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Generalized Well-Posedness and Stability of Solutions in Set Optimization

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Abstract The aim of this paper is to investigate the well-posedness and stability in set optimization. The notion of generalized well-posedness for set optimization problems is introduced using the embedding technique for the first time. Some criteria and characterizations of this type of well-posedness are derived. Sufficient conditions are also given for this type of well-posedness. Moreover, by virtue of a generalized Gerstewitz's function, the equivalent relation between this type of well-posedness and the generalized well-posedness of a scalar optimization problem is established. Finally, the upper semi-continuity and lower semi-continuity of weak efficient solution mappings for parametric set optimization problems are investigated under some suitable conditions.

Keywords well-posedness; stability; set optimization; Gerstewitz's function; upper semi-continuity; lower semi-continuity

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1. Introduction

During the last few years, set-valued optimization which is a generalization of vector optimization, has received much attention due to its wide applications in many areas such as fuzzy optimization, game theory, control theory and mathematical economics. For more details we refer to [1].

It is well known that there are two types of criteria of solutions for set-valued optimization problem: vector criterion [2, 3] and set optimization criterion [4, 5]. Vector criterion consists of finding the solutions that give efficient points of image set of the objective set-valued map. Set optimization was introduced by Kuroiwa [6]. This criterion depends on comparisons among values of the set-valued map and this concept requires order relations to compare sets. The last criterion seems to be more natural and is used in this paper.

Well-posedness plays a crucial role in the study of stability theory in optimization. The study of well-posedness of an optimization problem is to investigate the behavior of the variable when the corresponding objective function value is close to the optimal value. The notion of

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well-posedness for optimization problems was firstly introduced by Tykhonov [7] in 1966. Since then, many authors have extended the notion of well-posedness for scalar and vector optimization problems, we refer the readers to [8–13]. Recently, many authors have studied well-posedness in set optimization. Zhang et al. [14] introduced a notion of pointwise well-posedness and two notions of global well-posedness in set optimization and obtained their scalar characterization. Using the scalar results, the authors derived some criteria and characterizations for all the three types of well-posedness. This research was generalized by Gutierrez et al. [15] under assumptions of cone properness. Long and Peng [16] studied various types of well-posedness in the sense of Bednarczuk in set optimization. Crespi et al. [17] linked some already existing notions of well-posedness with the upper semicontinuity and compactness of solution maps. Vui et al. [18] studied various types of Levitin-Polyak (LP) well-posedness in set optimization. Gupta and Srivastava [19] defined new types of well-posedness in set optimization and derived their necessary and sufficient conditions. To the best of our knowledge, however, there is still no paper concerning the well-posedness under perturbation in set optimization.

On the other hand, the stability of solutions is a very interesting topic in the study of set optimization. Recently, Xu and Li [20] obtained the upper semi-continuity and lower semi-continuity of the minimal solution and weak minimal solution mappings for a parametric set optimization problem by using converse u-property of objective mappings. Han and Huang [21] studied the upper semi-continuity and lower semi-continuity of minimal solution mappings for parametric set optimization problems by using the level mappings. Khoshkhabar-amiranloo [22] discussed the upper semi-continuity, lower semi-continuity and compactness of minimal solution mappings for parametric set optimization problems whose objective values are not necessarily compact. Zhang and Huang [23] studied the upper semi-continuity, lower semi-continuity and compactness of minimal solution mappings for parametric set optimization involving the cone Lipschitz continuous set-valued mapping.

Motivated by these works, in this paper, we aim to investigate the well-posedness under perturbation for set optimizations and stability of solution mappings for a new parametric set optimization problem. We introduce the notion of generalized well-posedness by embedding the original set optimization problem in a family of perturbed problems depending on a parameter. It is worth mentioning that generalized well-posedness includes the extended k_0-well-posedness studied in [14] as a special case. Some criteria and characterizations of this type of well-posedness are derived. Sufficient conditions are also given for this type of well-posedness. Moreover, using a generalized version of so-called nonlinear scalarization functional [24], we establish the equivalent relation between this type of well-posedness and the generalized well-posedness of a suitable scalar optimization problem. Finally, the upper semi-continuity and lower semi-continuity of weak efficient solution mappings are investigated under some suitable conditions.

2. Preliminaries

Let X and Y be normed spaces. Denote by $\mathcal{B}_X$ and $\mathcal{B}_Y$ the closed unit balls, respectively, in X and Y. The family of all nonempty subsets of Y is denoted by $P_0(Y)$. The space Y is ordered by a convex closed and pointed cone $K \subset Y$ with its topological interior $\operatorname{int} K \neq \emptyset$, in the following way:

$$x \leq y \Leftrightarrow y - x \in K,$$
$$x < y \Leftrightarrow y - x \in \operatorname{int} K.$$

Throughout we assume $e \in \operatorname{int} K$ to be a fixed element. $A \subset Y$ is said to be K-closed if $A + K$ is a closed set, K-bounded if for each neighborhood U of 0 in Y there is some positive number t such that $A \subset tU + K$ and K-compact if any cover of A of the form $\{U_\alpha + K |\ U_\alpha$ are open$\}$ admits a finite subcover.

Assume that P is a metric space, p^* is a fixed point in P, and that L is a closed ball in P centered at p^* with a positive radius. Let S be a nonempty subset of X, let $J : S \rightrightarrows Y$ and $I : S \times L \rightrightarrows Y$ be set-valued mappings such that

$$I(x, p^*) = J(x), \quad \forall\, x \in S.$$

The problem is set as follows. The original set optimization problem is described by:

$$(S, J) : \min_{x \in S} J(x).$$

The parametric set optimization problem is described by:

$$(S, I(\cdot, p)) : \min_{x \in S} I(x, p).$$

Note that the original problem (S, J) is consistent with problem $(S, I(\cdot, p^*))$. $(S, I(\cdot, p))$ (for short (p) if no confusion arises) is called the perturbed problem of the original problem corresponding to the parameter $p \in L$.

To study the set optimization problem (p), we will consider the set relation introduced by Kuroiwa [6] as follows.

Definition 2.1 *Let $A, B \in P_0(Y)$. The lower set less relation $\prec^l_K$ is defined by*

$$A \prec^l_K B \Leftrightarrow B \subset A + \operatorname{int} K \Leftrightarrow \forall\, b \in B, \exists\, a \in A \text{ s.t. } a < b.$$

The negation of $A \prec^l_K B$ is denoted by $A \not\prec^l_K B$, that is, $B \not\subset A + \operatorname{int} K$. In general, $\prec^l_K$ is not a preorder since it is not reflexive.

In particular, if A and B are singletons, then $\prec^l_K$ and $<$ have the same meaning.

Using the set relation $\prec^l_K$, Kuroiwa [6] defined the concept of weak efficient solutions to set optimization problems.

Definition 2.2 *An element $x_0 \in S$ is said to be a weak efficient solution of (p) iff $I(x, p) \not\prec^l_K I(x_0, p)$, for all $x \in S$.*

We denote the set of all weak efficient solutions of (p) by $\mathrm{WEff}(p)$. Throughout we assume that $\mathrm{WEff}(p) \neq \emptyset$ for all $p \in L$. Note that $\mathrm{WEff}(p^*)$ is the set of all weak efficient solutions of (S, J).

Let $F : S \rightrightarrows Y$ be a set-valued mapping. The graph of F, denoted by $\operatorname{graph} F$, is defined as

$$\operatorname{graph} F := \{(x, y) \in S \times Y : y \in F(x)\}.$$

Definition 2.3 ([14]) *The mapping F is said to be bounded-valued (respectively closed-valued, compact-valued, open-valued, K-bounded-valued, K-closed-valued, K-compact-valued and so on) if for any $x \in S$, the set $F(x)$ is a bounded set (respectively a closed set, a compact set, an open set, a K-bounded set, a K-closed set, a K-compact set and so on).*

We now recall the notion of compactness of set-valued mappings.

Definition 2.4 ([17]) *The mapping F is said to be compact at $x_0 \in S$ if for every sequence $\{(x_n, y_n)\} \subset \operatorname{graph} F$ with $x_n \to x_0$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0 \in F(x_0)$.*

Also, we say that F is compact on S if F is compact at every $x_0 \in S$.

We next recall the concepts of semi-continuity of set-valued mappings.

Definition 2.5 ([1]) *The mapping F is said to be*

(i) Upper semi-continuous (for short, u.s.c.) at $x_0 \in S$ if for any neighborhood V of $F(x_0)$, there exists a neighborhoood U of x_0 such that $F(x) \subset V$ for all $x \in U \cap S$;

(ii) Lower semi-continuous (for short, l.s.c.) at $x_0 \in S$ if for any $y \in F(x_0)$ and any neighborhood V of y, there exists a neighborhoood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U \cap S$;

(iii) Hausdorff lower semi-continuous (for short, H-l.s.c.) at $x_0 \in S$ if for any neighborhood V of $0 \in Y$, there exists a neighborhoood U of x_0 such that $F(x_0) \subset F(x) + V$ for all $x \in U \cap S$.

Remark 2.6 F is compact at $x_0 \in S$ iff $F(x_0)$ is compact and F is u.s.c. at x_0.

Definition 2.7 ([25]) *The mapping F is said to be K-upper semicontinuous (for short, K-u.s.c.) at $x_0 \in S$ if for any $d \in \operatorname{int} K$, there exists a neighborhood U of x_0 such that $F(x_0) + d \subset F(x) + \operatorname{int} K$ for all $x \in U \cap S$.*

Definition 2.8 *We say that the mapping F has the $\prec_K^l$-continuous property at $x_0 \in S$ with respect to $y_0 \in S$ if, either $F(y_0) \prec_K^l F(x_0)$ or for any sequence $\{x_n\}, \{y_n\} \subset S$ with $x_n \to x_0, y_n \to y_0$, there exists $n \in \mathbb{N}$ such that $F(y_n) \not\prec_K^l F(x_n)$.*

Also, we say that F has the $\prec_K^l$-continuous property on S if F has the $\prec_K^l$-continuous property at each $x_0 \in S$ with respect to each $y_0 \in S$.

We give the following example to illustrate Definition 2.8.

Example 2.9 Let $X = Y = \mathbb{R}$, $S = [0, 2]$, $K = \mathbb{R}_+$ and $F : S \rightrightarrows Y$ be defined by

$$F(x) = (-1, x], \quad \forall\, x \in [0, 2].$$

Then, it is easy to check that F has the $\prec_K^l$-continuous property on S.

3. Generalized well-posedness

In this section, we will introduce the notion of generalized well-posedness for (S, J) and investigate its characterizations and sufficient conditions.

Definition 3.1 *Let $\varepsilon \geq 0$. A point $x_\varepsilon \in S$ is said to be an ε-weak efficient solution of (p), if there is no $x \in S$ such that $I(x,p) + \varepsilon e \prec_K^l I(x_\varepsilon, p)$.*

We denote the set of all ε-weak efficient solutions of (p) by ε-WEff(p). If $\varepsilon = 0$, then the set ε-WEff(p) is consistent with WEff(p).

Remark 3.2 Clearly, if $0 \leq \varepsilon_1 \leq \varepsilon_2$, then for all $p \in L$,

$$\mathrm{WEff}(p) \subset \varepsilon_1\text{-}\mathrm{WEff}(p) \subset \varepsilon_2\text{-}\mathrm{WEff}(p).$$

Definition 3.3 *Let $\{p_n\} \subset L$ be such that $p_n \to p^*$. A sequence $\{x_n\} \subset S$ is said to be a generalized e-minimizing sequence corresponding to p_n iff there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that*

$$I(x, p_n) + \varepsilon_n e \not\prec_K^l I(x_n, p_n), \quad \forall\, x \in S.$$

Remark 3.4 (i) Let $\{p_n\} \subset L$ be such that $p_n \to p^*$. Then $\{x_n\} \subset S$ is a generalized e-minimizing sequence corresponding to p_n iff there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that $x_n \in \varepsilon_n$-WEff(p_n);

(ii) If $p_n \equiv p^*$, then $\{x_n\}$ is an extended e-minimizing sequence in [14].

Definition 3.5 *(S, J) is said to be generalized well-posed iff for any $\{p_n\} \subset L$ such that $p_n \to p^*$, each generalized e-minimizing sequence corresponding to p_n has a subsequence that converges to an element of* WEff(p^*).

Remark 3.6 (i) Definition 3.5 generalizes Definition 2.3 of [14] to the perturbed case;

(ii) (S, J) is generalized well-posed iff WEff(p^*) is compact and for any $\{p_n\} \subset L$ such that $p_n \to p^*$, each generalized e-minimizing sequence corresponding to p_n has a subsequence $\{x_{n_k}\}$ such that $d(x_{n_k}, \mathrm{WEff}(p^*)) \to 0$.

We now give the following example to illustrate Definition 3.5.

Example 3.7 Let $X = P = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = (0, +\infty)$, $K = \mathbb{R}^2_+$, $e = (1, 1)$, $p^* = 0$, $L = [-\frac{1}{2}, \frac{1}{2}]$ and let $J : S \rightrightarrows Y$ and $I : S \times L \rightrightarrows Y$ be defined by

$$J(x) = (x - n, n) + [0, 1] \times [0, 1], \quad x \in (n, n + 1],\ n = 0, 1, \ldots$$

and

$$I(x, p) = (x - n, n) + [p^2, 1 - p^2] \times [p^2, 1 - p^2], \quad x \in (n, n + 1],\ n = 0, 1, \ldots.$$

Then, the problem (S, J) is generalized well-posed.

Now we consider some criteria and characterizations of generalized well-posedness for (S, J). Given a set $A \subset S$, the Kuratowski measure of noncompactness of A is defined as follows:

$$\alpha(A) = \inf\{k > 0 : A \text{ has a finite cover of sets with diameter } < k\}.$$

When p is near p^*, let $T(\varepsilon) = \bigcup\{\varepsilon\text{-WEff}(p) : \rho(p, p^*) < \varepsilon\}$ for any $\varepsilon > 0$. We need the following condition:

$$\alpha(T(\varepsilon)) \to 0, \quad \text{as } \varepsilon \to 0 \tag{3.1}$$

Theorem 3.8 *Assume that:*

(i) $T(\varepsilon)$ is closed for all $\varepsilon > 0$;

(ii) J is open-valued and compact-valued;

(iii) $I(x, \cdot)$ is K-u.s.c. at p^* for all $x \in S$;

(iv) $I(x, p) \subset J(x)$ for all $(x, p) \in S \times L$.

Then (S, J) is generalized well-posed iff (3.1) holds.

Proof First, it follows from Remark 3.2 that $T(\varepsilon) \neq \emptyset$ for all $p \in L$. By (i), similar to Theorem 3.2 in [11], we only need to verify

$$\mathrm{WEff}(p^*) = \bigcap_{\varepsilon>0} T(\varepsilon).$$

As $\mathrm{WEff}(p^*) \subset T(\varepsilon)$, for every $\varepsilon > 0$, it is clear that $\mathrm{WEff}(p^*) \subset \bigcap_{\varepsilon>0} T(\varepsilon)$. Now we show that $\bigcap_{\varepsilon>0} T(\varepsilon) \subset \mathrm{WEff}(p^*)$. Let $x^* \in \bigcap_{\varepsilon>0} T(\varepsilon)$. Thus there exists a sequence $\{p_n\} \subset L$ satisfying $\rho(p_n, p^*) < \varepsilon_n$, where $\varepsilon_n \downarrow 0$, such that $x^* \in \varepsilon_n\text{-WEff}(p_n)$ for all $n \in \mathbb{N}$. Note that we also have $p_n \to p^*$. Let $x \in S$, then

$$I(x^*, p_n) \not\subset I(x, p_n) + \varepsilon_n e + \operatorname{int} K.$$

Thus there exists $y_n \in I(x^*, p_n)$ such that

$$y_n - \varepsilon_n e \notin I(x, p_n) + \operatorname{int} K.$$

Using (iii), for any $d \in \operatorname{int} K$, there exists $n_0 \in \mathbb{N}$ such that

$$I(x, p^*) + d \subset I(x, p_n) + \operatorname{int} K, \quad \forall\, n > n_0.$$

This implies that

$$y_n - \varepsilon_n e \notin J(x) + d, \quad \forall\, n > n_0. \tag{3.2}$$

From (iv) we obtain $y_n \in J(x^*)$ and by (ii) we have $J(x^*)$ is compact. Thus there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y^* \in J(x^*)$ such that $y_{n_k} \to y^*$. Also, by (ii) we have $J(x) + d$ is open. Hence by taking limits in (3.2) it follows that

$$y^* \notin J(x) + d. \tag{3.3}$$

Since d is arbitrary, it follows that (3.3) holds for all $d \in \operatorname{int} K$. Therefore,

$$y^* \notin J(x) + \operatorname{int} K,$$

which implies

$$J(x^*) \not\subseteq J(x) + \operatorname{int} K, \quad \forall\, x \in S.$$

That is, $x^* \in \mathrm{WEff}(p^*)$. The proof is completed. □

Remark 3.9 Theorem 3.2 in [11] uses the Kuratowski noncompactness measure to study the characterization of strongly extended well-posedness of set-valued vector optimization problems. In this paper, Theorem 3.8 uses the same tool to give the characterization of generalized well-posedness for set optimization problems. The proof method is different from [11].

Now we define approximate solution mapping $M : \mathbb{R}_+ \times L \rightrightarrows S$ by

$$M(\varepsilon, p) = \varepsilon\text{-WEff}(p), \quad \forall\, (\varepsilon, p) \in \mathbb{R}_+ \times L.$$

Proposition 3.10 *Suppose that* $\mathrm{WEff}(p^*) \subset \mathrm{WEff}(p)$ *for all* $p \in L$, *then the approximate solution mapping* $M(\cdot,\cdot)$ *is H-l.s.c. at* $(0, p^*)$.

Proof Suppose the contrary that $M(\cdot,\cdot)$ is not H-l.s.c. at $(0, p^*)$. Then there exists $\delta > 0$ such that for any neighborhood $U \times W$ of $(0, p^*)$, there exists $(\varepsilon, p) \in U \times W$ satisfying

$$M(0, p^*) \not\subset M(\varepsilon, p) + B(0, \delta). \tag{3.4}$$

Observe that $M(0, p^*) = \mathrm{WEff}(p^*) \subset \mathrm{WEff}(p)$, it follows from Remark 3.1 that

$$M(0, p^*) \subset M(\varepsilon, p) \subset M(\varepsilon, p) + B(0, \delta),$$

which contradicts (3.4). Therefore, $M(\cdot,\cdot)$ is H-l.s.c. at $(0, p^*)$. □

Theorem 3.11 (S, J) *is generalized well-posed iff* $\mathrm{WEff}(p^*)$ *is compact and the approximate solution mapping* $M(\cdot,\cdot)$ *is u.s.c. at* $(0, p^*)$.

Proof Suppose that (S, J) is generalized well-posed. Let $\{x_n\}$ be a sequence in $\mathrm{WEff}(p^*)$. Since $\mathrm{WEff}(p^*) \subset \varepsilon\text{-WEff}(p^*)$, there exists $\varepsilon_n \downarrow 0$ such that $x_n \in \varepsilon_n\text{-WEff}(p^*)$. It follows from Remark 3.4 (i) that $\{x_n\}$ is a generalized e-minimizing sequence corresponding to $p_n \equiv p^*$. Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x^* \in \mathrm{WEff}(p^*)$ such that $x_{n_k} \to x^*$. This implies $\mathrm{WEff}(p^*)$ is compact. It remains to verify $M(\cdot,\cdot)$ is u.s.c. at $(0, p^*)$. Suppose not, then there exists a neighborhood V_0 of $M(0, p^*)$ such that for any neighborhood $U \times W$ of $(0, p^*)$, there exists $(\varepsilon, p) \in U \times W$ satisfying

$$M(\varepsilon, p) \not\subset V_0.$$

Thus, we can choose $(\varepsilon_n, p_n) \to (0, p^*)$, which satisfies $\exists\, x_n \in M(\varepsilon_n, p_n)$ such that $x_n \notin V_0$. Then it follows from Remark 3.4 (i) that $\{x_n\}$ is a generalized e-minimizing sequence corresponding to p_n, hence there exists a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \to x_0 \in \mathrm{WEff}(p^*) \subset V_0.$$

This gives a contradiction to $x_{n_k} \notin V_0$.

Conversely, let $p_n \to p^*$ and $\{x_n\}$ be a generalized e-minimizing sequence corresponding to p_n. It follows from Remark 3.4 (i) that there exists $\varepsilon_n \to 0$ such that $x_n \in M(\varepsilon_n, p_n)$. Since $M(\cdot,\cdot)$ is u.s.c. at $(0, p^*)$ and $(\varepsilon_n, p_n) \to (0, p^*)$, it follows that for any neighborhood V of $0 \in X$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n \in M(\varepsilon_n, p_n) \subset \mathrm{WEff}(p^*) + V, \quad \forall\, n > n_0.$$

By the compactness of $\mathrm{WEff}(p^*)$, we obtain that (S, J) is generalized well-posed. The proof is completed. □

Corollary 3.12 *(S, J) is generalized well-posed iff $M(\cdot,\cdot)$ is compact at $(0, p^*)$.*

We next give sufficient conditions for generalized well-posedness.

Theorem 3.13 *Assume that:*

(i) S is a compact set;

(ii) J is open-valued and compact on S;

(iii) $I(x,\cdot)$ is K-u.s.c. at p^* for all $x \in S$;

(iv) $I(x,p) \subset J(x)$ for all $(x,p) \in S \times L$.

Then (S, J) is generalized well-posed.

Proof Let $p_n \to p^*$ and $\{x_n\}$ be a generalized e-minimizing sequence corresponding to p_n. Then it follows from Remark 3.4 (i) that there exists $\varepsilon_n \to 0$ such that $x_n \in \varepsilon_n$-WEff(p_n). Since S is compact, there exist a subsequence $\{x_{n_k}\}$ and $x^* \in S$ such that $x_{n_k} \to x^*$. Let $x \in S$. Now $x_{n_k} \in \varepsilon_{n_k}$-WEff$(p_{n_k})$, thus there exists $y_{n_k} \in I(x_{n_k}, p_{n_k})$ such that

$$y_{n_k} - \varepsilon_{n_k} e \notin I(x, p_{n_k}) + \operatorname{int} K.$$

Use (iii), for any $d \in \operatorname{int} K$, there exists $k_0 \in \mathbb{N}$ such that

$$I(x, p^*) + d \subset I(x, p_{n_k}) + \operatorname{int} K, \quad \forall\, k > k_0.$$

This implies that

$$y_{n_k} - \varepsilon_{n_k} e \notin J(x) + d, \quad \forall\, k > k_0. \tag{3.5}$$

Using (iv), we obtain $y_{n_k} \in J(x_{n_k})$ for all $k \in \mathbb{N}$. Since J is compact at x^*, there exist a subsequence of $\{y_{n_k}\}$ and $y^* \in J(x^*)$ such that it converges to y^*. Also, by (ii) we have $J(x) + d$ is open. Hence by taking limits in (3.5) it follows that

$$y^* \notin J(x) + d. \tag{3.6}$$

Since d is arbitrary, it follows that (3.6) holds for all $d \in \operatorname{int} K$. Therefore,

$$y^* \notin J(x) + \operatorname{int} K,$$

which implies

$$J(x^*) \nsubseteq J(x) + \operatorname{int} K, \quad \forall\, x \in S.$$

That is, $x^* \in$ WEff(p^*). This completes the proof. □

The following example shows that the compactness of S cannot be dropped in Theorem 3.13.

Example 3.14 Let $X = P = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = (0, 1]$, $K = \mathbb{R}^2_+$, $e = (1, 1)$, $p^* = 1$, $L = [0, 2]$ and let $J : S \rightrightarrows Y$ and $I : S \times L \rightrightarrows Y$ be defined by

$$J(x) = \begin{cases} \{(x, x)\} + \mathbb{R}^2_{++}, & \text{if } x \in (0, 1); \\ \mathbb{R}^2_{++}, & \text{if } x = 1; \end{cases}$$

and

$$I(x, p) = \begin{cases} \{(x + p, x + p)\} + \mathbb{R}^2_{++}, & \text{if } x \in (0, 1); \\ \mathbb{R}^2_{++}, & \text{if } x = 1 \end{cases}, \quad \forall\, p \neq 1.$$

It can be seen that $\mathrm{WEff}(p^*) = \{1\}$. Here, (ii)–(iv) hold but S is not compact. If $p_n = 1 + \frac{1}{n}$ and $x_n = \varepsilon_n = \frac{1}{n}$, then $\{x_n\}$ is a generalized e-minimizing sequence corresponding to p_n and $x_n \to 0$, but $0 \notin \mathrm{WEff}(p^*)$. Thus, (S, J) is not generalized well-posed.

Theorem 3.15 *Assume that:*

(i) X *is a finite dimensional normed space;*

(ii) $\mathrm{WEff}(p^*)$ *is a compact set;*

(iii) J *is open-valued and compact on* S*;*

(iv) $I(x,\cdot)$ *is* K*-u.s.c. at* p^* *for all* $x \in S$*;*

(v) $I(x,p) \subset J(x)$ *for all* $(x,p) \in S \times L$*;*

(vi) $\mathrm{WEff}(p^*) \subset \mathrm{WEff}(p)$ *for all* $p \in L$*;*

(vii) There exists $\delta > 0$ *such that* ε*-WEff(*p*) is connected for every* $(\varepsilon, p) \in (0,\delta) \times B(p^*, \delta)$.

Then (S, J) *is generalized well-posed.*

Proof Suppose (S, J) is not generalized well-posed. Since $\mathrm{WEff}(p^*)$ is compact, it follows from Remark 3.6 (ii) there exist a sequence $\{p_n\} \subset L$ with $p_n \to p^*$ and a generalized e-minimizing sequence $\{x_n\}$ corresponding to p_n such that $d(x_n, \mathrm{WEff}(p^*)) \not\to 0$. Since $\{x_n\}$ is a generalized e-minimizing sequence corresponding to p_n, it follows from Remark 3.4 (i) that there exists $\varepsilon_n \to 0$ such that $x_n \in \varepsilon_n$-$\mathrm{WEff}(p_n)$. As $d(x_n, \mathrm{WEff}(p^*)) \not\to 0$ it follows that there exist a subsequence $\{x_{n_k}\}$ and $\alpha > 0$ such that

$$x_{n_k} \notin \mathrm{WEff}(p^*) + \alpha\mathcal{B}_X.$$

Therefore,

$$x_{n_k} \in \varepsilon_{n_k}\text{-}\mathrm{WEff}(p_{n_k}) \cap (\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X)^c,$$

which implies

$$\varepsilon_{n_k}\text{-}\mathrm{WEff}(p_{n_k}) \cap (\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X)^c \neq \emptyset. \tag{3.7}$$

Also, by (vi) we have $\mathrm{WEff}(p^*) \subset \mathrm{WEff}(p_{n_k}) \subset \varepsilon_{n_k}$-$\mathrm{WEff}(p_{n_k})$. Thus,

$$\varepsilon_{n_k}\text{-}\mathrm{WEff}(p_{n_k}) \cap \mathrm{int}\,(\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X) \neq \emptyset. \tag{3.8}$$

Since $(\varepsilon_{n_k}, p_{n_k}) \to (0, p^*)$, it follows that for $\delta > 0$ given in (vii), there exists $k_0 \in \mathbb{N}$, such that for any $k > k_0$, we have $\varepsilon_{n_k} < \delta$ and $\rho(p_{n_k}, p^*) < \delta$. Now we claim that

$$\varepsilon_{n_k}\text{-}\mathrm{WEff}(p_{n_k}) \cap \partial(\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X) \neq \emptyset, \quad \forall\, k > k_0.$$

Suppose the contrary, then there exists $k > k_0$ such that

$$\varepsilon_{n_k}\text{-}\mathrm{WEff}(p_{n_k}) \subset (\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X)^c \cup \mathrm{int}\,(\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X). \tag{3.9}$$

From (3.7), (3.8) and (3.9), this arrives at a contradiction to the fact that ε_{n_k}-$\mathrm{WEff}(p_{n_k})$ is connected. Therefore, there exists a sequence $\{\omega_{n_k}\}$ such that $\omega_{n_k} \in \varepsilon_{n_k}$-$\mathrm{WEff}(p_{n_k}) \cap \partial(\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X)$ for all $k > k_0$. Since $\partial(\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X)$ is a compact set, there exists a subsequence, which we again denote by $\{\omega_{n_k}\}$ such that $\omega_{n_k} \to \omega^*$ and

$$\omega^* \in \partial(\mathrm{WEff}(p^*) + \alpha\mathcal{B}_X). \tag{3.10}$$

Let $x \in S$. For $k > k_0$, as $\omega_{n_k} \in \varepsilon_{n_k}$-WEff$(p_{n_k})$, there exists $y_{n_k} \in I(\omega_{n_k}, p_{n_k})$ such that

$$y_{n_k} - \varepsilon_{n_k} e \notin I(x, p_{n_k}) + \operatorname{int} K.$$

Proceeding as in Theorem 3.13, we derive that $\omega^* \in \mathrm{WEff}(p^*)$. It is a contradiction with (3.10) and the proof is completed. □

Remark 3.16 It may be observed in Example 3.14 that for every $(\varepsilon, p) \in (0,1) \times [0,2]$, the set ε-WEff$(p) = (0, \varepsilon] \cup \{1\}$ is not connected. Since (S, J) is not generalized well-posed, therefore (vii) cannot be dropped in Theorem 3.15.

Remark 3.17 Both Theorems 3.13 and 3.15 give sufficient conditions for generalized well-posedness, but their hypotheses are slightly different. Moreover, the proof method used in the two theorems are different.

4. Scalarization results

In this section, we will discuss the equivalent relation between the generalized well-posedness for (S, J) and the generalized well-posedness for a suitable scalar optimization problem by using a nonlinear scalarization function.

First we recall the definition of generalized well-posedness for a scalar optimization problem in [26]. Let $f : S \to \mathbb{R}$ be a real-valued function. Consider the following scalar optimization problem:

$$(S, f) : \min_{x \in S} f(x).$$

We denote the set of all minimizers of (S, f) by $\operatorname{argmin}(S, f)$. Then (S, f) is said to be generalized well-posed in the scalar sense iff $\operatorname{argmin}(S, f)$ is not empty, and every sequence $\{x_n\} \subset S$ such that $f(x_n) \to \inf f(S)$ has some subsequence $\{x_{n_k}\}$ converging to a minimizer of (S, f).

Based on the Gerstewitz's nonconvex separation function studied in [24], Hernández and Rodríguez-Marín [27] introduced a generalization of the Gerstewitz's function.

Definition 4.1 ([27]) *Let the function $G_e(\cdot, \cdot) : P_0(Y) \times P_0(Y) \to \mathbb{R} \cup \{\infty\}$ defined by setting*

$$G_e(A, B) = \sup_{b \in B} \phi_{e,A}(b), \quad \text{for} \quad (A, B) \in P_0(Y) \times P_0(Y),$$

where the function $\phi_{e,A} : Y \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\phi_{e,A}(y) = \inf\{t \in \mathbb{R} : y \in -te + A + K\}, \quad \text{for} \quad y \in Y.$$

Note that when $A = \{a\}$ and $B = \{y\}$, the function $G_e(A, B)$ reduces to the Gerstewitz's function $\phi_{e,a}(y)$.

We now recall the following important properties of $G_e(\cdot, \cdot)$.

Lemma 4.2 ([14]) *Assume $A, B \in P_0(Y)$, $r \in \mathbb{R}$, A is K-closed and B is K-bounded. Then*

(i) $G_e(A, A) = 0$;

(ii) $G_e(A, B + \varepsilon e) = G_e(A, B) - \varepsilon$, *for all* $\varepsilon \geq 0$;

(iii) $G_e(A, B) < r \Leftrightarrow B \subset A - re + \operatorname{int} K$;

(iv) *If* B_1 *and* B_2 *are* K*-compact sets and* $B_1 \prec_K^l B_2$*, then* $G_e(A, B_2) < G_e(A, B_1)$.

Proposition 4.3 *Suppose that* $I(\cdot, p)$ *is* K*-compact-valued for all* $p \in L$*. Then*

$$\bigcup_{p\in L} \mathrm{WEff}(p) = \operatorname{argmin}\Big(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p))\Big).$$

Proof Let $x^* \in \bigcup_{p\in L} \mathrm{WEff}(p)$. Then $x^* \in \mathrm{WEff}(p_0)$ for some p_0. We claim that for every $x \in S$, $G_e(I(x,p_0), I(x^*,p_0)) \geq 0$. Suppose on the contrary that there exist $x_0 \in S$ and $r < 0$ such that $G_e(I(x_0,p_0), I(x^*,p_0)) < r$. Then it follows from Lemma 4.2 (iii) that

$$I(x^*,p_0) \subset I(x_0,p_0) - re + \operatorname{int} K \subset I(x_0,p_0) + \operatorname{int} K,$$

that is, $I(x_0,p_0) \prec_K^l I(x^*,p_0)$, which gives a contradiction to $x^* \in \mathrm{WEff}(p_0)$. Therefore, $\inf_{x\in S} G_e(I(x,p_0), I(x^*,p_0)) \geq 0$. It follows that

$$\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x^*,p)) \geq 0. \tag{4.1}$$

Also, using Lemma 4.2 (i) we obtain for any $y \in S$

$$G_e(I(y,p), I(y,p)) = 0, \quad \forall\, p \in L.$$

This implies that

$$\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(y,p)) \leq 0. \tag{4.2}$$

Along with (4.1) we obtain for all $y \in S$

$$\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(y,p)) \geq -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x^*,p)), \tag{4.3}$$

i.e., $x^* \in \operatorname{argmin}(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p)))$. This implies

$$\bigcup_{p\in L} \mathrm{WEff}(p) \subset \operatorname{argmin}\Big(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p))\Big).$$

Now we prove the converse case. Let $x^* \in \operatorname{argmin}(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p)))$. Then (4.3) holds. Combining with (4.2), we have (4.1) holds. Therefore, there exists $p_0 \in L$ such that

$$\inf_{x\in S} G_e(I(x,p_0), I(x^*,p_0)) \geq 0.$$

Thus, for any $x \in S$, we have

$$G_e(I(x,p_0), I(x^*,p_0)) \geq 0 = G_e(I(x,p_0), I(x,p_0)).$$

Then it follows from Lemma 4.2 (iv) that $I(x,p_0) \not\prec_K^l I(x^*,p_0)$, which implies $x^* \in \mathrm{WEff}(p_0)$. This completes the proof. □

Theorem 4.4 *Suppose that* $I(\cdot,p)$ *is* K*-compact-valued and* $\mathrm{WEff}(p) \subset \mathrm{WEff}(p^*)$ *for all* $p \in L$*. Then* (S, J) *is generalized well-posed iff the scalar problem* $(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p)))$ *is generalized well-posed in the scalar sense.*

Proof First, using Proposition 4.3 and the fact that $\mathrm{WEff}(p) \subset \mathrm{WEff}(p^*)$ for all $p \in L$, we obtain

$$\mathrm{WEff}(p^*) = \mathrm{argmin}\Big(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p))\Big). \tag{4.4}$$

Let $p_n \to p^*$ and $\{x_n\}$ be a generalized e-minimizing sequence corresponding to p_n. Then there exists $\varepsilon_n \to 0$ such that for all $x \in S$

$$I(x_n,p_n) \not\subset I(x,p_n) + \varepsilon_n e + \mathrm{int}\, K.$$

From Lemma 4.2 (iii), we have

$$G_e(I(x,p_n), I(x_n,p_n)) \geq -\varepsilon_n, \quad \forall\, x \in S,$$

which implies

$$-\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x_n,p)) \leq \varepsilon_n.$$

Also, note that $G_e(I(x_n,p), I(x_n,p)) = 0$ for any $p \in L$. It follows that

$$-\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x_n,p)) \geq 0.$$

Hence, for all n

$$0 \leq -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x_n,p)) \leq \varepsilon_n.$$

Thus, $-\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x_n,p)) \to 0$. Since $(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p)))$ is well-posed, there exists a subsequence $\{x_{n_k}\}$ converging to a point

$$x^* \in \mathrm{argmin}\Big(S, -\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(\cdot,p))\Big).$$

From (4.4), $x^* \in \mathrm{WEff}(p^*)$. Therefore, (S,J) is generalized e-well-posed.

Conversely, assume that $\{x_n\}$ is a sequence which satisfies that $\exists\, \varepsilon_n \to 0$ such that

$$-\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x_n,p)) < \inf_{y\in S}\Big[-\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(y,p))\Big] + \varepsilon_n.$$

Thus, for all $y \in S$, we have

$$\sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(x_n,p)) > \sup_{p\in L}\inf_{x\in S} G_e(I(x,p), I(y,p)) - \varepsilon_n.$$

So for all $p \in L$, there exists $p_n \to p^*$ such that

$$\inf_{x\in S} G_e(I(x,p_n), I(x_n,p_n)) > \inf_{x\in S} G_e(I(x,p), I(y,p)) - \varepsilon_n.$$

Especially, we have

$$\inf_{x\in S} G_e(I(x,p_n), I(x_n,p_n)) > \inf_{x\in S} G_e(I(x,p_n), I(y,p_n)) - \varepsilon_n.$$

Thus, for any $y \in S$ and n, there exists $z_{y,n} \in S$ such that

$$\begin{aligned} G_e(I(z_{y,n},p_n), I(x_n,p_n)) &> G_e(I(z_{y,n},p_n), I(y,p_n)) - \varepsilon_n \\ &= G_e(I(z_{y,n},p_n), I(y,p_n) + \varepsilon_n e). \end{aligned}$$

It follows from Lemma 4.2 (iv) that $I(y,p_n)+\varepsilon_n e \not\preceq^l_K I(x_n,p_n)$ for all $y\in S$ and n. Hence, $\{x_n\}$ is a generalized e-minimizing sequence corresponding to p_n. Since (S,J) is generalized e-well-posed, there exists a subsequence $\{x_{n_k}\}$ converging to a point $x^*\in \mathrm{WEff}(p^*)$. From (4.4), $x^*\in \mathrm{argmin}(S,-\sup_{p\in L}\inf_{x\in S}G_e(I(x,p),I(\cdot,p)))$. Therefore, $(S,-\sup_{p\in L}\inf_{x\in S}G_e(I(x,p),I(\cdot,p)))$ is generalized well-posed in the scalar sense. The proof is completed. □

5. Stability of the solution mappings

Let $W: L\rightrightarrows S$ denote weak efficient solution mapping of the parametric set optimization problem (p), i.e.,

$$W(p)=\mathrm{WEff}(p)=\{x_0\in S: I(x,p)\not\preceq^l_K I(x_0,p),\forall\, x\in S\},\quad \forall p\in L.$$

In this section, we establish the upper semi-continuity and lower semi-continuity of W.

Theorem 5.1 *Assume that:*

(i) S is a compact set;

(ii) J is open-valued and compact on S;

(iii) $I(x,\cdot)$ is K-u.s.c. at p^* for all $x\in S$;

(iv) $I(x,p)\subset J(x)$ for all $(x,p)\in S\times L$.

Then W is u.s.c. at p^*.

Proof Suppose to the contrary that W is not u.s.c. at p^*. Then there exists a neighborhood V_0 of $W(p^*)$ such that for any neighborhood U of p^*, there exists $p\in U$ satisfying

$$W(p)\not\subset V_0.$$

Thus, we can choose $p_n\to p^*$, which satisfies $\exists x_n\in W(p_n)$ such that $x_n\notin V_0$. Since S is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x^*\in S$ such that $x_{n_k}\to x^*$. Let $x\in S$. Now $x_{n_k}\in W(p_{n_k})$, thus there exists $y_{n_k}\in I(x_{n_k},p_{n_k})$ such that

$$y_{n_k}\notin I(x,p_{n_k})+\mathrm{int}\,K.$$

Using (iii), for any $d\in \mathrm{int}\,K$, there exists $k_0\in\mathbb{N}$ such that

$$I(x,p^*)+d\subset I(x,p_{n_k})+\mathrm{int}\,K,\quad \forall\, k>k_0.$$

This implies that

$$y_{n_k}\notin J(x)+d,\quad \forall\, k>k_0.$$

Proceeding as in Theorem 3.13, we derive that $x^*\in\mathrm{WEff}(p^*)$. Thus, it follows that $x_{n_k}\to x^*\in V_0$, which gives a contradiction to $x_{n_k}\notin V_0$. Therefore, W is u.s.c. at p^*. □

Theorem 5.2 *Assume that:*

(i) S is a compact set;

(ii) $I(\cdot,\cdot)$ is compact on $S\times\{p^*\}$;

(iii) $J(x)\subset I(x,p)$ for all $(x,p)\in S\times L$.

Then W is u.s.c. at p^*.

Proof Suppose to the contrary that W is not u.s.c. at p^*. Then there exists a neighborhood V_0 of $W(p^*)$ such that for any neighborhood U of p^*, there exists $p \in U$ satisfying

$$W(p) \not\subset V_0.$$

Thus, we can choose $p_n \to p^*$, which satisfies $\exists\, x_n \in W(p_n)$ such that $x_n \notin V_0$. Since S is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x^* \in S$ such that $x_{n_k} \to x^*$. Without loss of generality, we can assume that $x_n \to x^*$.

Now we claim that $x^* \in W(p^*)$. In fact, suppose that $x^* \notin W(p^*)$. Then it follows that there exists $y \in S$ such that $J(y) \prec_K^l J(x^*)$, that is,

$$J(x^*) \subset J(y) + \operatorname{int} K. \tag{5.1}$$

From (5.1), we claim that there exists $n_0 \in \mathbb{N}$ such that

$$I(x_n, p_n) \subset I(y, p_n) + \operatorname{int} K, \quad \forall\, n > n_0. \tag{5.2}$$

In fact, if not, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $I(x_{n_k}, p_{n_k}) \not\subset I(y, p_{n_k}) + \operatorname{int} K$. Without loss of generality, we can assume that $I(x_n, p_n) \not\subset I(y, p_n) + \operatorname{int} K$. Thus, there exists $v_n \in I(x_n, p_n)$ such that

$$v_n \notin I(y, p_n) + \operatorname{int} K.$$

From (iii) we obtain $J(y) \subset I(y, p_n)$ for all $n \in \mathbb{N}$, it follows that

$$v_n \notin J(y) + \operatorname{int} K. \tag{5.3}$$

Since $I(\cdot,\cdot)$ is compact at (x^*, p^*) and $(x_n, p_n) \to (x^*, p^*)$, it follows that there exist $v^* \in J(x^*)$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to v^*$. Without loss of generality, we can assume that $v_n \to v^*$. It follows from (5.1) that

$$v^* \in J(y) + \operatorname{int} K.$$

This implies that $v_n \in J(y) + \operatorname{int} K$ for n large enough, which contradicts (5.3). Then it follows from (5.2) that $I(y, p_n) \prec_K^l I(x_n, p_n)$ when $n > n_0$, which contradicts the fact that $x_n \in W(p_n)$. Thus, $x^* \in W(p^*)$. We can see that $x_n \to x^* \in V_0$, which gives a contradiction to $x_n \notin V_0$. Therefore, W is u.s.c. at p^*. $\square$

Remark 5.3 The hypotheses of Theorems 5.1 and 5.2 are different, while we obtain the same stability result concerned with the upper semi-continuity for set optimization problems by using different methods. Moreover, we would like to mention that the results we obtained are new since the parametric set optimization problem and the solution mapping discussed in this section are different from those in [20–23].

Theorem 5.4 *Assume that:*

(i) S is a compact set;

(ii) $I(\cdot,\cdot)$ has the $\prec_K^l$-continuous property on $S \times L$;

(iii) $W(p^*) \subset S'$, where S' denotes the derived set of S.

Then W is l.s.c. at p^*.

Proof Suppose the contrary that W is not l.s.c. at p^*. Then there exist $x^* \in W(p^*)$ and a neighborhood V_0 of $0 \in X$ such that for any neighborhood U of p^*, there exists $p \in U$ satisfying $(x^* + V_0) \cap W(p) = \emptyset$. Thus there exists a sequence $\{p_n\}$ with $p_n \to p^*$ such that

$$(x^* + V_0) \cap W(p_n) = \emptyset, \quad \forall\, n \in \mathbb{N}. \tag{5.4}$$

From (iii) we obtain x^* is a limit point of S and hence there exits a sequence $\{x_n\} \subset S \setminus \{x^*\}$ such that $x_n \to x^*$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$x_n \in x^* + V_0, \quad \forall\, n > n_0. \tag{5.5}$$

Now we claim that there exists $n_1 \in \mathbb{N}$ such that $x_n \in W(p_n)$ when $n > n_1$. Indeed, if not, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin W(p_{n_k})$ for $k \in \mathbb{N}$. Without loss of generality, we can assume that $x_n \notin W(p_n)$ for $n \in \mathbb{N}$. Then, there exists $y_n \in S$ such that $I(y_n, p_n) \prec^l_K I(x_n, p_n)$. Noting that S is compact, there exist $y^* \in S$ and a subsequence $\{y_{n_l}\}$ of $\{y_n\}$ such that $y_{n_l} \to y$. Without loss of generality, we can assume that $y_n \to y^*$. From (ii), $I(\cdot,\cdot)$ has the $\prec^l_K$-continuous property at (x^*, p^*) with respect to (y^*, p^*). Then it follows from $I(y_n, p_n) \prec^l_K I(x_n, p_n)$ that $J(y^*) \prec^l_K J(x^*)$, which contradicts the fact that $x^* \in W(p^*)$. Therefore, $x_n \in W(p_n)$ when $n > n_1$. Along with (5.5), we obtain $x_n \in (x^* + V_0) \cap W(p_n)$ when $n > \max\{n_0, n_1\}$. This gives a contradiction to (5.4) and hence W is l.s.c. at p^*. □

Finally, we give the following example to illustrate Theorem 5.4.

Example 5.5 Let $X = P = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0,1]$, $K = \mathbb{R}^2_+$, $p^* = 1$, $L = [0,2]$ and let $J : S \rightrightarrows Y$ and $I : S \times L \rightrightarrows Y$ be defined by

$$J(x) = [-1, x] \times [0,1], \quad \forall\, x \in [0,1];$$

and

$$I(x,p) = [-1, px] \times [0,1], \quad \forall\, (x,p) \in [0,1] \times [0,2].$$

Then it is easy to check that all assumptions of Theorem 5.4 are satisfied. Moreover, we can see that $W(p) = [0,1]$ for all $p \in [0,2]$. Thus, W is l.s.c. at 1.

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