mathrm{d}\sigma_z$$
$$\leq \int_{\mathbb{D}} |A_1(w)|(1-|\varphi_a(w)|^2)\mathrm{d}\sigma_w + 4\int_{D} |A_2(w)|\frac{1-|\varphi_a(w)|^2}{1-|w|^2}\mathrm{d}\sigma_w + O(1).$$

The condition (1.8) implies

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B_1(z)|(1-|z|^2)\mathrm{d}\sigma_z < \infty.$$

Finally, from(1.8) and Lemma 2.3, we can deduce that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B_2(z)|\mathrm{d}\sigma_z \leq \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_2(\varphi_a(z))||\varphi_a'(z)|\mathrm{d}\sigma_z + 3\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\frac{\varphi_a''(z)}{\varphi_a'(z)}|\mathrm{d}\sigma_z$$
$$= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_2(w)|\frac{1-|\varphi_a(w)|^2}{1-|w|^2}\mathrm{d}\sigma_w + 3\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{2|a|}{|1-\overline{a}z|}\mathrm{d}\sigma_z < \infty.$$

Let $\mathcal{L}$ be the sequence of two-fold zeros of f. As above, by applying the proof of [26, Lemma 4.6], there exists an absolute constant $C_2 > 0$, such that

$$\sum_{\substack{z_k\in\mathcal{L}\\ 0<|\varphi_a(z_k)|<r}} \log\frac{r}{|\varphi_a(z_k)|} \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+|g(re^{i\theta})|\mathrm{d}\theta - \log|g''(0)| + \log\frac{2}{r^2}$$
$$\leq C_1\Big(\int_0^{2\pi} \log^+ \int_0^r |B_3(se^{i\theta})|(1-s)^2\mathrm{d}s\mathrm{d}\theta +$$
$$\sum_{j=1}^{2}\sum_{n=0}^{j} \int_0^{2\pi}\int_0^r |B_j^{(n)}(se^{i\theta})|1-s)^{2-j+n}\mathrm{d}s\mathrm{d}\theta\Big) - \log|g''(0)| + \log\frac{2}{r^2}$$
$$\leq C_1\Big(\int_0^{2\pi}\int_0^r |B_3(se^{i\theta})|(1-s)^2\mathrm{d}s\mathrm{d}\theta +$$
$$\sum_{j=1}^{2}\sum_{n=0}^{j} \int_0^{2\pi}\int_0^r |B_j^{(n)}(se^{i\theta})|1-s)^{2-j+n}\mathrm{d}s\mathrm{d}\theta\Big) - \log|g''(0)| + \log\frac{2}{r^2}$$
$$\leq C_2\Big(\int_{\mathbb{D}} |B_3(z)|(1-|z|^2)^2\mathrm{d}\sigma_z + \sum_{j=1}^{2}\sum_{n=0}^{j}\int_{\mathbb{D}} |B_j^{(n)}(z)|1-|z|^2)^{2-j+n}\mathrm{d}\sigma_z\Big) -$$
$$\log|g''(0)| + \log\frac{2}{r^2}.$$

Using the estimates (3.8) and (3.9), and letting $r \to 1^-$, we get

$$\sup_{a\in\mathcal{L}} \sum_{z_k\in\mathcal{L}\setminus\{a\}} (1-|\varphi_a(z_k)|^2) < \infty.$$

Consequently, the assertion of Theorem 1.2 (ii) follows from Lemma 2.1 (ii). □

Proof of Theorem 1.5 If $r \in (R, 1)$, it follows from (1.10) that

$$(\log(1-r)^{-l})^{-1}\int_R^r \sum_{j=0}^{k-1} |A_j(te^{i\theta})|^{\frac{1}{k-j}}\mathrm{d}t$$

$$\leq \int_R^r (\log(1-t)^{-l})^{-1} \sum_{j=0}^{k-1} |A_j(te^{i\theta})|^{\frac{1}{k-j}}\,\mathrm{d}t \leq \frac{1}{n_c},$$

so,

$$n_c \int_R^r \sum_{j=0}^{k-1} |A_j(te^{i\theta})|^{\frac{1}{k-j}}\,\mathrm{d}t \leq \log\frac{1}{(1-r)^l}. \tag{3.10}$$

Let f be a non-constant solution of (1.3). By the Cauchy integral formula for derivatives and Lemma 2.4, we have

$$|f''(re^{i\varphi})| \leq \frac{1}{\pi}\int_{|\zeta|=\frac{1+r}{2}} \frac{|f(\zeta)|}{|\zeta - re^{i\varphi}|^3}|\mathrm{d}\zeta| = O(\frac{M(\frac{1+r}{2}, f)}{(1-r)^2}).$$

Now an application of (3.10) and (2.1) yields,

$$|f''(re^{i\varphi})| = O(\frac{1}{(1-r)^{l+2}}),$$

that is, $f \in \mathcal{Z}^{l+2}$. $\square$

Proof of Theorem 1.6 Let f be a non-constant solution of (1.3). By Lemma 2.6,

$$\begin{aligned}|f''(re^{i\theta})| \leq & C\Big(\sup_{\nu\leq x\leq\frac{1+r}{2}}(n_c \max_{j=0,\dots,k-1}|A_j(xe^{i\theta})|^{\frac{1}{k-j}})\Big)^2 \times \\ & \exp\Big(n_c\int_\nu^r \max_{j=0,\dots,k-1}|A_j(te^{i\theta})|^{\frac{1}{k-j}}\,\mathrm{d}t\Big),\end{aligned}$$

for all $r \in (\nu, R)$. Here in after we use C to denote a positive constant which need not be the same at each occurrence. Then (1.11) gives

$$\begin{aligned}|f''(re^{i\theta})| \leq & C\Big(\sup_{\nu\leq x\leq\frac{1+r}{2}}(\max_{j=0,\dots,k-1}(\frac{1}{1-x})^{1-\frac{1}{k-j}})\Big)^2 \times \\ & \exp\Big(C\int_\nu^r \max_{j=0,\dots,k-1}(\frac{1}{1-t})^{1-\frac{1}{k-j}}\,\mathrm{d}t\Big) \\ \leq & C\Big(\sup_{\nu\leq x\leq\frac{1+r}{2}}(\frac{1}{1-x})^{1-\frac{1}{k}}\Big)^2 \exp\Big(C\int_0^r (\frac{1}{1-t})^{1-\frac{1}{k}}\,\mathrm{d}t\Big) \\ \leq & C(\frac{2}{1-r})^{2-\frac{2}{k}} \exp(Ck(1-(1-r)^{\frac{1}{k}})) \leq C(\frac{1}{1-r})^{2-\frac{2}{k}}.\end{aligned}$$

It follows that

$$\sup_{z\in\mathbb{D}}|f''(z)|(1-|z|^2)^{2-\frac{2}{k}} < \infty,$$

which implies that $f \in \mathcal{Z}^{2-\frac{2}{k}}$. $\square$

Proof of Theorem 1.7 Let f be a nonconstant solution of (1.3). An application of Lemma 2.6 yields,

$$\begin{aligned}|f''(re^{i\theta})| \leq & C\Big(\sup_{\nu\leq x\leq\frac{1+r}{2}}(n_c \max_{j=0,\dots,k-1}|A_j(xe^{i\theta})|^{\frac{1}{k-j}})\Big)^2 \times \\ & \exp(n_c\int_\nu^r \max_{j=0,\dots,k-1}|A_j(te^{i\theta})|^{\frac{1}{k-j}}\,\mathrm{d}t),\end{aligned}$$

for all $r \in (\nu, R)$. By (1.12), we can obtain

$$\begin{aligned}|f''(re^{i\theta})| \le & C\Big(\sup_{\nu\le x\le \frac{1+r}{2}} \max_{j=0,\ldots,k-1}(\log \frac{e}{1-x})^{\frac{1}{k-j}}\Big)^2 \times \\ & \exp\Big(C\int_0^r \max_{j=0,\ldots,k-1}(\log\frac{e}{1-t})^{\frac{1}{k-j}}\mathrm{d}t\Big) \\ \le & C\Big(\sup_{\nu\le x\le \frac{1+r}{2}}(\log\frac{e}{1-x})\Big)^2 \exp\Big(C\int_0^r \log\frac{e}{1-t}\mathrm{d}t\Big) \\ \le & C(\log\frac{2e}{1-r})^2 \exp\Big(C\int_0^r \log\frac{e}{1-t}\mathrm{d}t\Big) \\ \le & C(\log\frac{1}{1-r})^2. \end{aligned} \tag{3.11}$$

Taking any $\alpha \in (0, +\infty)$, multiplying $(1-r^2)^\alpha$ on both sides of (3.11) and letting $r \to 1$, we obtain

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f''(z)| = \lim_{r\to 1}(1-r^2)^\alpha |f''(e^{i\theta})| = 0,$$

which implies that $f \in \mathcal{Z}_o^\alpha$. Since $\alpha \in (0, +\infty)$ is arbitrary, the conclusion follows. □

Proof of Corollary 1.8 From $A_j(z) \in \mathcal{Z}^2$, we can get

$$|A_j''(z)| \le \frac{\|A_j\|_{\mathcal{Z}^2}}{(1-|z|^2)^2}, \quad j = 0, \ldots, k-1,$$

which yields

$$\begin{aligned}|A_j'(z) - A_j'(0)| = & \Big|z\int_0^r A_j''(zt)\mathrm{d}t\Big| \le |z|\|A_j\|_{\mathcal{Z}^2}\int_0^1 \frac{1}{(1-|z|^2t^2)^2}\mathrm{d}t \\ & \le |z|\|A_j\|_{\mathcal{Z}^2}\int_0^1 \frac{1}{(1-|z|t)^2}\mathrm{d}t = \|A_j\|_{\mathcal{Z}^2}\frac{|z|}{1-|z|} \\ & \le \|A_j\|_{\mathcal{Z}^2}\frac{1}{1-|z|}.\end{aligned}$$

This gives

$$|A_j'(z)| \le |A_j'(0)| + \|A_j\|_{\mathcal{Z}^2}\frac{1}{1-|z|} \le 2\|A_j\|_{\mathcal{Z}^2}\frac{1}{1-|z|}.$$

Using the same procedure as above, we have

$$\begin{aligned}|A_j(z) - A_j(0)| = & \Big|z\int_0^r A_j'(zt)\mathrm{d}t\Big| \le 2|z|\|A_j\|_{\mathcal{Z}^2}\int_0^1 \frac{1}{(1-|z|t)}\mathrm{d}t \\ & = 2\|A_j\|_{\mathcal{Z}^2}\log\frac{1}{1-|z|},\end{aligned}$$

which implies

$$|A_j(z)| \le |A_j(0)| + 2\|A_j\|_{\mathcal{Z}^2}\log\frac{1}{1-|z|} \le 2\|A_j\|_{\mathcal{Z}^2}\log\frac{e}{1-|z|}.$$

Therefore, all solutions of (1.3) belong to space $\bigcap_{0<\alpha<\infty}\mathcal{Z}_o^\alpha$ by Theorem 1.7. □

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