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On the Haagerup Property of C^*-Dynamical Systems

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Abstract Let A be a unital C^*-algebra with a state τ and G be a discrete group that acts on A through a τ-preserving action α. We first generalize the Haagerup property of dynamical systems by considering states and prove that the dynamical system has the Haagerup property if and only if the reduced crossed product does. Then we introduce the quasi-amenable action of G on A with respect to τ. Finally, using the above results, we prove that if α is a quasi-amenable action of G on A with respect to τ, then (A,τ) has the Haagerup property if and only if $(A\rtimes_{\alpha,r}G,\tau')$ does, where τ' is the induced state on $A\rtimes_{\alpha,r}G$. As a consequence, our main results improve some well known results in the classical situation.

Keywords C^*-dynamical system; Haagerup property; quasi-amenable action

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1. Introduction

Approximation theory has always been a hot research topic in group theory and operator algebra theory. Over the past few decades, many mathematicians have studied many different approximation properties, such as amenability which was first introduced by von Neumann [1] in response to the famous Banach-Tarski paradox, weak amenability which was defined formally by Cowling and Haagerup in [2], the Haagerup property which was first introduced for groups by Haagerup in [3] as a weaker version of amenability, the weak Haagerup property which was introduced by Knudby [4] in order to study the relation between weak amenability and the Haagerup property, and so on.

The Haagerup property has been considered for von Neumann algebras [5–8]. In [9], Dong introduced the Haagerup property for a C^*-algebra with a faithful tracial state. Suzuki [10] proved that the Haagerup property for C^*-algebras does depend on the faithful tracial state. In [11], the authors generalized the Haagerup property to arbitrary C^*-algebras.

Herz-Schur multiplier is an important notion in operator algebra theory, since a number of approximation properties of groups and group C^*-algebras depend on it (see [3, 4, 12] for more information). In [13], Mckee, Todorov and Turowska extended the notion to the setting of non-commutative dynamical systems. Let G be a discrete group which acts on a unital C^*-algebra A through an action α. We denote by $CB(A)$ the space of all completely bounded linear maps

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from A into itself, and $C_c(G,A)$ the space of all finitely supported functions on G with values in A. In fact, we have known that $C_c(G,A)$ is a dense subalgebra of the reduced crossed product $A \rtimes_{\alpha,r} G$ and a typical element $x \in C_c(G,A)$ is written as a finite sum $x = \sum_{s\in G} a_s s$.

Definition 1.1 ([13]) *Let (A,G,α) be a C^*-dynamical system. A bounded function $F : G \to CB(A)$ is called a Herz-Schur (A,G,α)-multiplier if the map $S_F : C_c(G,A) \to C_c(G,A)$ such that $S_F(\sum_{s\in G} a_s s) = \sum_{s\in G} F(s)(a_s)s$ is completely bounded.*

If F is a Herz-Schur (A,G,α)-multiplier, then the map S_F has a unique extension to a completely bounded map on $A\rtimes_{\alpha,r} G$, which is still denoted by S_F. We say that F is a completely positive Herz-Schur (A,G,α)-multiplier if the map $S_F : A \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G$ is completely positive. Note that the set of all Herz-Schur (A,G,α)-multipliers is an algebra with respect to the obvious operations and endowed with the norm $\|F\|_m = \|S_F\|_{cb}$. Let ρ be a faithful α-invariant tracial state on A. In [14], the authors introduced the Haagerup property of the dynamical system (A,G,α,ρ) and proved that (A,G,α,ρ) has the Haagerup property if and only if $(A \rtimes_{\alpha,r} G, \rho')$ has the Haagerup property, where ρ' is the induced faithful tracial state on $A \rtimes_{\alpha,r} G$.

In Section 2, we first recall some notations and results of Gelfand-Naimark-Segal construction with repect to a state τ on A. Next, we introduce the Haagerup property of (A,G,α,τ) and give some similar results as [14, Lemma 3.5-3.7]. Finally, we prove that (A,G,α,τ) has the Haagerup property if and only if $(A \rtimes_{\alpha,r} G, \tau')$ has the Haagerup property.

In Section 3, we first recall some notations and results about multiplicative domains. Next, we introduce the quasi-amenable action with respect to a state τ. Finally, we prove that if α is a quasi-amenable action of G on A with respect to τ, then (A,τ) has the Haagerup property if and only if $(A \rtimes_{\alpha,r} G, \tau')$ has the Haagerup property, where τ' is the induced state on $A \rtimes_{\alpha,r} G$. As a consequence, our main results improve some well known results in the classical situation.

2. Haagerup property of dynamical systems

In this paper, G is a discrete group with the unit e, A is a unital C^*-algebra with a state τ, α is a τ-preserving action of G on A, A^+ is the cone of positive elements in A, $N_\tau = \{a \in A | \tau(a^*a) = 0\}$ and $\Lambda_\tau(A) = A/N_\tau$. By the Gelfand-Naimark-Segal construction, τ defines a Hilbert space $L^2(A,\tau)$. We denote by $\langle \Lambda_\tau(a), \Lambda_\tau(b)\rangle = \tau(b^*a)$ the associated inner product and $\|\Lambda_\tau(a)\|_{2,\tau} = \tau(a^*a)^{1/2}$ the associated Hilbert norm for any $a, b \in A$. We say a linear map $\Phi : A \to A$ is L^2-bounded if there exists a constant $C > 0$ such that $\|\Lambda_\tau(\Phi(a))\|_{2,\tau} \le C\|\Lambda_\tau(a)\|_{2,\tau}$ for any $a \in A$.

To simplify notations, we use c.p. to abbreviate “completely positive”, u.c.p. for “unital completely positive” and c.c.p. for “contractive completely positive”.

Lemma 2.1 *If $\Phi : A \to A$ is L^2-bounded, then there exists a bounded operator $T_\Phi : L^2(A,\tau) \to L^2(A,\tau)$ determined by $\Lambda_\tau(a) \mapsto \Lambda_\tau(\Phi(a)), a \in A$.*

Proof Since Φ is L^2-bounded, there is a constant $C > 0$ such that

$$\|\Lambda_\tau(\Phi(a))\|_{2,\tau} \leq C\|\Lambda_\tau(a)\|_{2,\tau}.$$

Let $\tilde{\Phi} : \Lambda_\tau(A) \to \Lambda_\tau(A)$ such that

$$\Lambda_\tau(a) \mapsto \Lambda_\tau(\Phi(a)).$$

First, we show that $\tilde{\Phi}$ is well-defined. Indeed, if $a, b \in A$ such that $\Lambda_\tau(a) = \Lambda_\tau(b)$, i.e., $a-b \in N_\tau$, we have

$$0 \leq \tau(\Phi(a-b)^*\Phi(a-b)) = \|\Lambda_\tau(\Phi(a-b))\|_{2,\tau}^2 \leq C^2\|\Lambda_\tau(a-b)\|_{2,\tau}^2 = 0.$$

Hence, $\tilde{\Phi}$ is well-defined and it is obvious that $\tilde{\Phi}$ extends to a bounded operator $T_\Phi : L^2(A,\tau) \to L^2(A,\tau)$ with norm at most C. $\square$

Note that if Φ is a c.c.p. map and satisfies $\tau \circ \Phi \leq \tau$, then Φ is L^2-bounded and T_Φ is a contraction. We say that $\Phi : A \to A$ is L^2-compact if T_Φ is a compact operator on $L^2(A,\tau)$.

Definition 2.2 ([11]) *We say that (A,τ) has the Haagerup property if there exists a net $(\Phi_i)_{i\in I}$ of c.c.p. maps from A into itself such that*

(1) $\tau \circ \Phi_i \leq \tau$ and Φ_i is L^2-compact for every $i \in I$;

(2) $\{T_{\Phi_i}\}_{i\in I}$ converges to the identity map in the strong operator topology.

Remark 2.3 Since A is a unital C^*-algebra and τ is a state on A, we have the following statements.

(1) By [11, Remark 3.4], the above c.c.p. maps can be replaced by u.c.p. maps.

(2) Condition (2) is equivalent to

$$\|\Lambda_\tau(\Phi_i(a)) - \Lambda_\tau(a)\|_{2,\tau} \to 0, \quad a \in A.$$

Although τ is only a state here, the proof of the following lemma is similar to [14, Lemma 3.2], we omit it.

Lemma 2.4 *Let $F : G \to CB(A)$ be a c.p. Herz-Schur (A,G,α)-multiplier such that $\tau\circ F(e) \leq \tau$ and $F(e)(1_A) = 1_A$. Then $F(t)$ is L^2-bounded and $T_{F_i(t)}$ is a contraction for each $t \in G$.*

Now we introduce the Haagerup property of the dynamical system (A,G,α,τ).

Definition 2.5 *We say that (A,G,α,τ) has the Haagerup property if there is a net $(F_i)_{i\in I}$ of c.p. Herz-Schur (A,G,α)-multipliers such that*

(1) $F_i(e)$ is unital and $\tau \circ F_i(e) \leq \tau, i \in I$;

(2) $F_i(t)$ is L^2-compact, $t \in G$;

(3) The function $s \to \|T_{F_i(s)}\|$ vanishes at infinity, $i \in I$;

(4) $\|\Lambda_\tau(F_i(t)(a) - a)\|_{2,\tau} \to 0$ for all $t \in G$ and all $a \in A$.

Let $A \rtimes_{\alpha,r} G$ be the reduced crossed product of the dynamical system (A,G,α) and $\mathcal{E} : A \rtimes_{\alpha,r} G \to A$ be the canonical faithful conditional expectation. Assume that τ' is the induced state on $A \rtimes_{\alpha,r} G$, i.e.,

$$\tau'(x) = \tau \circ \mathcal{E}(x), \quad x \in A \rtimes_{\alpha,r} G.$$

We denote by $L^2(\tau)$ the Hilbert space $L^2(A,\tau)$, and $L^2(\tau')$ the Hilbert space $L^2(A \rtimes_{\alpha,r} G, \tau')$. For any $t \in G$, we denote by L_t the subspace $\{at : a \in A\}$ of $A \rtimes_{\alpha,r} G$, and $L_t^2(\tau')$ the closure of $\Lambda_{\tau'}(L_t)$ in the norm $\|\cdot\|_{2,\tau'}$. Next we will give some results which are generalization of [14, Lemma 3.5-3.7].

Lemma 2.6 *The following statements are true.*

(1) We have an orthogonal decomposition

$$L^2(\tau') = \bigoplus_{t\in G} L_t^2(\tau').$$

(2) For each $t \in G$, the map $\Lambda_\tau(a) \mapsto \Lambda_{\tau'}(at)$ extends to a unitary V_t from $L^2(\tau)$ to $L_t^2(\tau')$.

(3) Let P_t be the orthogonal projection of $L^2(\tau')$ onto $L_t^2(\tau')$. Then the map $V_t^*P_t : L^2(\tau') \to L^2(\tau)$ satisfies

$$V_t^*P_t(\Lambda_{\tau'}(z)) = \Lambda_\tau(\mathcal{E}(zt^{-1})), \quad z \in A \rtimes_{\alpha,r} G.$$

(4) $F : G \to CB(A)$ is a c.p. Herz-Schur (A, G, α)-multiplier, then $\tau \circ F(e) \le \tau$ if and only if $\tau' \circ S_F \le \tau'$.

Proof (1) For all $s, t \in G$ and $a, b \in A$, we have

$$\langle \Lambda_{\tau'}(as), \Lambda_{\tau'}(bt)\rangle = \tau'((bt)^*(as)) = \tau'(t^{-1}b^*as) = \tau'(b^*ast^{-1}),$$

where the last equality is due to [11, Lemma 3.15]. It follows that $\Lambda_{\tau'}(L_s) \perp \Lambda_{\tau'}(L_t)$ whenever $s \neq t$. Since $\{L_t^2(\tau') : t \in G\}$ is a collection of pairwise orthogonal subspaces of $L^2(\tau')$, we have

$$\bigoplus_{t\in G} L_t^2(\tau') = \overline{\mathrm{span}\{\Lambda_{\tau'}(at) : a \in A, t \in G\}}^{\|\cdot\|_{2,\tau'}}.$$

Hence, we have

$$\bigoplus_{t\in G} L_t^2(\tau') = \overline{\Lambda_{\tau'}(C_c(G, A))}^{\|\cdot\|_{2,\tau'}} = L^2(\tau').$$

(2) For each $t \in G$, we define a map $V_t : \Lambda_\tau(A) \to L_t^2(\tau')$ by

$$\Lambda_\tau(a) \mapsto \Lambda_{\tau'}(at), \quad a \in A.$$

First, we prove that V_t is well-defined. Indeed, if $\Lambda_\tau(a) = \Lambda_\tau(b)$, i.e., $a - b \in N_\tau$, we have

$$0 \le \tau'((at - bt)^*(at - bt)) = \tau'(t^{-1}(a - b)^*(a - b)t) = \tau((a - b)^*(a - b)) = 0.$$

It follows that V_t is well-defined. Since

$$\|V_t(\Lambda_\tau(a))\|_{2,\tau'}^2 = \|\Lambda_{\tau'}(at)\|_{2,\tau'}^2 = \|\Lambda_\tau(a)\|_{2,\tau}^2,$$

it can be extended to an isometry from $L^2(\tau)$ to $L_t^2(\tau')$ denoted still by V_t. It is obvious that $V_t : L^2(\tau) \to L_t^2(\tau')$ is surjective.

(3) Since $\mathcal{E} : A \rtimes_{\alpha,r} G \to A$ is a conditional expectation and $\tau' = \tau \circ \mathcal{E}$, it follows from [11, Lemma 3.1] that $T_{\mathcal{E}} : L^2(\tau') \to L^2(\tau)$, $\Lambda_{\tau'}(x) \mapsto \Lambda_\tau(\mathcal{E}(x))$, $x \in A \rtimes_{\alpha,r} G$ exists and is a contraction. For $z = bt$ $(b \in A)$, since $V_t^*(\Lambda_{\tau'}(bt)) = \Lambda_\tau(b)$, we have

$$V_t^*P_t(\Lambda_{\tau'}(z)) = V_t^*(\Lambda_{\tau'}(bt)) = \Lambda_\tau(b) = T_{\mathcal{E}}(\Lambda_{\tau'}(zt^{-1})) = \Lambda_\tau(\mathcal{E}(zt^{-1})).$$

Considering the linearity and continuity of involved maps and the fact that

$$\|\Lambda_{\tau'}(x)\|_{2,\tau'} \leq \|x\|$$

for any $x \in A \rtimes_{\alpha,r} G$, it follows that the claim is true for all $z \in A \rtimes_{\alpha,r} G$.

(4) ($\Rightarrow$). For any positive element $\sum_{s\in G} a_s s \in C_c(G, A)$, since $\mathcal{E}$ is a c.p. map, we have $\mathcal{E}(\sum_{s\in G} a_s s) = a_e \geq 0$. Hence

$$\tau' \circ S_F\Big(\sum_{s\in G} a_s s\Big) = \tau'\Big(\sum_{s\in G} F(s)(a_s)s\Big) = \tau(F(e)(a_e)) \leq \tau(a_e) = \tau'\Big(\sum_{s\in G} a_s s\Big).$$

($\Leftarrow$). For any $a \in A^+$,

$$\tau \circ F(e)(a) = \tau' \circ S_F(ae) \leq \tau'(ae) = \tau(a).$$

This completes the proof. $\square$

Lemma 2.7 *Let $F : G \to CB(A)$ be a c.p. Herz-Schur (A, G, α)-multiplier such that $\tau\circ F(e) \leq \tau$ and $F(e)(1_A) = 1_A$. Then T_{S_F} is a contraction and*

$$T_{S_F} = \bigoplus_{t\in G} T_{F(t)}.$$

Proof Since $\tau \circ F(e) \leq \tau$, it follows from Lemma 2.6 (4) that $\tau' \circ S_F \leq \tau'$. Hence, S_F is L^2-bounded and T_{S_F} is a contraction. Clearly, for every $t \in G$, T_{S_F} leaves the space $\Lambda_{\tau'}(L_t)$ invariant. By Lemma 2.4, $T_{F(t)}$ exists for all $t \in G$. Hence, after identifying $L^2_t(\tau')$ with $L^2(\tau)$ by Lemma 2.6 (2), we have that the restriction of T_{S_F} to $\Lambda_{\tau'}(L_t)$ coincides with $T_{F(t)}$. After identifying $L^2(\tau')$ with $\bigoplus_{t\in G} L^2_t(\tau')$ by Lemma 2.6 (1), it follows that T_{S_F} coincides with $\bigoplus_{t\in G} T_{F(t)}$ in the dense subspace $\Lambda_{\tau'}(C_c(G, A))$. So it is obvious that $T_{S_F} = \bigoplus_{t\in G} T_{F(t)}$ on $L^2(\tau')$. $\square$

Lemma 2.8 *If $(F_i)_{i\in I}$ is a net of c.p. Herz-Schur (A, G, α)-multiplier such that $\tau \circ F_i(e) \leq \tau$ and $F_i(e)(1_A) = 1_A$ for every $i \in I$, then the following are equivalent:*

(1) $\|\Lambda_{\tau'}(S_{F_i}(x) - x)\|_{2,\tau'} \to 0, x \in A \rtimes_{\alpha,r} G$;

(2) $\|\Lambda_\tau(F_i(t)(a) - a)\|_{2,\tau} \to 0$ for all $t \in G, a \in A$.

Proof (1)$\Rightarrow$(2). For any $a \in A$ and $t \in G$, we have

$$\|\Lambda_\tau(F_i(t)(a) - a)\|_{2,\tau} = \|\Lambda_{\tau'}(F_i(t)(a)t - at)\|_{2,\tau'} = \|\Lambda_{\tau'}(S_{F_i}(at) - at)\|_{2,\tau'} \to 0.$$

(2)$\Rightarrow$(1). By the above equation and the fact that $T_{S_{F_i}}$ is a contraction for each $i \in I$, so for any $\varepsilon > 0$, a routine $\varepsilon/3$-argument shows the convergence for all $x \in A \rtimes_{\alpha,r} G$. $\square$

Let Φ be a c.p. map on $A \rtimes_{\alpha,r} G$, we define a function $h_\Phi : G \to B(A)$ by

$$h_\Phi(s)(a) = \mathcal{E}(\Phi(as)s^{-1})$$

for all $s \in G$ and $a \in A$. It follows from [14, Proposition 3.4] that h_Φ is a c.p. Herz-Schur (A, G, α)-multiplier.

Theorem 2.9 *The following statements are equivalent:*

(1) (A, G, α, τ) *has the Haagerup property;*

(2) $(A \rtimes_{\alpha,r} G, \tau')$ *has the Haagerup property.*

Proof (1)$\Rightarrow$ (2). Let $(F_i)_{i\in I}$ be a net of c.p. Herz-Schur (A, G, α)-multipliers witnessing the Haagerup property of (A, G, α, τ). It follows from Lemma 2.6 (4) that $\tau' \circ S_{F_i} \le \tau'$. By Lemma 2.8, $\|\Lambda_{\tau'}(S_{F_i}(x) - x)\|_{2,\tau'} \to 0, x \in A \rtimes_{\alpha,r} G$. The rest proof is similar to [14, Theorem 3.8].

(2)$\Rightarrow$ (1). Let $(\Phi_i)_{i\in I}$ be a net of u.c.p. maps on $A \rtimes_{\alpha,r} G$ witnessing the Haagerup property of $(A \rtimes_{\alpha,r} G, \tau')$. Let $F_i = h_{\Phi_i}, i \in I$. We only prove that $F_i(t)$ is L^2-compact for any $t \in G$, the rest proof is similar to [14, Theorem 3.8]. Let P_t be the orthogonal projection from $L^2(\tau')$ to $L^2_t(\tau')$, we have $T_{F_i(t)} = V_t^* P_t T_{\Phi_i} P_t V_t$. Indeed, for any $a \in A$,

$$\begin{aligned} T_{F_i(t)}(\Lambda_\tau(a)) &= \Lambda_\tau(F_i(t)(a)) = \Lambda_\tau(h_{\Phi_i}(t)(a)) = \Lambda_\tau(\mathcal{E}(\Phi_i(at)t^{-1})) \\ &= V_t^* P_t(\Lambda_{\tau'}(\Phi_i(at))) = V_t^* P_t T_{\Phi_i}(\Lambda_{\tau'}(at)) = V_t^* P_t T_{\Phi_i} P_t V_t(\Lambda_\tau(a)). \end{aligned}$$

Since $T_{\Phi_i} : L^2(\tau') \to L^2(\tau')$ is a compact operator, we conclude that $T_{F_i(t)}$ is a compact operator on $L^2(\tau)$. $\square$

3. Quasi-amenable actions with respect to a state

In this section, $Z(A)$ is the center of A, $Z(A)^+$ is the cone of positive elements in $Z(A)$. First we recall some notations and results about multiplicative domains.

Proposition 3.1 ([12]) *Let A and B be C^*-algebras and $\varphi : A \to B$ be a c.c.p. map.*

(1) (Schwarz Inequality) The inequality $\varphi(a)^*\varphi(a) \le \varphi(a^*a)$ holds for every $a \in A$.

(2) (Bimodule Property) Given $a \in A$, if $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ and $\varphi(aa^*) = \varphi(a)\varphi(a)^*$, then $\varphi(ba) = \varphi(b)\varphi(a)$ and $\varphi(ab) = \varphi(a)\varphi(b)$, for all $b \in A$.

(3) The subspace $A_\varphi = \{a \in A : \varphi(a^*a) = \varphi(a)^*\varphi(a)$ and $\varphi(aa^*) = \varphi(a)\varphi(a)^*\}$ is a C^*-subalgebra of A.

Definition 3.2 ([12]) *The C^*-subalgebra A_φ in Proposition 3.1 is called the multiplicative domain of φ.*

Next, we introduce the quasi-amenable action of G on A with respect to τ.

Definition 3.3 *We say that an action α of G on a unital C^*-algebra A with a state τ is quasi-amenable with respect to τ if there exists a net $\{T_i\}_{i\in I}$ of finitely supported functions $T_i : G \to A^+ \cap A_\tau$ such that $\sum_{t\in G} T_i(t)^2 = 1_A$ and*

$$\Big\| \sum_{s\in G} T_i(s) a \alpha_t(T_i(t^{-1}s)) - a \Big\| \to 0 \tag{3.1}$$

for all $t \in G$ and $a \in A$.

Remark 3.4 If there exists a net $\{T_i\}_{i\in I}$ of finitely supported functions $T_i : G \to Z(A)^+ \cap A_\tau$ such that $\sum_{t\in G} T_i(t)^2 = 1_A$ and

$$\Big\| \sum_{s\in G} (T_i(s) - \alpha_t(T_i(t^{-1}s)))^*(T_i(s) - \alpha_t(T_i(t^{-1}s))) \Big\| \to 0$$

for all $t \in G$, then it follows from [12, Lemma 4.3.2] that the action α is quasi-amenable with respect to τ.

Before the following theorem, we note a fact that for each $a \in A$, we can define a map $L_a : A \to A$ by $L_a(b) = ab$ for all $b \in A$. Indeed,

$$\|\Lambda_\tau(L_a(b))\|_{2,\tau} = \|\Lambda_\tau(ab)\|_{2,\tau} = \tau(b^*a^*ab)^{1/2} \leq \|a\|\tau(b^*b)^{1/2} = \|a\|\|\Lambda_\tau(b)\|_{2,\tau}.$$

Similarly, for any $c \in A_\tau$, we can define $R_c : A \to A$ by $R_c(b) = bc$. Indeed,

$$\|\Lambda_\tau(R_c(b))\|_{2,\tau} = \|\Lambda_\tau(bc)\|_{2,\tau} = \tau(c^*b^*bc)^{1/2} = \tau(c^*c)^{1/2}\tau(b^*b)^{1/2} \leq \|c\|\|\Lambda_\tau(b)\|_{2,\tau}.$$

Hence, there exist bounded operators T_{L_a} and T_{R_c} on $L^2(A,\tau)$ with the norm at most $\|a\|$ and $\|c\|$, respectively, such that

$$T_{L_a}(\Lambda_\tau(b)) = \Lambda_\tau(L_a(b)) = \Lambda_\tau(ab),$$
$$T_{R_c}(\Lambda_\tau(b)) = \Lambda_\tau(R_c(b)) = \Lambda_\tau(bc)$$

for all $b \in A$.

Theorem 3.5 *If α is a quasi-amenable action of G on A with respect to τ, then (A,τ) has the Haagerup property if and only if $(A \rtimes_{\alpha,r} G, \tau')$ has the Haagerup property.*

Proof $(\Rightarrow)$. Suppose that $\{\Phi_j\}_{j\in J}$ is a net of u.c.p. maps witnessing the Haagerup property of (A,τ) and $\{T_i\}_{i\in I}$ is a net as in Definition 3.3 witnessing the quasi-amenability of α with respect to τ, where T_i is supported on F_i. Now we define a map $F_{i,j}(s)$ $(i \in I, j \in J, s \in G)$ given by

$$F_{i,j}(s)(a) = \sum_{p\in G} T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_s(T_i(s^{-1}p)), \quad a \in A.$$

Using the same argument from [14, Corollary 4.6], we get $F_{i,j}$ is a c.p. Herz-Schur (A,G,α)-multiplier. Since

$$F_{i,j}(e)(a) = \sum_{p\in G} T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(a)))T_i(p), \quad a \in A,$$

we have

$$F_{i,j}(e)(1_A) = \sum_{p\in G} T_i(p)^2 = 1_A.$$

For any $a \in A^+$,

$$\begin{aligned}(\tau \circ F_{i,j}(e))(a) &= \tau\Big(\sum_{p\in G} T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(a)))T_i(p)\Big) = \sum_{p\in G} \tau(T_i(p))\tau(\alpha_p(\Phi_j(\alpha_p^{-1}(a))))\tau(T_i(p))\\ &= \sum_{p\in G} \tau(\alpha_p(\Phi_j(\alpha_p^{-1}(a))))\tau(T_i(p)^2) \leq \sum_{p\in G} \tau(a)\tau(T_i(p)^2) = \tau(a),\end{aligned}$$

where we use bimodule property of multiplicative domain and $\tau \circ \Phi_j \leq \tau$.

Since $\alpha_s(T_i(s^{-1}p)) \in A_\tau$ and

$$L_{T_i(p)}R_{\alpha_s(T_i(s^{-1}p))} = R_{\alpha_s(T_i(s^{-1}p))}L_{T_i(p)},$$

for any $s, p \in G$, we have

$$\begin{aligned}F_{i,j}(s)(a) &= \sum_{p\in G} T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_s(T_i(s^{-1}p)) \\ &= \sum_{p\in G} L_{T_i(p)} R_{\alpha_s(T_i(s^{-1}p))}\alpha_p\Phi_j\alpha_p^{-1}(a)\end{aligned}$$

for all $a \in A$. For every $t \in G$, we get

$$\|\Lambda_\tau(\alpha_t(a))\|_{2,\tau}^2 = \tau(\alpha_t(a)^*\alpha_t(a)) = \tau(a^*a) = \|\Lambda_\tau(a)\|_{2,\tau}^2$$

for all $a \in A$. Hence, there exists an isometry $\hat{\alpha}_t$ on $L^2(A,\tau)$ such that $\hat{\alpha}_t(\Lambda_\tau(a)) = \Lambda_\tau(\alpha_t(a))$ for all $a \in A$. Therefore, we have

$$T_{F_{i,j}(s)}(\Lambda_\tau(a)) = \sum_{p\in G} T_{L_{T_i(p)}} T_{R_{\alpha_s(T_i(s^{-1}p))}} \hat{\alpha}_p T_{\Phi_j} \hat{\alpha}_{p^{-1}}(\Lambda_\tau(a)). \tag{3.2}$$

It follows that $T_{F_{i,j}(s)}$ is a compact operator on $L^2(A,\tau)$ by the fact that the space $K(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$ is a closed ideal of $B(\mathcal{H})$.

By Eq. (3.2), we have

$$\|T_{F_{i,j}(s)}\| \le \sum_{p\in G} \|T_i(p)\|\|\alpha_s(T_i(s^{-1}p))\|\|\hat{\alpha}_p \circ T_{\Phi_j} \circ \hat{\alpha}_{p^{-1}}\|.$$

Since T_i is supported on a finite set F_i, it is easy to see that

$$\Big\{s \in G : \sum_{p\in G} \|T_i(p)\|\|\alpha_s(T_i(s^{-1}p))\|\|\hat{\alpha}_p \circ T_{\Phi_j} \circ \hat{\alpha}_{p^{-1}}\| \neq 0\Big\}$$

is finite. Hence for any $\varepsilon > 0$, the set $\{s \in G : \|T_{F_{i,j}(s)}\| > \varepsilon\}$ is finite, i.e., $\|T_{F_{i,j}(s)}\| \to 0$ as $s \to \infty$, for any $i \in I, j \in J$.

Finally, we show that $\|\Lambda_\tau(F_{i,j}(s)(a) - a)\|_{2,\tau} \to 0$ for all $s \in G, a \in A$. Indeed,

$$\begin{aligned}\|\Lambda_\tau(F_{i,j}(s)(a) - a)\|_{2,\tau} =& \Big\|\Lambda_\tau\Big(\sum_{p\in G} T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(a)))\alpha_s(T_i(s^{-1}p)) - a\Big)\Big\|_{2,\tau} \\ \le& \Big\|\Lambda_\tau\Big(\sum_{p\in G} T_i(p)\alpha_p(\Phi_j(\alpha_p^{-1}(a)) - \alpha_p^{-1}(a))\alpha_s(T_i(s^{-1}p))\Big)\Big\|_{2,\tau} + \\ & \Big\|\Lambda_\tau\Big(\sum_{p\in G} T_i(p)a\alpha_s(T_i(s^{-1}p)) - a\Big)\Big\|_{2,\tau} \\ \le& \|T_{F_{i,j}(s)}(\Lambda_\tau(\Phi_j(\alpha_p^{-1}(a)) - \alpha_p^{-1}(a)))\|_{2,\tau} + \\ & \Big\|\sum_{p\in G} T_i(p)a\alpha_s(T_i(s^{-1}p)) - a\Big\|.\end{aligned}$$

The former converges to zero as $\|\Lambda_\tau(\Phi_j(x) - x)\|_{2,\tau} \to 0$ for all $x \in A$, the latter converges to zero by Eq. (3.1). Hence, (A, G, α, τ) has the Haagerup property. It follows from Theorem 2.9 that $(A \rtimes_{\alpha,r} G, \tau')$ has the Haagerup property.

($\Leftarrow$). It follows from [11, Corollary 3.14]. □

As a special case of the above result, we give a new proof of [11, Theorem 3.19]. If $F \subseteq G$ is a finite subset, we denote by $|F|$ the number of elements in F and χ_F the characteristic function over F. Let $E, F \subseteq G$ be two subsets, and the symmetric difference of them be

$E \triangle F = E \cup F \backslash E \cap F$.

Corollary 3.6 *Let A be a unital C^*-algebra with a state τ and G be a discrete amenable group acting on A through a τ-preserving action α. If (A, τ) has the Haagerup property, then $(A \rtimes_{\alpha,r} G, \tau')$ has the Haagerup property.*

Proof Let F_n be a Følner sequence. Define $T_n : G \to A$ by

$$T_n(t) = \frac{1_A}{\sqrt{|F_n|}} \chi_{F_n}(t)$$

for all $t \in G$. It is obvious that T_n is finitely supported, $T_n(t) \in Z(A)^+ \cap A_\tau$ for all $t \in G$ and $\sum_{t\in G} T_n(t)^2 = 1_A$. We just need to check that

$$\Big\| \sum_{s\in G} (T_n(s) - \alpha_t(T_n(t^{-1}s)))^* (T_n(s) - \alpha_t(T_n(t^{-1}s))) \Big\| \to 0$$

for every $t \in G$. Indeed,

$$\begin{aligned} \Big\| \sum_{s\in G} \Big(\frac{1_A}{\sqrt{|F_n|}} \chi_{F_n}(s) - \frac{1_A}{\sqrt{|F_n|}} \chi_{F_n}(t^{-1}s)\Big)^2 \Big\| &= \Big| 2 - \frac{2}{|F_n|} \sum_{s\in G} \chi_{F_n}(s) \cdot \chi_{F_n}(t^{-1}s) \Big| \\ &= |2 - 2\frac{|F_n \cap tF_n|}{|F_n|}| = \frac{|tF_n \triangle F_n|}{|F_n|} \to 0 \end{aligned}$$

for every $t \in G$. Therefore, it follows from Remark 3.4 and Theorem 3.5 that $(A \rtimes_{\alpha,r} G, \tau')$ has the Haagerup property. □

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