Journal of Mathematical Research with Applications
Nov., 2022, Vol. 42, No. 6, pp. 653–657
DOI:10.3770/j.issn:2095-2651.2022.06.009
Http://jmre.dlut.edu.cn

An Application of Set Theory in Quasi-Paracompactness

Xiangeng ZHOU, Shou LIN*
Institute of Mathematics, Ningde Normal University, Fujian 352100*, P. R. China*

Abstract In 1977, Yingming Liu introduced quasi-paracompactness and proved that under $2^{\omega_1} > 2^{\omega}$ every separable normal quasi-paracompact space is a paracompact space, which is a result of set-theoretic topology. In this paper we further prove that hypothesis "$2^{\omega_1} > 2^{\omega}$" is equivalent to that every separable normal quasi-paracompact space is a paracompact space, which gives an independent result of quasi-paracompactness.

Keywords quasi-paracompact space; weak continuum hypothesis; irreducible space; ω_1-compact space; Stone-Čech compactification

MR(2020) Subject Classification 54A35; 54D20; 54D65; 03E35

1. Questions

Metrizability and compactness are the heart and soul of general topology. Besides metrizability and compactness, there are a few other concepts which are fundamental in general topology, for examples, generalized metric spaces and covering properties [1, 2]. A topological property is called a covering property if it can be characterized by every open cover of a space having a certain refinement, for examples, compactness, Lindelöfness, paracompactness, subparacompactness, etc.

In 1977, Liu [3] introduced and studied special covering properties containing weak paracompactness and subparacompactness: quasi-paracompact spaces and strongly quasi-paracompact spaces. In the book "Selected Topics in General Topology", Jiang pointed out [4, p. 197]: "In recent years, it has been found that paracompactness, subparacompactness, metacompactness and submetacompactness can be decomposed into factors of strong quasi-paracompactness. It shows that paracompact, subparacompact, metacompact and submetacompact spaces can be characterized by strongly quasi-paracompact spaces, which further displays the theoretical value of this kind of new spaces."

Jiang and Zhang obtained a relationship between paracompactness and quasi-paracompactness under set-theoretic hypothesis, i.e., under hypothesis "$V = L$" in set theory, every normal locally compact quasi-paracompact space is a paracompact space [5, Theorem 2.1]. Hence, they partially answered the famous Arhangel'skiĭ-Tall problem [6, p. 44]: Is every normal locally compact

Received January 19, 2022; Accepted May 8, 2022
Supported by the National Natural Science Foundation of China (Grant No. 12171015) and the Natural Science Foundation of Fujian Province (Grant Nos. 2020J01428; 2020J05230).
* Corresponding author
E-mail address: 56667400@qq.com (Xiangeng ZHOU); shoulin60@163.com (Shou LIN)

space a paracompact space? Now, it is known that the problem is undecidable in ZFC axiom system [7, p. 2]. This demonstrates the special role of quasi-paracompactness in covering properties. In this paper we will continue to discuss the role of quasi-paracompactness in set-theoretic topology and promulgates the importance of this property.

Set-theoretic topology is an interdisciplinary research of topology and mathematical logic, which emerged in the 1960–1970s (see [8,9]). With the achievements of mathematical logic, some problems in topology have been solved, in which set-theoretic hypothesis played an important role. Recently, Hrušák and Ramos-García proved that every separable Fréchet topological group being metrizable is compatible with ZFC axiom system [10], which solved the Malykhin problem in 1978 (see [10, p. 194]). One of the main sources of general topology, which used to be called point-set topology, is the set theory founded by Cantor. Later, with the help of mathematical logic, set-theoretic topology was generated. This is a good case of separation, intersection and integration of some branches of mathematics.

In 1937, Jones proved that if $2^{\omega} < 2^{\omega_1}$, then every separable normal Moore space is ω_1-compact, and therefore metrizable [11], and also posed the famous Normal Moore Space Problem [11, p. 676]: Is every normal Moore space metrizable? Related to the well-known conjecture of normal Moore spaces, McAuley proposed the following problem [12]: Is every separable normal semimetric space paracompact? It is known that McAuley's problem was undecidable in ZFC axiom system [3]. Furthermore, by using the weak continuum hypothesis and quasi-paracompactness, Liu found a way to solve the above problem, i.e., he proved the following result [3]: Under $2^{\omega_1} > 2^{\omega}$, every separable normal quasi-paracompact space is paracompact. "$2^{\omega_1} > 2^{\omega}$" is a set-theoreric hypothesis which is independent of ZFC axiom system [13]. Liu's theorem was the first result of applying set-theoretic hypothesis to solving problems of general topology in China, which expanded the research of set-theoretic topology.

This paper discusses the set-theoretic hypothesis in Liu's theorem. There is the following question.

Question 1.1 Is hypothesis $2^{\omega_1} > 2^{\omega}$ equivalent to that every separable normal quasi-paracompact space is a paracompact space?

The following result is related to set-theoretic hypothesis for separable normal spaces.

Lemma 1.2 ([13, Lemma 2.1 and Example E]) *Hypothesis $2^{\omega_1} > 2^{\omega}$ is equivalent to that every separable normal space is a ω_1-compact space.*

In this paper, we prove that the answer to Question 1.1 is affirmative, hence the following question is undecidable in ZFC: Is every separable normal quasi-paracompact space a paracompact space in ZFC axiom system?

2. The main results

First, recall some related concepts and results. In this paper, all spaces are T_1-topological spaces. The readers may refer to [14] for notation and terminology not explicitly given here.

The sets of the first infinite ordinal and the first uncountable ordinal are denoted by ω and

ω_1, respectively. Continuum hypothesis (CH, i.e., $2^\omega = \omega_1$) implies hypothesis $2^{\omega_1} > 2^\omega$, hence hypothesis $2^{\omega_1} > 2^\omega$ is also called weak continuum hypothesis [15, Definition 14.5].

Let $[\omega]^\omega = \{A \subseteq \omega : |A| = \omega\}$, where $|A|$ is the cardinality of the set A. A subset E of $[\omega]^\omega$ is called a family of independent sets [15, Definition 11.2] if, whenever $A_1, \dots, A_m$, $B_1, \dots, B_n$ in E are different from each other, there holds

$$\Big|\bigcap_{i=1}^{m} A_i \cap \bigcap_{j=1}^{n} (\omega \setminus B_j)\Big| = \omega.$$

Lemma 2.1 ([15, Theorem 11.3]) *There is a family of independent sets with cardinality* 2^ω.

A topological space X is called an ω_1-compact space, if every uncountable subset of X has an accumulation point. Both Lindelöf spaces and countably compact spaces are ω_1-compact spaces. In order to make our main results more general, the following concepts are introduced. A topological space X is called an iso-Lindelöf space, if every ω_1-compact and closed subspace of X is Lindelöf. This space can be compared with the iso-compact space as follows: A topological space X is called an iso-compact space [16], if every countably compact and closed subspace of X is compact. Every iso-Lindelöf space is iso-compact [1].

Worrell and Wicke proved that a space is developable if and only if it is a submetacompact space having a base of countable order [17]. The search for weak covering properties was going strong from this point on. The following theorem is the main result of the present paper, which involves quasi-paracompact spaces and irreducible spaces. A topological space X is called a quasi-paracompact space [3], if every open cover of X has a σ-relatively discrete refinement. A topological space X is called an irreducible space [18, Definition 6.6.2], if every open cover of X has the minimal open refinement.

Theorem 2.2 *The following statements are equivalent.*

(1) $2^\omega < 2^{\omega_1}$.

(2) Every separable normal iso-Lindelöf space is paracompact.

(3) Every separable normal quasi-paracompact space is paracompact.

(4) Every separable normal irreducible space is paracompact.

Proof (1)⇒(2). Let X be a separable normal iso-Lindelöf space. By Lemma 1.2 and the definition of iso-Lindelöf spaces, X is a Lindelöf space. And since X is a regular space, X is a paracompact space [14, Theorem 5.1.2].

Since every quasi-paracompact space is iso-Lindelöf [3, Theorem 6] and irreducible [19, Corollary 1], (2)⇒(3) and (4)⇒ (3) hold.

(3)⇒(1). Suppose that $2^\omega = 2^{\omega_1}$. Using a subspace of Stone-Čech compactification $\beta\omega$ of ω, we should construct a separable normal quasi-paracompact space which is not a paracompact space. It is well known that Stone-Čech compactification $\beta\omega$ of ω is the family of all ultrafilters (i.e., maximal filters) endowed with the topology generated by all $\{A^* : A \subseteq \omega\}$ as a subbase, where $A^* = \{p \in \beta\omega : A \in p\}$ (see [20, p. 111]).

[1] Let X be an iso-Lindelöf space. If A is a countably compact and closed subspace of X, then A is ω_1-compact and closed, thus A is Lindelöf, so then A is compact. Hence, X is iso-compact.

By Lemma 2.1, let $\{A_\alpha : \alpha < 2^\omega\}$ be a family of independent sets of ω. The power set $\mathcal{P}(\omega_1)$ is denoted by the family $\{H_\alpha : \alpha < 2^\omega\}$. For every $\xi < \omega_1$, put

$$U_\xi = \{A_\alpha : \xi \in H_\alpha \text{ and } \alpha < 2^\omega\} \cup \{\omega \setminus A_\alpha : \xi \notin H_\alpha \text{ and } \alpha < 2^\omega\}.$$

By the definition of independent sets, U_ξ is of the finite intersection property. And by Zorn's Lemma, there exists an ultrafilter p_ξ of ω with $U_\xi \subseteq p_\xi$. Since $\{A_\alpha : \alpha < 2^\omega\}$ is a family of independent sets, we have that $p_\xi \in \beta\omega \setminus \omega$. If $\xi_1 \neq \xi_2 < \omega_1$, then there exists $\alpha < 2^\omega$ such that $\xi_1 \in H_\alpha$ and $\xi_2 \notin H_\alpha$ (for example, pick $H_\alpha = \{\xi_1\}$). Hence $A_\alpha \in U_{\xi_1}$ and $\omega \setminus A_\alpha \in U_{\xi_2}$, and therefore $p_{\xi_1} \neq p_{\xi_2}$.

Let $X = \omega \cup \{p_\xi : \xi < \omega_1\} \subset \beta\omega$ endowed with the subspace topology of Stone-Čech compactification $\beta\omega$.

(a) Since ω is a dense subset of $\beta\omega$, ω is dense in X. Hence, X is a separable space.

(b) X is a normal space. In fact, let A, B be disjoint closed subsets of X. There exists $\Lambda \subseteq \omega_1$ such that $A \setminus \omega = \{p_\xi : \xi \in \Lambda\}$, and then there exists $\alpha_0 < 2^\omega$ such that $H_{\alpha_0} = \Lambda$. For each $\xi \in \Lambda = H_{\alpha_0}$, we have that $A_{\alpha_0} \in U_\xi \subseteq p_\xi$, hence $p_\xi \in A^*_{\alpha_0}$, and thus $A \setminus \omega \subset A^*_{\alpha_0}$. For each $p_\eta \in B$, we have that $\eta \notin \Lambda = H_{\alpha_0}$, hence $\omega \setminus A_{\alpha_0} \in U_\eta \subseteq p_\eta$, thus $p_\eta \in (\omega \setminus A_{\alpha_0})^*$, and therefore $B \setminus \omega \subseteq (\omega \setminus A_{\alpha_0})^*$. And it is easy to verify that $A^*_{\alpha_0} \cap (\omega \setminus A_{\alpha_0})^* = \varnothing$.

Set

$$U = [(\omega \cap A) \cup A^*_{\alpha_0}] \cap X \setminus B, \quad V = [(\omega \cap B) \cup (\omega \setminus A_{\alpha_0})^*] \cap X \setminus A.$$

Then U and V are disjoint open sets in X containing A and B, respectively.

(c) $\{p_\xi : \xi < \omega_1\}$ is a discrete subset of X. In fact, for each $\xi < \omega_1$, there exists $\alpha_0 < 2^\omega$ such that $H_{\alpha_0} = \{\xi\}$, thus $A_{\alpha_0} \in U_\xi \subseteq p_\xi$, and hence $p_\xi \in A^*_{\alpha_0}$. If $\eta < \omega_1$ and $\eta \neq \xi$, then $\eta \notin H_{\alpha_0}$, thus $\omega \setminus A_{\alpha_0} \in U_\eta \subseteq p_\eta$, hence $A_{\alpha_0} \notin p_\eta$, and therefore $p_\eta \notin A^*_{\alpha_0}$.

Since ω is an open subset of X, $\{p_\xi : \xi < \omega_1\}$ is a closed discrete subset of X. Thus X is not an ω_1-compact space. It follows from the fact that every separable paracompact space is a Lindelöf space [14, Corollary 5.1.26] that X is not a paracompact space. And since $X = \omega \cup \{p_\xi : \xi < \omega_1\}$ is σ-closed discrete and every discrete family is relatively discrete, the space X is a quasi-paracompact space.

(1)$\Rightarrow$(4). Let X be a separable normal irreducible space. By Lemma 1.2, X is an ω_1-compact irreducible space, hence X is a Lindelöf space [18, Theorem 6.6.13], and therefore X is a paracompact space. □

Since paracompactness is equivalent to Lindelöf property in separable regular spaces, we have the following corollary.

Corollary 2.3 *The following statements are equivalent.*

(1) $2^\omega < 2^{\omega_1}$.

(2) Every separable normal iso-Lindelöf space is Lindelöf.

(3) Every separable normal quasi-paracompact space is Lindelöf.

(4) Every separable normal irreducible space is Lindelöf.

Remark 2.4 The conditions of Theorem 2.2 and Corollary 2.3 are explained below.

(1) Iso-Lindelöf spaces are a kind of very extensive spaces. Among the main covering properties, paracompact spaces $\Rightarrow$ metacompact spaces (or subparacompact spaces if the spaces are T_2) $\Rightarrow$ submetacompact spaces $\Rightarrow$ quasi-paracompact spaces $\Rightarrow$ weakly submetacompact spaces and irreducible spaces; weakly submetacompact spaces (or spaces which each closed subspace is irreducible) $\Rightarrow$ iso-Lindelöf spaces [18, Chapter 6].

(2) There is a normal metacompact space satisfying the countable chain condition (i.e., CCC) which is not a paracompact space [13, Example J].

(3) There is a separable normal space which is not a paracompact space [21, Theorem].

By Theorem 2.2, if every separable normal iso-compact space is paracompact, then $2^{\omega} < 2^{\omega_1}$.

Question 2.5 Is every separable normal iso-compact space a paracompact space under $2^{\omega} < 2^{\omega_1}$?

Acknowledgements We would like to express our gratitude to the reviewer for valuable comments.

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