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Boundedness of the Intrinsic Square Function on Grand Variable Herz Spaces

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Abstract We prove that the intrinsic square function and its commutator are bounded on the Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$, where the exponents α and q are variable. The results are still new even when $\alpha(\cdot) \equiv \alpha$ is constant.

Keywords the intrinsic square function; commutator; variable exponent; grand space; Herz space

MR(2020) Subject Classification 42B20; 42B35

1. Introduction

For $0 < \beta \leq 1$, let $\mathcal{C}_\beta$ be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi's$ support is contained in $\{x \in \mathbb{R}^n : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 0$ and for all $x_1, x_2 \in \mathbb{R}^n$,

$$|\phi(x_1) - \phi(x_2)| \leq |x_1 - x_2|^\beta. \tag{1.1}$$

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$, we set

$$A_\beta(f)(y,t) = \sup_{\phi\in\mathcal{C}_\beta} |f * \phi_t(y)| = \sup_{\phi\in\mathcal{C}_\beta} \Big| \int_{\mathbb{R}^n} \phi_t(y-z) f(z)\mathrm{d}z \Big|. \tag{1.2}$$

Then the intrinsic square function of f (of order β) is defined by

$$S_\beta(f)(x) = \Big(\int\int_{\Gamma(x)} (A_\beta(f)(y,t))^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \Big)^{\frac{1}{2}}, \tag{1.3}$$

where $\phi_t(x) = \frac{1}{t^n}\phi(\frac{x}{t})$ and $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

The intrinsic square function $S_\beta(f)$ was first introduced by Wilson in [1] to solve a conjecture proposed by Fefferman and Stein [2] on the boundedness of the Lusin area function from the weighted Lebesgue space $L^2_{M(\nu)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_\nu(\mathbb{R}^n)$, where ν is a non-negative, locally integrable function and M denotes the Hardy-Littlewood maximal operator. In [3], Wilson proved that if $\omega \in A_p$ (the class of Muckenhoupt weights), then the intrinsic square function $S_\beta(f)$ is bounded on $L^p_\omega(\mathbb{R}^n)$ for any $1 < p < \infty$. Moreover, Lerner [4] showed sharp $L^p_\omega(\mathbb{R}^n)$ norm inequalities for the $S_\beta(f)$ in terms of the A_p characteristic constant of ω

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for all $1 < p < \infty$. As for the boundedness of such intrinsic square function on function spaces including weighted Morrey spaces, weighted Hardy spaces and weighted Herz spaces, we refer to [5–7] and their references.

A function $b \in L^1_{\rm loc}(\mathbb{R}^n)$ is said to belong to space $\mathrm{BMO}(\mathbb{R}^n)$, if it satisfies

$$\|b\|_* := \sup_B \frac{1}{|B|}\int_B |b(y) - b_B|\mathrm{d}y < \infty,$$

where B denotes the ball centered at $x \in \mathbb{R}^n$ and radius of $r > 0$, and b_B denotes the average of b on B, i.e., $b_B := |B|^{-1}\int_B b(t)\mathrm{d}t$. For $b \in \mathrm{BMO}(\mathbb{R}^n)$, $[b, S_\beta]$, the commutator of the intrinsic square function, is then defined by

$$[b, S_\beta](f)(x) = \Big(\int\int_{\Gamma(x)} \sup_{\phi\in\mathcal{C}_\beta} \Big|\int_{\mathbb{R}^n} (b(x) - b(z))\phi_t(y - z)f(z)\mathrm{d}z\Big|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{\frac{1}{2}}. \tag{1.4}$$

Wang [6] proved that if $\omega \in A_p$ for some $1 < p < \infty$, then the commutator $[b, S_\beta]$ is bounded on the weighted Lebesgue space $L^p_\omega(\mathbb{R}^n)$, and is also bounded on the weighted Morrey space $L^{p,\kappa}_\omega(\mathbb{R}^n)$ for $0 < \kappa < 1$. Hu and Wang [8] obtained its boundedness on the classical weighted Herz spaces. More applications of such intrinsic square function were also given by Guliyev et al. [9] and Zhuo, Yang and Liang [10].

Recently, the theory of variable function spaces has been extensively studied by a significant number of authors. The variable spaces $L^{p(\cdot)}$ were introduced by Orlicz [11] in 1931 and were systematically studied by Kováčik and Rákosník [12] from 1991. These spaces and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}$ proved to be useful in application to PDE with non-standard growth conditions [13], fluid dynamics [14] and the calculus of variations [15]. Since then various other function spaces, e.g. Herz spaces, Hardy spaces and Morrey spces have been intensively studied in the variable exponent setting, see [7, 16–21] and their references.

Herz spaces $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponent q were first studied by Izuki in [22, 23]. Moreover, some boundedness properties of classical operators were also established. Subsequently, Almeida and Direhem [24] defined the Herz spaces $K^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$, where the exponent α is variable as well. Recently, a new type of Herz spaces were introduced by Samko in [25], where all the three main indices are variable exponents, and he proved the boundedness of a wide class of sublinear operators including maximal, potential and Calderón-Zygmund operators on such spaces.

It is well known that grand spaces play a key role in Harmonic Analysis. Grand variable Herz spaces $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ were introduced in [26, 27], where various operators of harmonic analysis were studied in these spaces, e.g., Marcinkiewicz integrals with rough kernel and sublinear operators satisfying the size conditions introduced by Soria and Weiss in [28].

Motivated by the work of [26, 27], the aim of this paper is to prove the boundedness of the intrinsic square function and its commutator on grand variable Herz spaces $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$, where α and q are variable. We note that, even when $\alpha(\cdot) \equiv \alpha$ is constant, the main results obtained (see Theorems 3.1 and 3.2 below) are completely new.

Throughout this paper, we denote by $\mathbb{Z}$ the set of all integer numbers, $\mathbb{N}$ the set of all natural

numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter C stands for a positive constant, which may vary from line to line. The expression $f \lesssim g$ means $f \leqslant Cg$, and $f \approx g$ means $f \lesssim g \lesssim f$. $p'(\cdot)$ denotes the conjugate exponent defined by $1/p(\cdot) + 1/p'(\cdot) = 1$.

2. Preliminaries

In this section, we recall some basic facts on variable function spaces, we also give some key technical lemmas needed in the proofs of the main results.

2.1. The variable Lebesgue space

We begin with a brief review of the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, see the monographs [29, 30] for futher details.

Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a Lebesgue measurable function. The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} \mathrm{d}x < \infty.$$

This is a Banach space when equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1\}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, the space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \Omega\}.$$

By $\mathcal{P}(\mathbb{R}^n)$ we denote the set of all Lebesgue measurable functions $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ with $1 < p_- \leq p(x) \leq p_+ < \infty$, where

$$p_- := \operatorname{ess} \inf_{x \in \mathbb{R}^n} p(x) \text{ and } p_+ := \operatorname{ess} \sup_{x \in \mathbb{R}^n} p(x).$$

For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the generalized Hölder's inequality [12, Theorem 2.1] holds in the form

$$\int_{\mathbb{R}^n} |f(x)g(x)| \mathrm{d}x \leq \big(1 + \frac{1}{p_-} - \frac{1}{p_+}\big) \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{2.1}$$

Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x \in \mathbb{R}^n, r > 0} r^{-n} \int_{B(x,r)} |f(y)| \mathrm{d}y,$$

where $B(x, r)$ is the ball of radius r centered at x. By $\mathcal{B}(\mathbb{R}^n)$ we denote the collection of all measurable functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We say that a function $\rho : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant $C = C(\rho) > 0$ such that

$$|\rho(x) - \rho(y)| \leq \frac{C}{-\log|x - y|}, \ |x - y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^n. \tag{2.2}$$

If, for some $\rho_\infty \in (1, +\infty)$, there holds

$$|\rho(x) - \rho_\infty| \leq \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n, \tag{2.3}$$

and

$$|\rho(x)-\rho(0)| \le \frac{C}{|\log|x||}, \quad |x| \le \frac{1}{2}, \tag{2.4}$$

then we say $\rho(\cdot)$ has a log decay at infinity and at the origin, respectively.

By $\mathcal{P}_\infty(\mathbb{R}^n)$ and $\mathcal{P}_0(\mathbb{R}^n)$ we denote the class of exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the conditions (2.3) and (2.4), respectively. $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ is the set of functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the conditions (2.2) and (2.3), with $p_\infty := \lim_{|x|\to\infty} p(x)$. Obviously, we have $\mathcal{P}_\infty^{\log}(\mathbb{R}^n) \subset \mathcal{P}_0(\mathbb{R}^n) \cap \mathcal{P}_\infty(\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ if and only if $p'(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$. In particular, we note that if $p(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, namely, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ (see [30, Theorem 4.3.8]).

2.2. Grand variable Herz space

We now recall the definition of the grand Lebesgue sequence space, which was introduced recently by Rafeiro, Samko and Umarkhadzhiev in [31]. The symbol $\mathbb{X}$ stands for one of the sets $\mathbb{Z}, \mathbb{N}$ and $\mathbb{N}_0$.

Definition 2.1 *Let $1 \le p < \infty$ and $\theta > 0$. The grand Lebesgue sequence space is defined as the set of seqences $\mathbf{x}$ with*

$$\|\mathbf{x}\|_{l^{p),\theta}(\mathbb{X})} := \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k\in\mathbb{X}} \|x_k\|^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|\mathbf{x}\|_{l^{p(1+\varepsilon)}(\mathbb{X})} < \infty,$$

where $\mathbf{x} = \{x_k\}_{k\in\mathbb{X}}$.

Note that the following basic embeddings hold:

$$l^{p(1-\varepsilon)} \hookrightarrow l^p \hookrightarrow l^{p),\theta_1} \hookrightarrow l^{p),\theta_2} \hookrightarrow l^{p(1+\delta)} \tag{2.5}$$

for $0 < \varepsilon < \frac{1}{p}, \delta > 0$ and $0 < \theta_1 \le \theta_2$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $R_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$.

Definition 2.2 *Let $0 < p \le \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous variable Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{\rm loc}^{q(\cdot)}(\mathbb{R}^n\backslash\{0\})$ such that*

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)} := \Big(\sum_{k\in\mathbb{Z}} \|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\Big)^{1/p} < \infty,$$

with the usual modification when $p = \infty$.

Obviously, if α and q are constants, then $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ is the classical Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ defined in [32].

Definition 2.3 *Let $1 \le p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n), \theta > 0$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous grand variable Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$ is defined as the set of all $f \in L_{\rm loc}^{q(\cdot)}(\mathbb{R}^n\backslash\{0\})$ such that*

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)} := \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k\in\mathbb{Z}} \|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}} < \infty.$$

In the following, we show that variable Herz space is contained in grand variable Herz space.

Proposition 2.4 *Let $1 \le p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n), \theta > 0$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. Then we have*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n) \subset \dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n).$$

Proof Let $f \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)$, it follows from (2.5) that

$$\begin{aligned}
\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)} &= \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k\in\mathbb{Z}} \|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&= \|\|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}}\|_{l^{p),\theta}} \\
&\le C\|\|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}}\|_{l^p} = C\Big(\sum_{k\in\mathbb{Z}} \|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}}^p\Big)^{\frac{1}{p}} \\
&= C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)}.
\end{aligned}$$

The proof is completed. □

2.3. Key lemmas

The following Lemma 2.5 appears in Samko [25], Lemmas 2.6 and 2.7 are due to Izuki [22].

Lemma 2.5 *Let $D > 1$ and $q(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$. Then we have*

$$\frac{1}{c_0} r^{\frac{n}{q(0)}} \le \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le c_0 r^{\frac{n}{q(0)}}, \quad 0 < r \le 1,$$

and

$$\frac{1}{c_\infty} r^{\frac{n}{q_\infty}} \le \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le c_\infty r^{\frac{n}{q_\infty}}, \quad r \ge 1,$$

where $c_0, c_\infty \ge 1$ depend on D, but do not depend on r.

Lemma 2.6 *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have for all measurable subsets $E \subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_E\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|E|}, \ \frac{\|\chi_E\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C(\frac{|E|}{|B|})^\delta,$$

where δ is constant with $0 < \delta < 1$.

Lemma 2.7 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $k > l$ $(k, l \in \mathbb{N})$. If $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then we have*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \approx \|b\|_*$$

and

$$\|(b - b_{B_l})\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C(k - l)\|b\|_*\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

3. Main results and their proofs

In this section, we show that the intrinsic square function and its commutator are bounded on grand variable Herz spaces. We do so by applying some properties of variable exponent and BMO norms. Our main results can be stated as follows.

Theorem 3.1 *Let $0<\beta\le 1$, $\theta>0$, $1<p<\infty$ and $\alpha(\cdot)$, $q(\cdot)\in\mathcal{P}_\infty^{log}(\mathbb{R}^n)$. If $-\frac{n}{q(0)}<\alpha(0)<\frac{n}{q'(0)}$ and $-\frac{n}{q_\infty}<\alpha_\infty<\frac{n}{q'_\infty}$, then we have for all $f\in\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$,*

$$\|S_\beta(f)\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}\le C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}.$$

Theorem 3.2 *Let $b\in\mathrm{BMO}(\mathbb{R}^n)$, $0<\beta\le 1$, $\theta>0$, $1<p<\infty$ and $\alpha(\cdot)$, $q(\cdot)\in\mathcal{P}_\infty^{log}(\mathbb{R}^n)$. If $-\frac{n}{q(0)}<\alpha(0)<\frac{n}{q'(0)}$ and $-\frac{n}{q_\infty}<\alpha_\infty<\frac{n}{q'_\infty}$, then we have for all $f\in\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$,*

$$\|[b,S_\beta](f)\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}\le C\|b\|_*\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}.$$

Remark 3.3 According to Proposition 2.4, we have $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)\subset\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$ for $p>1$. Therefore, the above Theorems 3.1 and 3.2 can be regarded as the generalization of [33, Theorem 2.1] and [34, Corollary 21], respectively.

Proof of Theorem 3.1 Let $f\in\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$, we decompose

$$f(x)=\sum_{l=-\infty}^{\infty}f(x)\chi_l(x)=\sum_{l=-\infty}^{\infty}f_l(x).$$

The Minkowski inequality implies that

$$\begin{aligned}\|S_\beta(f)\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}=&\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\|2^{k\alpha(\cdot)}S_\beta(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
\le&C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\Big(\sum_{l=-\infty}^{k-2}\|2^{k\alpha(\cdot)}S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\
&C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\Big(\sum_{l=k-1}^{k+1}\|2^{k\alpha(\cdot)}S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\
&C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\Big(\sum_{l=k+2}^{\infty}\|2^{k\alpha(\cdot)}S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
=:&U_1+U_2+U_3.\end{aligned}$$

For U_2, we have

$$\begin{aligned}U_2\le&C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}\Big(\sum_{l=k-1}^{k+1}\|2^{k\alpha(\cdot)}S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\
&C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=0}^{\infty}\Big(\sum_{l=k-1}^{k+1}\|2^{k\alpha(\cdot)}S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
=:&U_{21}+U_{22}.\end{aligned}$$

For U_{21}, using the boundedness of S_β on $L^{q(\cdot)}(\mathbb{R}^n)$ (see [34]) and the fact that $2^{k\alpha(x)}\approx 2^{k\alpha(0)}$, $k<0$, $x\in R_k$, we get

$$U_{21}\le C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)p(1+\varepsilon)}\Big(\sum_{l=k-1}^{k+1}\|S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}$$

$$\begin{aligned}
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&=C\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

To estimate U_{22}, we note that $2^{k\alpha(x)}\approx 2^{k\alpha_\infty}$ if $k\ge 0, x\in R_k$. Hence we have

$$\begin{aligned}
U_{22}&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_\infty p(1+\varepsilon)}\Big(\sum_{l=k-1}^{k+1}\|S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_\infty p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&= C\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Combining the estimates of U_{21} and U_{22}, we conclude that

$$U_2\le C\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.$$

Now we turn to estimate U_1. Noting that if $x\in R_k$, $(y,t)\in\Gamma(x)$, $z\in R_l\bigcap\{z:|y-z|\le t\}$ and $l\le k-2$, then we have

$$t\ge\frac{1}{2}(|x-y|+|y-z|)\ge\frac{1}{2}|x-z|\ge\frac{1}{2}(|x|-|z|)\ge\frac{1}{4}|x|.$$

From this and the generalized Hölder inequality, it follows that

$$\begin{aligned}
|S_\beta(f_l)(x)|&=\Big(\iint_{\Gamma(x)}\sup_{\phi\in\mathcal{C}_\beta}\Big|\int_{\mathbb{R}^n}\phi_t(y-z)f_l(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{\frac{1}{2}}\\
&\le C\Big(\int_{\frac{|x|}{4}}^{\infty}\int_{|x-y|<t}\Big|\frac{1}{t^n}\int_{R_l\bigcap\{z:|y-z|\le t\}}f_l(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{\frac{1}{2}}\\
&\le C\Big(\int_{R_l}|f_l(z)|\mathrm{d}z\Big)\Big(\int_{\frac{|x|}{4}}^{\infty}\frac{\mathrm{d}t}{t^{2n+1}}\Big)^{\frac{1}{2}}\\
&\le C2^{-kn}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.1}$$

Splitting U_1 by means of Minkowski's inequality, we get

$$\begin{aligned}
U_1\le&C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{0}\Big(\sum_{l=-\infty}^{k-2}\|2^{k\alpha(\cdot)}S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\
&C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}\|2^{k\alpha(\cdot)}S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
:=&U_{11}+U_{12}.
\end{aligned}$$

For U_{11}, by Lemma 2.5, we have

$$2^{-kn}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\le C2^{-kn}2^{\frac{kn}{q(0)}}2^{\frac{-ln}{q'(0)}}\le 2^{\frac{(l-k)n}{q'(0)}}.\tag{3.2}$$

Using (3.1) and (3.2), we get

$$U_{11} \le C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} \Big(\sum_{l=-\infty}^{k-2} 2^{k\alpha(0)} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{-kn} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\le C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} \Big(\sum_{l=-\infty}^{k-2} 2^{l\alpha(0)} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}}.$$

Since $\frac{n}{q'(0)} - \alpha(0) > 0$ and $2^{-p(1+\varepsilon)} < 2^{-p}$, by Hölder's inequality we have

$$\begin{aligned}
U_{11} \le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} \Big(\sum_{l=-\infty}^{k-2} 2^{l\alpha(0)p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))p(1+\varepsilon)/2} \Big) \times \\
& \Big(\sum_{l=-\infty}^{k-2} 2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))(p(1+\varepsilon))'/2} \Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=-\infty}^{-2} 2^{l\alpha(0)p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=l+2}^{0} 2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))p(1+\varepsilon)/2} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=-\infty}^{-2} 2^{l\alpha(0)p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=l+2}^{0} 2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))p/2} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=-\infty}^{\infty} \|2^{l\alpha(\cdot)} f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For U_{12}, the Minkowski inequality implies that

$$\begin{aligned}
U_{12} \le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} \Big(\sum_{l=-\infty}^{0} \|2^{k\alpha(\cdot)} S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} + \\
& C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} \Big(\sum_{l=1}^{k-2} \|2^{k\alpha(\cdot)} S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
:= & I + II.
\end{aligned}$$

The estimate for II follows in a similar manner to U_{11} with $q'(0)$ replaced by q'_∞ and using the fact that $\frac{n}{q'_\infty} - \alpha_\infty < 0$.

For I, by Lemma 2.5, we get

$$2^{-kn} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C 2^{-kn} 2^{\frac{kn}{q_\infty}} 2^{\frac{ln}{q'(0)}} \le C 2^{\frac{-kn}{q'_\infty}} 2^{\frac{ln}{q'(0)}}. \tag{3.3}$$

Using (3.1) and (3.3), we get

$$\begin{aligned}
I \le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} \Big(\sum_{l=-\infty}^{0} 2^{k\alpha_\infty} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{-kn} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} 2^{k(\alpha_\infty - \frac{n}{q'_\infty})p(1+\varepsilon)} \Big(\sum_{l=-\infty}^{0} 2^{l\alpha(0)} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{l(\frac{n}{q'(0)}-\alpha(0))} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}}.
\end{aligned}$$

Since $\alpha_\infty - \frac{n}{q'_\infty} < 0$ and $2^{-p(1+\varepsilon)} < 2^{-p}$, by Hölder's inequality we have

$$\begin{aligned}
I \leq & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} 2^{k(\alpha_\infty - \frac{n}{q'_\infty})p} \Big(\sum_{l=-\infty}^{0} 2^{l\alpha(0)} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{l(\frac{n}{q'(0)} - \alpha(0))} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\leq & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \Big(\sum_{l=-\infty}^{0} 2^{l\alpha(0)} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{l(\frac{n}{q'(0)} - \alpha(0))} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\leq & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \Big(\sum_{l=-\infty}^{0} 2^{l\alpha(0)p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big) \times \\
& \Big(\sum_{l=-\infty}^{0} 2^{l(\frac{n}{q'(0)} - \alpha(0))(p(1+\varepsilon))'} \Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\leq & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=-\infty}^{0} 2^{l\alpha(0)p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\leq & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=-\infty}^{\infty} \|2^{l\alpha(\cdot)} f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\leq & C \|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next, we estimate U_3. For $x \in R_k$, $(y,t) \in \Gamma(x)$, $z \in R_l \bigcap \{z : |y-z| \leq t\}$ and $l \geq k+2$, we have

$$t \geq \frac{1}{2}(|x-y| + |y-z|) \geq \frac{1}{2}|x-z| \geq \frac{1}{2}(|z| - |x|) \geq \frac{1}{4}|z|.$$

Hence, by Hölder's inequality, we have

$$\begin{aligned}
|S_\beta(f_l)(x)| = & \Big(\int\int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\beta} \Big| \int_{\mathbb{R}^n} \phi_t(y-z) f_l(z) \mathrm{d}z \Big|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \Big)^{\frac{1}{2}} \\
\leq & C \Big(\int_{\frac{|z|}{4}}^{\infty} \int_{|x-y|<t} \Big| t^{-n} \int_{R_l \bigcap \{z:|y-z|\leq t\}} f_l(z) \mathrm{d}z \Big|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \Big)^{\frac{1}{2}} \\
\leq & C \Big(\int_{R_l} |f_l(z)| \mathrm{d}z \Big) \Big(\int_{\frac{|z|}{4}}^{\infty} \frac{\mathrm{d}t}{t^{2n+1}} \Big)^{\frac{1}{2}} \\
\leq & C 2^{-ln} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.4}$$

Splitting U_3 by means of Minkowski's inequality, we get

$$\begin{aligned}
U_3 \leq & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} \Big(\sum_{l=k+2}^{\infty} \|2^{k\alpha(\cdot)} S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} + \\
& C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} \Big(\sum_{l=k+2}^{\infty} \|2^{k\alpha(\cdot)} S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
:= & U_{31} + U_{32}.
\end{aligned}$$

For U_{32}, by Lemma 2.5, we have

$$2^{-ln} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C 2^{-ln} 2^{\frac{kn}{q_\infty}} 2^{\frac{ln}{q'_\infty}} \leq 2^{\frac{(k-l)n}{q_\infty}}. \tag{3.5}$$

Using (3.4) and (3.5), we get

$$U_{32} \le C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} \Big(\sum_{l=k+2}^{\infty} 2^{k\alpha_\infty} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{-ln} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\le C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} \Big(\sum_{l=k+2}^{\infty} 2^{l\alpha_\infty} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}}.$$

Since $\frac{n}{q_\infty} + \alpha_\infty > 0$ and $2^{-p(1+\varepsilon)} < 2^{-p}$, by Hölder's inequality we have

$$\begin{aligned}
U_{32} \le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=1}^{\infty} \Big(\sum_{l=k+2}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)p(1+\varepsilon)/2} \Big) \times \\
& \Big(\sum_{l=k+2}^{\infty} 2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)(p(1+\varepsilon))'/2} \Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=3}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=1}^{l-2} 2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)p(1+\varepsilon)/2} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=0}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=1}^{l-2} 2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)p/2} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{l=-\infty}^{\infty} \|2^{l\alpha(\cdot)} f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
\le & C \|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For U_{31}, the Minkowski inequality implies that

$$\begin{aligned}
U_{31} \le & C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} \Big(\sum_{l=k+2}^{0} \|2^{k\alpha(\cdot)} S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} + \\
& C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} \Big(\sum_{l=1}^{\infty} \|2^{k\alpha(\cdot)} S_\beta(f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}} \\
:= & J + JJ.
\end{aligned}$$

The estimate for J follows in a similar manner to U_{32} with q_∞ replaced by $q(0)$ and using the fact that $\frac{n}{q(0)} + \alpha(0) > 0$.

For JJ, by Lemma 2.5, we get

$$2^{-ln} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C 2^{-ln} 2^{\frac{kn}{q(0)}} 2^{\frac{ln}{q'_\infty}} \le C 2^{\frac{kn}{q(0)}} 2^{\frac{-ln}{q_\infty}}. \tag{3.6}$$

Using (3.4) and (3.6), we get

$$JJ \le C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} \Big(\sum_{l=1}^{\infty} 2^{k\alpha(0)} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{-ln} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\le C \sup_{\varepsilon>0} \Big(\varepsilon^\theta \sum_{k=-\infty}^{0} 2^{k(\alpha(0)+\frac{n}{q(0)})p(1+\varepsilon)} \Big(\sum_{l=1}^{\infty} 2^{l\alpha_\infty} \|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{-l(\frac{n}{q_\infty}+\alpha_\infty)} \Big)^{p(1+\varepsilon)} \Big)^{\frac{1}{p(1+\varepsilon)}}.$$

Since $\frac{n}{q(0)}+\alpha(0)>0$ and $2^{-p(1+\varepsilon)}<2^{-p}$, by Hölder's inequality we have

$$\begin{aligned}
JJ &\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{0}2^{k(\alpha(0)+\frac{n}{q(0)})p}\Big(\sum_{l=1}^{\infty}2^{l\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-l(\frac{n}{q_\infty}+\alpha_\infty)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\Big(\sum_{l=1}^{\infty}2^{l\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-l(\frac{n}{q_\infty}+\alpha_\infty)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\Big(\sum_{l=1}^{\infty}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\Big(\sum_{l=1}^{\infty}2^{-l(\frac{n}{q_\infty}+\alpha_\infty)(p(1+\varepsilon))'}\Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{l=1}^{\infty}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{l=-\infty}^{\infty}\|2^{l\alpha(\cdot)}f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

The proof of Theorem 3.1 is completed. □

Proof of Theorem 3.2 We write

$$f(x)=\sum_{l=-\infty}^{\infty}f(x)\chi_l(x)=\sum_{l=-\infty}^{\infty}f_l(x).$$

Applying the Minkowski inequality, we obtain

$$\begin{aligned}
\|[b,S_\beta](f)\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}\|2^{k\alpha(\cdot)}[b,S_\beta](f)\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&\le C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}\Big(\sum_{l=-\infty}^{k-2}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\
&\quad C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}\Big(\sum_{l=k-1}^{k+1}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\
&\quad C\sup_{\varepsilon>0}\Big(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}\Big(\sum_{l=k+2}^{\infty}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
&=: V_1+V_2+V_3.
\end{aligned}$$

For V_1, noting that $x\in R_k$, $(y,t)\in\Gamma(x)$, $z\in R_l\bigcap\{z:|y-z|\le t\}$ and $l\le k-2$, we have

$$t\ge\frac{1}{2}(|x-y|+|y-z|)\ge\frac{1}{2}|x-z|\ge\frac{1}{2}(|x|-|z|)\ge\frac{1}{4}|x|.$$

Thus, we obtain

$$\begin{aligned}
|[b,S_\beta](f_l)(x)| &= \Big(\int\int_{\Gamma(x)}\sup_{\phi\in\mathcal{C}_\beta}\Big|\int_{\mathbb{R}^n}(b(x)-b(z))\phi_t(y-z)f_l(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{\frac{1}{2}}\\
&\le C\Big(\int_{\frac{|x|}{4}}^{\infty}\int_{|x-y|<t}\Big|t^{-n}\int_{R_l\bigcap\{z:|y-z|\le t\}}(b(x)-b(z))f_l(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\int_{R_l}|b(x)-b(z)||f_l(z)|\mathrm{d}z\Big)\Big(\int_{\frac{|x|}{4}}^{\infty}\frac{\mathrm{d}t}{t^{2n+1}}\Big)^{\frac{1}{2}}\\
&\le C2^{-kn}\int_{R_l}|b(x)-b(z)||f_l(z)|\mathrm{d}z\\
&\le C2^{-kn}\Big(|b(x)-b_{B_l}|\int_{R_l}|f_l(z)|\mathrm{d}z+\int_{R_l}|b_{B_l}-b(z)||f_l(z)|\mathrm{d}z\Big)\\
&\le C2^{-kn}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(|b(x)-b_{B_l}|\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_l}-b)\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big)
\end{aligned}\tag{3.7}
$$

By Lemmas 2.6 and 2.7, we have

$$
\begin{aligned}
&\|[b,S_\beta](f_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C2^{-kn}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\|(b-b_{B_l})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\\
&\qquad\|(b_{B_l}-b)\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\
&\quad\le C\|b\|_*2^{-kn}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big((k-l)\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\\
&\qquad\|\chi_{B_l}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\
&\quad\le C(k-l)\|b\|_*\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-kn}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C(k-l)\|b\|_*\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-kn}\frac{|B_k|}{|R_k|}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{|B_l|}{|R_l|}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C(k-l)\|b\|_*\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-kn}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.8}
$$

Splitting V_1 by means of Minkowski's inequality, we get

$$
\begin{aligned}
V_1\le& C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{0}\Big(\sum_{l=-\infty}^{k-2}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\
&C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
:=&V_{11}+V_{12}.
\end{aligned}
$$

For V_{11}, using (3.8) and (3.2), we get

$$
\begin{aligned}
V_{11}\le& C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{0}\Big(\sum_{l=-\infty}^{k-2}2^{k\alpha(0)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\times\\
&(k-l)2^{-kn}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\
\le& C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{0}\Big(\sum_{l=-\infty}^{k-2}2^{l\alpha(0)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(k-l)2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}.
\end{aligned}
$$

Since $\frac{n}{q'(0)}-\alpha(0)>0$ and $2^{-p(1+\varepsilon)}<2^{-p}$, by Hölder's inequality we have

$$
V_{11}\le C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{0}\Big(\sum_{l=-\infty}^{k-2}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))p(1+\varepsilon)/2}\Big)\times
$$

$$\Big(\sum_{l=-\infty}^{k-2}(k-l)^{(p(1+\varepsilon))'}2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))(p(1+\varepsilon))'/2}\Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\Big)^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-2}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=l+2}^{0}2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))p(1+\varepsilon)/2}\Big)^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-2}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=l+2}^{0}2^{(l-k)(\frac{n}{q'(0)}-\alpha(0))p/2}\Big)^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{\infty}\|2^{l\alpha(\cdot)}f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq C\|b\|_*\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.$$

For V_{12}, the Minkowski inequality implies that

$$\begin{aligned}V_{12}\leq& C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=-\infty}^{0}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+\\&C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{k-2}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\:=&E+F.\end{aligned}$$

The estimate for F follows in a similar manner to U_{11} with $q'(0)$ replaced by q'_∞ and using the fact that $\frac{n}{q'_\infty}-\alpha_\infty<0$.

For E, using (3.8) and (3.3), we get

$$\begin{aligned}E\leq& C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=-\infty}^{0}2^{k\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\times\\&(k-l)2^{-kn}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\\leq& C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}2^{k(\alpha_\infty-\frac{n}{q'_\infty})p(1+\varepsilon)}\times\\&\Big(\sum_{l=-\infty}^{0}2^{l\alpha(0)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(k-l)2^{l(\frac{n}{q'(0)}-\alpha(0))}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}.\end{aligned}$$

Since $\alpha_\infty-\frac{n}{q'_\infty}<0$ and $2^{-p(1+\varepsilon)}<2^{-p}$, by Hölder's inequality we have

$$\begin{aligned}E\leq& C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}2^{k(\alpha_\infty-\frac{n}{q'_\infty})p}\times\\&\Big(\sum_{l=-\infty}^{0}2^{l\alpha(0)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(k-l)2^{l(\frac{n}{q'(0)}-\alpha(0))}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\\\leq& C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\Big(\sum_{l=-\infty}^{0}2^{l\alpha(0)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(k-l)2^{l(\frac{n}{q'(0)}-\alpha(0))}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}\end{aligned}$$

$$
\begin{aligned}
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta\Big(\sum_{l=-\infty}^{0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\times \\
&\quad\Big(\sum_{l=-\infty}^{0}(k-l)^{(p(1+\varepsilon))'}2^{l(\frac{n}{q'(0)}-\alpha(0))(p(1+\varepsilon))'}\Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{\infty} \|2^{l\alpha(\cdot)}f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_*\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

As argued in the estimation of U_2, using the boundedness of $[b,S_\beta]$ on $L^{q(\cdot)}(\mathbb{R}^n)$ (see [33]) we have

$$V_2 \le C\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.$$

Next, we estimate V_3. For $x\in R_k$, $(y,t)\in\Gamma(x)$, $z\in R_l\bigcap\{z:|y-z|\le t\}$ and $l\ge k+2$, we have

$$t\ge\frac{1}{2}(|x-y|+|y-z|)\ge\frac{1}{2}|x-z|\ge\frac{1}{2}(|z|-|x|)\ge\frac{1}{4}|z|.$$

Hence, by Hölder's inequality, we have

$$
\begin{aligned}
&|[b,S_\beta](f_l)(x)| \\
&=\Big(\int\int_{\Gamma(x)}\sup_{\phi\in\mathcal{C}_\beta}\Big|\int_{\mathbb{R}^n}(b(x)-b(z))\phi_t(y-z)f_l(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{\frac{1}{2}} \\
&\le C\Big(\int_{\frac{|z|}{4}}^{\infty}\int_{|x-y|<t}\Big|t^{-n}\int_{R_l\bigcap\{z:|y-z|\le t\}}(b(x)-b(z))f_l(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{\frac{1}{2}} \\
&\le C\Big(\int_{R_l}|b(x)-b(z)||f_l(z)|\mathrm{d}z\Big)\Big(\int_{\frac{|z|}{4}}^{\infty}\frac{\mathrm{d}t}{t^{2n+1}}\Big)^{\frac{1}{2}} \\
&\le C2^{-ln}\int_{R_l}|b(x)-b(z)||f_l(z)|\mathrm{d}z \\
&\le C2^{-ln}\Big(|b(x)-b_{B_k}|\int_{R_l}||f_l(z)|\mathrm{d}z+\int_{R_l}|b_{B_k}-b(z)||f_l(z)|\mathrm{d}z\Big) \\
&\le C2^{-ln}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big(|b(x)-b_{B_k}|\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_k}-b)\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\big).
\end{aligned}
\tag{3.9}
$$

Using Lemmas 2.6 and 2.7, we have

$$
\begin{aligned}
&\|[b,S_\beta](f_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C2^{-ln}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big(\|(b-b_{B_k})\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+ \\
&\quad\|(b_{B_k}-b)\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big) \\
&\le C\|b\|_*2^{-ln}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big((l-k)\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+ \\
&\quad\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big) \\
&\le C(l-k)\|b\|_*2^{-ln}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}
\end{aligned}
$$

$$\leq C(l-k)\|b\|_*\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-ln}\frac{|B_k|}{|R_k|}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{|B_l|}{|R_l|}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}$$
$$\leq C(l-k)\|b\|_*\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{-ln}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \tag{3.10}$$

Splitting V_3 by means of Minkowski's inequality, we get

$$V_3 \leq C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{0}\Big(\sum_{l=k+2}^{\infty}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+$$
$$C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=k+2}^{\infty}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$:= V_{31}+V_{32}.$$

For V_{32}, using (3.10) and (3.5), we get

$$V_{32} \leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=k+2}^{\infty}2^{k\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\times$$
$$(l-k)2^{-ln}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=k+2}^{\infty}2^{l\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(l-k)2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}.$$

Since $\frac{n}{q_\infty}+\alpha_\infty>0$ and $2^{-p(1+\varepsilon)}<2^{-p}$, by Hölder's inequality we have

$$V_{32} \leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=1}^{\infty}\Big(\sum_{l=k+2}^{\infty}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)p(1+\varepsilon)/2}\Big)\times$$
$$\Big(\sum_{l=k+2}^{\infty}(l-k)^{(p(1+\varepsilon))'}2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)(p(1+\varepsilon))'/2}\Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=3}^{\infty}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{l-2}2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)p(1+\varepsilon)/2}\Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=0}^{\infty}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=1}^{l-2}2^{(k-l)(\frac{n}{q_\infty}+\alpha_\infty)p/2}\Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\leq C\|b\|_*\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{l=-\infty}^{\infty}\|2^{l\alpha(\cdot)}f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$\leq C\|b\|_*\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.$$

For V_{31}, the Minkowski inequality implies that

$$V_{31} \leq C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{0}\Big(\sum_{l=k+2}^{0}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}+$$
$$C\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=-\infty}^{0}\Big(\sum_{l=1}^{\infty}\|2^{k\alpha(\cdot)}[b,S_\beta](f\chi_l)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}$$
$$:= G+H.$$

The estimate for G follows in a similar manner to V_{32} with q_∞ replaced by $q(0)$ and using the fact that $\frac{n}{q(0)}+\alpha(0)>0$.

For H, using (3.10) and (3.6), we have

$$\begin{aligned}
H &\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k=-\infty}^{0}\Big(\sum_{l=1}^{\infty} 2^{k\alpha(0)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\times \\
&\qquad (l-k)2^{-ln}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k=-\infty}^{0} 2^{k(\alpha(0)+\frac{n}{q(0)})p(1+\varepsilon)}\times \\
&\qquad \Big(\sum_{l=1}^{\infty} 2^{l\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(l-k)2^{-l(\frac{n}{q_\infty}+\alpha_\infty)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}}.
\end{aligned}$$

Since $\frac{n}{q(0)}+\alpha(0)>0$ and $2^{-p(1+\varepsilon)}<2^{-p}$, by Hölder's inequality we have

$$\begin{aligned}
H &\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{k=-\infty}^{0} 2^{k(\alpha(0)+\frac{n}{q(0)})p}\Big(\sum_{l=1}^{\infty} 2^{l\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(l-k)2^{-l(\frac{n}{q_\infty}+\alpha_\infty)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta\Big(\sum_{l=1}^{\infty} 2^{l\alpha_\infty}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}(l-k)2^{-l(\frac{n}{q_\infty}+\alpha_\infty)}\Big)^{p(1+\varepsilon)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta\Big(\sum_{l=1}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\times \\
&\qquad \Big(\sum_{l=1}^{\infty}(l-k)^{(p(1+\varepsilon))'}2^{-l(\frac{n}{q_\infty}+\alpha_\infty)(p(1+\varepsilon))'}\Big)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{l=1}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_* \sup_{\varepsilon>0}\Big(\varepsilon^\theta \sum_{l=-\infty}^{\infty} \|2^{l\alpha(\cdot)}f\chi_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\|b\|_*\|f\|_{\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

The proof of Theorem 3.2 is completed. □

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