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A Note on the A_α-Spectral Radius of a Graph

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Abstract Let G be a simple graph of order n and $\rho_\alpha(G)$ be the $A_\alpha(G)$-spectral radius of G. In this note, for any vertex v_i of G, we establish the relationship between $\rho_\alpha(G)$ and $\rho_\alpha(G-v_i)$.

Keywords graph; A_α-spectral radius; Principal eigenvector

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. Its order is $|V(G)|$, denoted by n, and its *size* is $|E(G)|$, denoted by m. For $v_i\in V(G)$, let $d(v_i)$ (or d_i for short) and $N(v_i)$ be the degree and the set of neighbors of v_i, respectively, and $G-v_i$ be the graph obtained from G by removing the vertex v_i and its incident edges. Let $G_1\vee G_2$ be the graph obtained by joining graphs G_1 and G_2 with $|V(G_1)|\times|V(G_2)|$ edges. For any undefined notations, we refer to [1].

The adjacency matrix of G is defined to be $A(G)=(a_{ij})$, where $a_{ij}=1$ if v_i is adjacent to v_j and $a_{ij}=0$ otherwise. Let $D(G)$ be the diagonal matrix of the vertex degrees of G. The signless Laplacian matrix of G is $Q(G)=D(G)+A(G)$. For every $\alpha\in[0,1]$, the matrix $A_\alpha(G)$ of a graph G is defined by Nikiforov in [2] as

$$A_\alpha(G)=\alpha D(G)+(1-\alpha)A(G).$$

In particular,

$$A_0(G)=A(G),\quad A_{\frac{1}{2}}(G)=\frac{1}{2}Q(G),\quad A_1(G)=D(G).$$

Since $A_\alpha(G)$ is a real symmetric matrix, all the eigenvalues of $A_\alpha(G)$ (also called the A_α-eigenvalues of G) are real. We denote its eigenvalues in non-increasing order as $\lambda_1(A_\alpha(G))\ge\lambda_2(A_\alpha(G))\ge\cdots\ge\lambda_n(A_\alpha(G))$. The largest eigenvalue $\lambda_1(A_\alpha(G))$, denoted by $\rho_\alpha(G)$ is called the A_α-spectral radius of G. Note that $A_\alpha(G)$ is irreducible if and only if G is connected for $\alpha\in[0,1)$. Therefore, when G is a connected graph, the Perron-Frobenius Theorem implies that the multiplicity of $\rho_\alpha(G)$ is one and there exists a positive unit eigenvector $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$,

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which is called the Perron vector of $A_\alpha(G)$. We refer the reader to Brouwer and Haemers [3] and Cvetković et al. [4] for literature in this area.

The spectral radius of a graph contains lots of information about the graph. Many studies on this topic have been conducted [4]. In particular, establishing the relationship between the spectral radius of a graph and its subgraph is of interest. It is known that when an edge or a vertex is removed from a graph G, the spectral radius of G will not increase due to the Perron-Frobenius Theorem. Van Mieghem et al. [5] and Li et al. [6] obtained several results on behavior of the spectral radius of a graph G after removing edges or vertices from G. Recently, for any $v_i \in V(G)$, Guo et al. [7] presented the relationship between $\rho_0(G)$ and $\rho_0(G - v_i)$ as follows.

Theorem 1.1 ([7]) *Let G be a connected graph of order n. For any $v_i \in V(G)$, we have*

$$\rho_0(G) \leq \sqrt{\rho_0^2(G - v_i) + 1} + \sqrt{d_i - 1}. \tag{1.1}$$

Moreover, the equality holds if and only if $G \cong K_{1,n-1}$ and v_i is a pendant vertex of $K_{1,n-1}$.

Moreover, they further conjectured that for any $v_i \in V(G)$,

$$\rho_0(G) \leq \sqrt{\rho_0^2(G - v_i) + 2d_i - 1}.$$

Very recently, Sun and Das [8] confirmed this conjecture by proving the following result.

Theorem 1.2 ([8]) *Let G be a graph of order n. For any $v_i \in V(G)$ with $d_i \geq 1$, we have*

$$\rho_0(G) \leq \sqrt{\rho_0^2(G - v_i) + 2d_i - 1}. \tag{1.2}$$

The equality holds if and only if $G \cong K_{1,n-1}$ and v_i is a pendant vertex of $K_{1,n-1}$, or $G \cong K_n$.

Moreover, Wang and Guo [9] further deduced the following relationship between $\rho_0(G - v_i)$ and $\rho_0(G)$ for $v_i \in V(G)$.

Theorem 1.3 ([9]) *Let G be a graph of order n. For $v_i \in V(G)$, we have*

$$\rho_0(G) \leq \frac{\rho_0(G - v_i) + \sqrt{\rho_0^2(G - v_i) + 4d_i}}{2}. \tag{1.3}$$

Moreover, the equality holds if and only if $G \cong K_1 \vee H$ and v_i is the vertex of degree $n - 1$, where H is a regular graph of order $n - 1$.

Motivated by the above mentioned recent results. In this note, we further study the relationship between $\rho_\alpha(G)$ and $\rho_\alpha(G - v_i)$ for $v_i \in V(G)$ by using its Perron vector, and establish the following result, which generalizes Theorem 1.3 since $A_0(G) = A(G)$.

Theorem 1.4 *Let G be a graph of order n. For $v_i \in V(G)$ and $\alpha \in [0, 1)$, we have*

$$\rho_0(G) \leq \frac{\rho_0(G - v_i) + \sqrt{\rho_0^2(G - v_i) + 4d_i}}{2}, \text{ when } \alpha = 0; \tag{1.4}$$

$$\rho_\alpha(G) < \frac{\alpha d_i + \rho_\alpha(G - v_i) + \sqrt{(\alpha d_i - \rho_\alpha(G - v_i))^2 + 4(1 - \alpha)^2 d_i}}{2}, \text{ when } \alpha \neq 0. \tag{1.5}$$

Moreover, for $\alpha = 0$, the equality holds if and only if $G \cong K_1 \vee H$ and v_i is the vertex of degree $n - 1$, where H is a regular graph of order $n - 1$.

Note that (1.2) is always better than (1.1). We now give the following remark to illustrate that the bound in Theorem 1.3 is but not always, better than that in Theorems 1.1 and 1.2, respectively.

Remark 1.5 It is easy to check that

$$\sqrt{\rho_0^2(G-v_i)+1} < \frac{\rho_0(G-v_i)+\sqrt{\rho_0^2(G-v_i)+4}}{2}$$

when $d_i = 1$. Then (1.1) is better than (1.3) for $d_i = 1$. But for $d_i \geq 2$, note that

$$\begin{aligned}\rho_0(G-v_i)\sqrt{\rho_0^2(G-v_i)+4d_i} &\leq \rho_0^2(G-v_i)+4\rho_0(G-v_i)\sqrt{d_i-1}\\ &< \rho_0^2(G-v_i)+4\sqrt{\rho_0^2(G-v_i)+1}\sqrt{d_i-1}.\end{aligned}$$

Then we have

$$[\frac{\rho_0(G-v_i)+\sqrt{\rho_0^2(G-v_i)+4d_i}}{2}]^2 < [\sqrt{\rho_0^2(G-v_i)+1}+\sqrt{d_i-1}]^2.$$

This shows that (1.3) is better than (1.1) for $d_i \geq 2$. Thus, the bound of (1.3) is slightly better than (1.1).

Now we compare the upper bounds of (1.3) and (1.2) in the following. It is easy to check that

$$\frac{\rho_0(G-v_i)+\sqrt{\rho_0^2(G-v_i)+4d_i}}{2} \leq \sqrt{\rho_0^2(G-v_i)+2d_i-1}$$

when $d_i \geq 1+\rho_0(G-v_i)$. That is (1.3) is better than (1.2) when $d_i \geq 1+\rho_0(G-v_i)$.

We now give the following examples to illustrate the sharpness of above bounds.

(1) In Theorems 1.2 and 1.3, if $G = S_n$ is a star of order $n \geq 2$, and $v_1 \in V(S_n)$ is the vertex of degree $n-1$, then $\rho_0(G-v_1) = 0$. Thus it turns out that

$$\frac{\rho_0(G-v_1)+\sqrt{\rho_0^2(G-v_1)+4d_1}}{2} = \frac{\sqrt{4(n-1)}}{2} = \sqrt{n-1},$$

$$\sqrt{\rho_0^2(G-v_1)+2d_1-1} = \sqrt{2(n-1)-1} = \sqrt{2n-3}.$$

Note that $\sqrt{2n-3} \geq \sqrt{n-1}$ since $n \geq 2$. This implies that the bound of (1.3) is better than (1.2) in this case.

(2) When $G = K_1 \vee C_{n-1}$ is the wheel graph of order n and $v_1 \in V(K_1 \vee C_{n-1})$ is a vertex of degree $n-1$, then by Lemma 2.3, we have

$$\rho_0(K_1 \vee C_{n-1}) = 1+\sqrt{n}.$$

Moreover, $\rho_0(G-v_1) = 2$ since $G - v_1 = C_{n-1}$. By (1.3) and (1.2), we have

$$1+\sqrt{n} = \rho_0(G) \leq \frac{\rho_0(G-v_1)+\sqrt{\rho_0^2(G-v_1)+4d_1}}{2} = 1+\sqrt{n},$$

$$1+\sqrt{n} = \rho_0(G) < \sqrt{\rho_0^2(G-v_1)+2d_1-1} = \sqrt{2n+1}.$$

This shows that (1.3) is better than (1.2) in this case.

2. Preliminaries

To give the proof of our result, the following lemmas are needed. For an $n \times n$ real symmetric matrix M, its eigenvalues are denoted in non-increasing order as $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$. The following is a classical result.

Lemma 2.1 (Cauchy's interlacing theorem) *Let M be an $n \times n$ real symmetric matrix. For an integer m with $1 \leq m \leq n$, let N be an $m \times m$ principal submatrix of M. Then for $i = 1, 2, \ldots, m$,*

$$\lambda_i(M) \geq \lambda_i(N) \geq \lambda_{i+n-m}(M).$$

In particular, in Lemma 2.1, let $m = 1$ and $W_v(G)$ be the principal submatrices of $A_\alpha(G)$ obtained by removing the row and the column of $A_\alpha(G)$ that correspond to the vertex v_i. Then we have

Corollary 2.2 *Let G be a graph of order n. For any $v_i \subseteq V(G)$, then we have*

$$\rho_\alpha(G - v_i) \leq \lambda_1(W_{v_i}(G)).$$

Lemma 2.3 ([10]) *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ non-negative symmetric matrices with $a_{ij} \leq b_{ij}$ for $i, j \in [1, 2, \ldots, n]$. Suppose that there is a positive eigenvector $\mathbf{x}$ corresponding to its largest eigenvalue $\lambda_1(A)$. Then $\lambda_1(A) \leq \lambda_1(B)$, where equality holds if and only if $A = B$.*

Lemma 2.4 ([10]) *Let $\|\cdot\|$ be a 2-norm of an vector or a matrix. For any $n \times n$ non-negative symmetric matrix A and $\mathbf{x} \in \mathbf{R}^n$, we have $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$, the equality holds if and only if $\|A\|$ is the spectral radius of A and $\mathbf{x}$ is the eigenvector corresponding to $\|A\|$.*

Lemma 2.5 ([11]) *Let the Hermitian matrix A be partitioned as*

$$A = \begin{bmatrix} a & \mathbf{b}^T \\ \mathbf{b} & M \end{bmatrix}$$

and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector of A corresponding to the largest eigenvalue $\lambda_1(A)$. If $\lambda_1(A)$ is not an eigenvalue of M, then

$$|x_1|^2 = \frac{1}{1 + \|(\lambda I - M)^{-1}\mathbf{b}\|^2}.$$

Lemma 2.6 *Let G be a connected graph of order n. For $v_i \in V(G)$, $A_\alpha(G)$ is partitioned as*

$$A_\alpha(G) = \begin{bmatrix} \alpha d_i & \mathbf{b}^T \\ \mathbf{b} & M \end{bmatrix}.$$

Then $\rho_\alpha(G - v_i) = \lambda_1(M)$ when $\alpha = 0$; $\rho_\alpha(G - v_i) < \lambda_1(M)$ when $\alpha \neq 0$.

Proof For $\alpha = 0$, it is obvious that $\rho_\alpha(G - v_i) = \lambda_1(M)$ since $A_0(G - v_i) = M$; for $\alpha \neq 0$, we now consider the following two cases:

Case 1. $G - v_i$ is connected. Then $A_\alpha(G - v_i)$ has a positive eigenvector corresponding to $\rho_\alpha(G - v_i)$. Thus Lemma 2.3 implies that $\rho_\alpha(G - v_i) < \lambda_1(M)$.

Case 2. $G - v_i$ is disconnected. Let $G - v_i = G_1 \cup G_2 \cup \cdots \cup G_t$. This gives the corresponding partition of M as

$$M = \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & M_t \end{bmatrix}.$$

Assume that $\rho_\alpha(G - v_i) = \rho_\alpha(G_k)$ for some $k \in [1, t]$. Then similarly to Case 1, we have $\rho_\alpha(G_k) < \lambda_1(M_k)$. Thus

$$\rho_\alpha(G - v_i) = \rho_\alpha(G_k) < \lambda_1(M_k) \le \max\{\lambda_1(M_1), \lambda_1(M_2), \ldots, \lambda_1(M_t)\} = \lambda_1(M).$$

This completes the proof of Lemma 2.6. □

3. Proof of Theorem 1.4

In this section, we will give a proof of Theorem 1.4. For this, we need the following upper and lower bounds on the eigencomponent x_i of the Perron vector of $A_a(G)$ corresponding to the vertex v_i.

Lemma 3.1 *Let G be a connected graph of order n, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $A_a(G)$ corresponding to $\rho_\alpha(G)$. Then for every $1 \le i \le n$ and $\alpha \in [0, 1)$, we have*

$$x_i \le \frac{1}{\sqrt{1 + \frac{(\rho_\alpha(G) - \alpha d_i)^2}{(1-\alpha)^2 d_i}}}.$$

Moreover, the equality holds if and only if $G = K_1 \vee H$ and v_i is the vertex of degree $n - 1$, where H is a regular graph of order $n - 1$.

Proof Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $A_\alpha(G)$ corresponding to $\rho_\alpha(G)$. Based on $A_\alpha(G)\mathbf{x} = \rho_\alpha(G)\mathbf{x}$, we have

$$\rho_\alpha(G)x_i = \alpha d_i x_i + (1 - \alpha) \sum_{v_j \in N(v_i)} x_j.$$

Using the Cauchy-Schwarz inequality, we have

$$(\rho_\alpha(G) - \alpha d_i)x_i = (1 - \alpha) \sum_{v_j \in N(v_i)} x_j \le (1 - \alpha) \sqrt{d_i \sum_{v_j \in N(v_i)} x_j^2},$$

i.e.,

$$(\rho_\alpha(G) - \alpha d_i)^2 x_i^2 \le (1 - \alpha)^2 d_i \sum_{v_j \in N(v_i)} x_j^2.$$

That is

$$\sum_{v_j \in N(v_i)} x_j^2 \ge \frac{(\rho_\alpha(G) - \alpha d_i)^2 x_i^2}{(1 - \alpha)^2 d_i}.$$

Thus

$$1=\sum_{l=1}^{n}x_l^2\ge x_i^2+\sum_{v_j\in N(v_i)}x_j^2\ge x_i^2(\frac{(\rho_\alpha(G)-\alpha d_i)^2}{(1-\alpha)^2d_i}+1).$$

It follows that

$$x_i\le\frac{1}{\sqrt{1+\frac{(\rho_\alpha(G)-\alpha d_i)^2}{(1-\alpha)^2d_i}}}.$$

Equality is attained if and only if $x_k=x_l$ for $v_k,v_l\in N(v_i)$ and $x_j=0$ for $v_j\notin N(v_i)$. Note that the Perron vector $\mathbf{x}$ is positive since G is connected. It follows that every vertex different from v_i is adjacent to v_i. Hence, $G=v_i\vee H$, where H is a graph of order $n-1$. Moreover, recall that $x_k=x_l$ for $v_k,v_l\in N(v_i)$ and $\rho_\alpha(G)x_j=x_i+(d_j-1)x_j$ for $v_j\in N(v_i)$. Those imply that each vertex in H has the same degree. Therefore, H is a regular graph. This completes the proof of Lemma 3.1. □

Next, we give a lower bound on the eigencomponent x_i of the Perron vector $\mathbf{x}$ corresponding to the vertex v_i.

Lemma 3.2 *Let G be a connected graph of order n, and $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$ be the Perron vector of $A_a(G)$ corresponding to $\rho_\alpha(G)$. Then for every $1\le i\le n$ and $\alpha\in[0,1)$, we have*

$$x_i\ge\frac{1}{\sqrt{1+\frac{d_i}{(\rho_0(G)-\rho_0(G-v_i))^2}}},\ \textit{when } \alpha=0;$$

$$x_i>\frac{1}{\sqrt{1+\frac{(1-\alpha)^2d_i}{(\rho_\alpha(G)-\rho_\alpha(G-v_i))^2}}},\ \textit{when } \alpha\ne0.$$

Moreover, for $\alpha=0$, the equality holds if $G=K_1\vee H$ and v_i is the vertex of degree $n-1$, where H is a regular graph of order $n-1$.

Proof Partition $V(G)$ as $\{v_i\}\cup V(G)\setminus\{v_i\}$, which gives a partition of $A_\alpha(G)$ as

$$A_\alpha(G)=\begin{bmatrix}\alpha d_i & \mathbf{b}^T\\ \mathbf{b} & M\end{bmatrix}.$$

By Corollary 2.2 and Lemma 2.6, we then have $\rho_\alpha(G)>\lambda_1(M)\ge\rho_\alpha(G-v_1)$ (More precisely, $\lambda_1(M)=\rho_0(G-v_1)$ and $\lambda_1(M)>\rho_\alpha(G-v_1)$ when $\alpha\ne0$). This means that $\rho_\alpha(G)$ is not an eigenvalue of M. Thus by Lemmas 2.4 and 2.5, we have

$$|x_i|^2=\frac{1}{1+\|(\rho_\alpha(G)I-M)^{-1}\mathbf{b}\|^2}\ge\frac{1}{1+\|(\rho_\alpha(G)I-M)^{-1}\|^2\|\mathbf{b}\|^2}. \tag{3.1}$$

Note that

$$\|(\rho_\alpha(G)I-M)^{-1}\|=\lambda_{\max}((\rho_\alpha(G)I-M)^{-1})=\frac{1}{\lambda_{\min}(\rho_\alpha(G)I-M)}=\frac{1}{\rho_\alpha(G)-\lambda_{\max}(M)}.$$

Moreover, by Lemma 2.6, we have $\lambda_{\max}(M)=\rho_0(G)$ when $\alpha=0$ and $\lambda_{\max}(M)>\rho_\alpha(G-v_i)$ when $\alpha\ne0$. Thus, note that $\|b\|^2=(1-\alpha)^2d_i$. By (3.1), we then have

$$|x_i|^2\ge\frac{1}{1+\frac{d_i}{(\rho_0(G)-\rho_0(G-v_i))^2}},\ \text{when } \alpha=0;$$

$$|x_i|^2 > \frac{1}{1+\frac{(1-\alpha)^2 d_1}{(\rho_\alpha(G)-\rho_\alpha(G-v_1))^2}}, \text{ when } \alpha \neq 0.$$

That is

$$x_i \geq \frac{1}{\sqrt{1+\frac{d_i}{(\rho_0(G)-\rho_0(G-v_i))^2}}};$$

$$x_i > \frac{1}{\sqrt{1+\frac{(1-\alpha)^2 d_i}{(\rho_\alpha(G)-\rho_\alpha(G-v_i))^2}}}, \text{ when } \alpha \neq 0.$$

Moreover, for $\alpha = 0$, if $G = K_1 \vee H$ and v_i is the vertex of degree $n-1$, where H is a regular graph of order $n-1$, then we have $b = (\underbrace{1, 1, \ldots, 1}_{n-1})^T$ and $M = A_0(H)$. It follows that

$$(\rho_0(G)I - M)^{-1}\mathbf{b} = \frac{1}{\rho_0(G) - \rho_0(H)}\mathbf{b}$$

and $\frac{1}{\rho_0(G)-\rho_0(H)}$ is the largest eigenvalue of $(\rho_0(G)I - M)^{-1}$. That is

$$\|(\rho_0(G)I - M)^{-1}\mathbf{b}\|^2 = \|(\rho_0(G)I - M)^{-1}\|^2\|\mathbf{b}\|^2.$$

Thus the equality in (3.1) holds, which completes the proof of Lemma 3.2. □

Combining Lemmas 3.1 and 3.2, we now give a proof of Theorem 1.4 as follows.

Proof of Theorem 1.4 By Lemmas 3.1 and 3.2, we have

$$\frac{1}{\sqrt{1+\frac{(1-\alpha)^2 d_i}{(\rho_0(G)-\rho_0(G-v_i))^2}}} \leq x_i \leq \frac{1}{\sqrt{1+\frac{\rho_0^2(G)}{d_i}}}, \text{ when } \alpha = 0;$$

$$\frac{1}{\sqrt{1+\frac{(1-\alpha)^2 d_i}{(\rho_\alpha(G)-\rho_\alpha(G-v_i))^2}}} < x_i \leq \frac{1}{\sqrt{1+\frac{(\rho_\alpha(G)-\alpha d_i)^2}{(1-\alpha)^2 d_i}}}, \text{ when } \alpha \neq 0.$$

It follows that

$$\frac{\rho_0^2(G)}{d_i} \leq \frac{d_i}{(\rho_0(G) - \rho_0(G-v_i))^2};$$

$$\frac{(\rho_\alpha - \alpha d_i)^2}{(1-\alpha)^2 d_i} < \frac{(1-\alpha)^2 d_i}{(\rho_\alpha(G) - \rho_\alpha(G-v_i))^2}, \text{ when } \alpha \neq 0,$$

i.e.,

$$\rho_0(G)(\rho_0(G) - \rho_0(G-v_i)) \leq d_i;$$

$$(\rho_\alpha(G) - \rho_\alpha(G-v_i))(\rho_\alpha(G) - \alpha d_i) < (1-\alpha)^2 d_i, \text{ when } \alpha \neq 0.$$

Those imply that

$$\rho_0^2(G) - \rho_0(G)\rho_0(G-v_i) - d_i \leq 0;$$

$$\rho_\alpha^2(G) - (\alpha d_i + \rho_\alpha(G-v_i))\rho_\alpha(G) + \alpha d_i \rho_\alpha(G-v_i) - (1-\alpha)^2 d_i < 0, \text{ when } \alpha \neq 0.$$

Thus

$$\rho_0(G) \leq \frac{\rho_0(G-v_i) + \sqrt{\rho_0^2(G-v_i) + 4d_i}}{2};$$

$$\rho_\alpha(G) < \frac{\alpha d_i + \rho_\alpha(G-v_i) + \sqrt{(\alpha d_i - \rho_\alpha(G-v_i))^2 + 4(1-\alpha)^2 d_i}}{2}, \text{ when } \alpha \neq 0.$$

Moreover, by Lemmas 3.1 and 3.2, for $\alpha = 0$, the equality holds if and only if $G \cong K_1 \vee H$ and v_i is the vertex of degree $n-1$, where H is a regular graph of order $n-1$. This completes the proof of Theorem 1.4. $\square$

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References

[1] J. BONDY, U. MURTY. *Graph Theory with Applications*. London, Macmillan, 1976.

[2] V. NIKIFOROV. *Merging the A-and Q-spectral theories*. Appl. Anal. Discrete Math., 2017, **11**(1): 81–107.

[3] A. BROUWER, W. HAEMERS. *Spectra of Graphs*. Springer, New York, 2012.

[4] D. CVETKOVIĆ, P. ROWLINSON, S. SIMIĆ. *An Introduction to the Theory of Graph Spectra*. Cambridge University Press, Cambridge, 2010.

[5] P. VAN MIEGHEM, D. STEVANOVIĆ, F. KUIPERS, et al. *Decreasing the spectral radius of a graph by link removals*. Phys. Rev. E, 2011, **84**(1): 016101.

[6] Cong LI, Huijuan WANG, P. VAN MIEGHEM. *Bounds for the spectral radius of a graph when nodes are removed*. Linear Algebra Appl., 2012, **437**(1): 319–323.

[7] Jiming GUO, Zhiwen WANG, Xin LI. *Sharp upper bounds of the spectral radius of a graph*. Discrete Math., 2019, **342**(9): 2559–2563.

[8] Shaowei SUN, K. C. DAS. *A conjecture on the spectral radius of graphs*. Linear Algebra Appl., 2020, **588**: 74–80.

[9] Zhiwen WANG. *Research on the Eigenvalues of Matrices for Graphs*. Ph. D Thesis, East China University of Science and Technology, Shanghai, 2020. (in Chinese)

[10] R. HORN, C. JOHNSON. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[11] T. TAO, V. VU. *Random matrices: universality of local eigenvalue statistics*. Acta Math., 2011, **206**(1): 127–204.