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Partitioning Planar Graphs with Girth at Least 6 into Bounded Size Components

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Abstract An $(\mathcal{O}_{k_1}, \mathcal{O}_{k_2})$-partition of a graph G is the partition of $V(G)$ into two non-empty subsets V_1 and V_2, such that $G[V_1]$ and $G[V_2]$ are graphs with components of order at most k_1 and k_2, respectively. In this paper, we consider the problem of partitioning the vertex set of a planar graph with girth restriction such that each part induces a graph with components of bounded order. We prove that every planar graph with girth at least 6 and i-cycle is not intersecting with j-cycle admits an $(\mathcal{O}_2, \mathcal{O}_3)$-partition, where $i \in \{6, 7, 8\}$ and $j \in \{6, 7, 8, 9\}$.

Keywords planar graph; face; girth; vertex partition; discharging procedure

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1. Introduction

In this paper, we only consider finite simple graphs. Given a graph G, let $V(G)$, $E(G)$, and $F(G)$ denote the vertex set, edge set and face set, respectively. We say that two cycles are intersecting if they have at least one common vertex. We use $g(G)$ to denote the girth of G, which is the length of a shortest cycle in G. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is called a plane graph.

For each $i \in \{1, 2, \ldots, m\}$, let $\mathcal{G}_i$ be the class of graphs satisfying some special properties. Given a graph G, a $(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m)$-partition of G is the partition of $V(G)$ into m sets $V_1, V_2, \ldots, V_m$, such that V_i induces a graph in $\mathcal{G}_i$ for each $i \in \{1, 2, \ldots, m\}$.

The following are notations of some graph classes.

$\mathcal{I}$: the class of edgeless graphs;

$\mathcal{F}$: the class of forests;

$\mathcal{O}_k$: the class of graphs whose components have order at most k;

$\mathcal{P}_k$: the class of graphs whose components are paths of order at most k;

$\mathcal{F}_d$: the class of forests with maximum degree d;

Δ_d: the class of graphs with maximum degree d.

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A k-vertex, k^+-vertex and k^--vertex are a vertex of degree k, at least k and at most k, respectively. A k-neighbour of a vertex is a neighbour that is k-vertex, and k^+-neighbour and k^--neighbour are defined analogously. A k-face, k^+-face, and k^--face are defined in the same way. We use $N(v)$ to denote the set of the neighbours of v. Let $N[v]$ denote $N(v) \cup \{v\}$. For a vertex $v \in V(G)$ and a $f \in F(G)$, we use $d(v)$ to denote the degree of v and use $d(f)$ to denote the size of f . We use $d_k(f)$ to denote the number of k-vertices incident with f. We write $f = [v_1 v_2 \ldots v_m]$ if $v_1, v_2, \ldots, v_m$ are all vertices of f in cyclic order. An $(\ell_1, \ell_2, \ldots, \ell_k)$-face is a k-face $[v_1 v_2 \cdots v_k]$ with $d(v_i) = \ell_i$ for every $i \in \{1, 2, \ldots, k\}$.

For an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G, we suppose that $V(G)$ is partitioned into two parts $\mathcal{O}_2$ and $\mathcal{O}_3$ where $\mathcal{O}_2$ and $\mathcal{O}_3$ induce graphs whose components have order at most 2, at most 3, respectively. We also call the sets $\mathcal{O}_2$ and $\mathcal{O}_3$ color classes, and a vertex in $\mathcal{O}_2$ and $\mathcal{O}_3$ is said to have color $\mathcal{O}_2$ and $\mathcal{O}_3$, respectively.

There are many results on partitions of planar graphs. The celebrated Four Color Theorem [1, 2] implies that every planar graph has an $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$-partition. Poh [3] showed that every planar graph admits an $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$-partition. Sittitrai and Nakprasit [4] showed that there does not exist an integer k such that every planar graph without 4-cycles and 5-cycles has a (Δ_1, Δ_k)-partition. They also showed that every planar graph without 4-cycles and 5-cycles has a (Δ_4, Δ_4)-partition, a (Δ_3, Δ_5)-partition, and a (Δ_2, Δ_9)-partition. Liu and Lv [5] proved that every planar graph without 4-cycles and 5-cycles has a (Δ_2, Δ_6)-partition. Dross, Montassier, Pinlou [6] proved that every triangle-free planar graph admits an $(\mathcal{F}_5, \mathcal{F})$-partition.

We are interested in the partition of planar graphs with girth restrictions. Montassier and Ochem [7] constructed graphs with girth 4 that do not admit $(\Delta_{d_1}, \Delta_{d_2})$-partition for each $d_1, d_2 \geq 0$. Borodin and Glebov [8] showed that every planar graph with girth 5 admits an $(\mathcal{I}, \mathcal{F})$-partition. Havet and Sereni [9] proved that graphs with girth 5 admit a (Δ_4, Δ_4)-partition. Choi and Raspaud [10] proved that graphs with girth 5 admit a (Δ_3, Δ_5)-partition. Axenovich, Ueckerdt and Weiner [11] showed that a planar graph with girth at least 6 has a $(\mathcal{P}_{15}, \mathcal{P}_{15})$-partition. Borodin and Ivanova [12] proved that every planar graph with girth at least 7 has a $(\mathcal{P}_3, \mathcal{P}_3)$-partition. Choi, Dross and Ochem [13] proved that every planar graph with girth at least 9 admits an $(\mathcal{I}, \mathcal{O}_9)$-partition. They also showed that every planar graph with girth at least 10 has an $(\mathcal{I}, \mathcal{P}_3)$-partition.

Our main result is stated as follows.

Theorem 1.1 *Every planar graph with girth at least* 6 *and i-cycle not intersecting with j-cycle admits an $(\mathcal{O}_2, \mathcal{O}_3)$-partition, where $i \in \{6, 7, 8\}$ and $j \in \{6, 7, 8, 9\}$.*

2. Structure properties of minimum counterexample

In order to prove Theorem 1.1, we use the discharging technique. Let G be the counterexample to Theorem 1.1 with minimal number of $|V(G)| + |E(G)|$. G is a plane graph. Clearly, the graph G is connected. According to the minimality of G, G has no $(\mathcal{O}_2, \mathcal{O}_3)$-partitions but every proper subgraph of G has. For an i-cycle with $i = 6, 7, 8$ or a j-cycle with $j = 6, 7$

with a hanging 1-vertex, it is obvious that it has an $(\mathcal{O}_2, \mathcal{O}_3)$-partition. Therefore, they are not minimal counterexamples. Furthermore, if i-cycle is not intersecting with j-cycle in graph G, we can deduce that i-face is not intersecting with j-face, where $i \in \{6, 7, 8\}$ and $j \in \{6, 7, 8, 9\}$.

Lemma 2.1 *Every vertex in G has degree at least 2.*

Proof Let v be a 1-vertex in G and $G' = G - v$. According to the minimality of G, G' has an $(\mathcal{O}_2, \mathcal{O}_3)$-partition. We can obtain an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G by giving v the color distinct from its neighbour, which is a contradiction. □

Lemma 2.2 *Every vertex v with $2 \le d(v) \le 4$ in G has at least one 3^+-neighbour.*

Proof Suppose to the contrary that every neighbour of v has degree 2. Let $G' = G - N[v]$. Since the girth of graph G is at least 6, the neighbours of each 2-neighbour of v are different and can only be in G'. Let $v_1, \ldots, v_m$ with $2 \le m \le 4$ be the 2-neighbours of v. According to the minimality of G, G' has an $(\mathcal{O}_2, \mathcal{O}_3)$-partition. We color v_i with $i = 1, \ldots, m$ with the color different from that of their neighbours in G', respectively. Then, if at least two of v_i are colored $\mathcal{O}_2$, then we assign $\mathcal{O}_3$ to v, otherwise we assign $\mathcal{O}_2$ to v. Therefore, we can obtain an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G, which is a contradiction. □

Lemma 2.3 *There are no adjacent 2-vertices in graph G.*

Proof Suppose to the contrary that v_1 and v_2 are two adjacent 2-vertices. Let $G' = G - \{v_1, v_2\}$. By the minimality of G, G' has an $(\mathcal{O}_2, \mathcal{O}_3)$-partition. We color v_1 and v_2 with the color different from that of their neighbours in G', respectively. In this way, we get an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G, which is a contradiction. □

In graph G, if a path is the longest induced path whose internal vertices all have degree 2, then we call it a chain. A chain is a k-chain if it has k internal 2-vertices. According to Lemma 2.3, we know there are no adjacent 2-vertices in graph G, so G has only 1-chains.

We give some interpretations here. In all the following tables, if the position of the vertices is symmetrical, we only list one coloring method.

If v is incident with one 1-chain and has two 3^+-neighbours, then we call it a good 3-vertex; if v is incident with two 1-chains and has one 3^+-neighbour, then we call it a weak 3-vertex; if v has three 3^+-neighbours, then we call it a best 3-vertex. According to Lemmas 2.2 and 2.3, we can know that there are only the above types of 3-vertices in G.

Lemma 2.4 *Let v_1 and v_2 be two adjacent 3-vertices. If v_1 is a weak 3-vertex, then v_2 cannot be a weak 3-vertex.*

Proof Suppose to the contrary that v_2 is a weak 3-vertex. Let u_1 and u_2 be two 2-neighbours of v_1. Let u_3 and u_4 be two 2-neighbours of v_2. Let $G' = G - \{v_1, v_2, u_1, u_2, u_3, u_4\}$. By the minimality of G, G' has an $(\mathcal{O}_2, \mathcal{O}_3)$-partition. Firstly, we color u_i with $i = 1, 2, 3, 4$ with the color different from that of their 3^+-neighbours in G', respectively. According to the colors of u_i

with $i = 1, 2, 3, 4$, we use the coloring methods in Table 1 to color v_1 and v_2. Therefore, we can obtain an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G, which is a contradiction.□

u_1	u_2	u_3	u_4	v_1	v_2
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$

Table 1 Coloring method 1

Lemma 2.5 *Let v_1 and v_2 be two adjacent 3-vertices. If v_1 is a good 3-vertex, then v_2 cannot be a weak 3-vertex.*

Proof Suppose to the contrary that v_2 is a weak 3-vertex. Let u_1, u_2 be the 2-neighbours of v_2 and u_3 be the 2-neighbour of v_1. Let $G' = G - \{v_1, v_2, u_1, u_2, u_3\}$. By the minimality of G, G' has an $(\mathcal{O}_2, \mathcal{O}_3)$-partition. Firstly, we color v_1 and u_i with $i = 1, 2, 3$ with the color different from that of their 3^+-neighbours in G', respectively. According to the colors of u_1, u_2, v_1 and u_3, we use the coloring methods in Table 2 to color v_2. Therefore, we can obtain an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G, which is a contradiction. □

u_1	u_2	v_1	u_3	v_2
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_2$
$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_2$
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$

Table 2 Coloring method 2

Lemma 2.6 *Let v_1 and v_2 be two 3-vertices and v_3 be the common 2-neighbour of v_1 and v_2. Then v_1 and v_2 cannot both be weak 3-vertices.*

Proof Suppose to the contrary that v_1 and v_2 are both weak 3-vertices. Let w_1 and w_2 be the 2-neighbours of v_1 and v_2, respectively. Let $G' = G - \{v_1, v_2, v_3, w_1, w_2\}$. By the minimality of G, G' has an $(\mathcal{O}_2, \mathcal{O}_3)$-partition. Firstly, we color v_i and w_i with $i = 1, 2$ with the colors different from that of their 3^+-neighbours in G', respectively. According to the colors of w_1, w_2, v_1 and v_2, we use the coloring methods in Table 3 to color v_3. Therefore, we can obtain an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G, which is a contradiction. □

w_1	w_2	v_1	v_2	v_3
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$

Table 3 Coloring method 3

Lemma 2.7 *For a* $(3,3,2,3,2,3)$*-face* $f=[v_1v_2v_3v_4v_5v_6]$, v_1 *can only be best* 3*-vertex.*

Proof Suppose to the contrary that v_1 has a 2-neighbour z. By Lemma 2.6, we know v_2 and v_6 are good 3-vertices. Let graph G' be a graph obtained from G by deleting z and all vertices on f. By the minimality of G, G' has an $(\mathcal{O}_2,\mathcal{O}_3)$-partition. Firstly, we color v_2, v_4, v_6 and z with the color different from that of their neighbours in G', respectively. According to the colors of v_2, v_4, v_6 and z, we use the coloring methods in Table 4 to color v_1, v_3 and v_5. Therefore, we can obtain an $(\mathcal{O}_2, \mathcal{O}_3)$-partition of G, which is a contradiction. □

v_2	v_4	v_6	z	v_1	v_3	v_5
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$
$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$
$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$
$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_3$
$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_3$	$\mathcal{O}_2/\mathcal{O}_3$	$\mathcal{O}_2$	$\mathcal{O}_2$	$\mathcal{O}_2$

Table 4 Coloring method 4

3. Discharging procedure

In order to reach the final contradiction, we will apply a discharging procedure. According to Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$, and $\sum_{v\in V} d(v)=\sum_{f\in F} d(f)=2|E|$, we get:

$$\sum_{v\in V(G)}(2d(v)-5)+\sum_{f\in F(G)}(\frac{1}{2}d(f)-5)=-10. \tag{3.1}$$

For all $x \in V(G) \cup F(G)$, let $2d(v) - 5$ and $\frac{1}{2}d(f) - 5$ be its initial charge $\omega(v)$ and $\omega(f)$, respectively. Let $\tau(v \to f)$ denote the charge v sends to f. From the above formula, we can know that the total initial charge is negative. Then we can design appropriate discharging rules and redistribute weights. Finally, we will prove that each $x \in V(G) \cup F(G)$ has final charge $\omega'(v) \geq 0$ and $\omega'(f) \geq 0$ by keeping the total sum of charges unchanged in discharging process. It leads to a contradiction that

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega'(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -10, \tag{3.2}$$

and thus such counterexample does not exist. Our discharging rules are defined as follows.

(R1) Every 2-vertex gets charge $\frac{1}{2}$ from each of its 3^+-neighbour.

For each 3^+-vertex v, let $\alpha(v)$ be the remaining charge of v after rule (R1).

(R2) Every 3^+-vertex v sends charge $\alpha(v)$ to incident $d(f)$-face $(6 \leq d(f) \leq 8)$.

(R3) Every $d(v)$-vertex v sends charge $\frac{\alpha(v)}{d(v)}$ to each incident 9-face $(d(v) \geq 3)$.

In the following, we will prove that $\omega'(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Lemma 3.1 *For each $v \in V(G)$, the final charge $\omega'(v) \geq 0$.*

Proof Let v be a 2-vertex. We know $\omega'(v) = -1 + \frac{1}{2} \times 2 = 0$ by (R1).

By the discharging rules, we only need to show that $\alpha(v) \geq 0$ for 3^+-vertex.

Let v be a 3-vertex. By Lemma 2.2, we know v has at least one 3^+-neighbour. If v is a weak 3-vertex, then $\alpha(v) = 1 - \frac{1}{2} \times 2 = 0$ by (R1); if v is a good 3-vertex, then $\alpha(v) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$; if v is a best 3-vertex, then $\alpha(v) = 1$ by (R1).

Let v be a 4-vertex. By Lemma 2.2, we know v has at least one 3^+-neighbour. So $\alpha(v) \geq 3 - \max\{\frac{1}{2} \times 3, \frac{1}{2} \times 2, \frac{1}{2} \times 1, 0\} = \frac{3}{2}$ by (R1).

Let v be a 5^+-vertex. We know $\alpha(v) \geq 2d(v) - 5 - d(v) \times \frac{1}{2} = \frac{3}{2}d(v) - 5 \geq \frac{5}{2}$ by (R1). □

Lemma 3.2 *For each $f \in F(G)$, the final charge $\omega'(f) \geq 0$.*

Proof Let f be a 6-face. If f is incident with at least two 4^+-vertices, then $\omega'(f) \geq \frac{1}{2} \times 6 - 5 + \frac{3}{2} \times 2 = 1$ by (R2). If f is incident with a 5^+-vertex, then $\omega'(f) \geq \frac{1}{2} \times 6 - 5 + \frac{5}{2} = \frac{1}{2}$ by (R2). Therefore, we only need to consider the case that there is at most one 4-vertex on the 6-face, and the rest are 2-vertices and 3-vertices.

Case 1. $d_2(f) = 0$.

For $(3, 3, 3, 3, 3, 4)$-face, we know $\omega'(f) \geq -2 + \frac{1}{2} \times 5 + 2 = \frac{5}{2}$ by (R2).

For $(3, 3, 3, 3, 3, 3)$-face, we know $\omega'(f) \geq -2 + \frac{1}{2} \times 6 = 1$ by (R2).

Case 2. $d_2(f) = 1$.

For $(3, 3, 3, 3, 3, 2)$-face, we know v_1 and v_5 cannot be weak 3-vertices at the same time by Lemma 2.6. So $\omega'(f) \geq -2 + \frac{1}{2} \times 4 = 0$ by (R2).

For $(4, 3, 3, 3, 3, 2)$-face, we know $\tau(v_1 \to f) + \tau(v_2 \to f) + \tau(v_3 \to f) \geq \frac{3}{2} + \frac{1}{2} + \frac{1}{2} = \frac{5}{2}$ by (R2). So $\omega'(f) \geq -2 + \frac{5}{2} = \frac{1}{2}$.

For $(4, 3, 3, 3, 2, 3)$-face and $(4, 3, 3, 2, 3, 3)$-face, we know $\tau(v_1 \to f) \geq 2$ by (R2). So $\omega'(f) \geq$

$-2+2=0$.

Case 3. $d_2(f)=2$.

For $(3,3,2,3,2,3)$-face, v_1 can only be best 3-vertex by Lemma 2.7. By Lemma 2.6, we know v_2 and v_6 are good 3-vertices. So $\omega'(f)\geq -2+1+\frac{1}{2}\times 2=0$ by (R2).

For $(4,3,2,3,2,3)$-face, we know $\tau(v_1\to f)\geq 2$ by (R2). So $\omega'(f)\geq -2+2=0$.

For $(3,4,2,3,2,3)$-face, we know v_6 is a good 3-vertex by Lemma 2.6. So $\tau(v_2\to f)+\tau(v_6\to f)\geq \frac{3}{2}+\frac{1}{2}=2$ by (R2). So $\omega'(f)\geq -2+2=0$.

For $(3,3,2,4,2,3)$-face, we know $\tau(v_1\to f)+\tau(v_4\to f)\geq \frac{1}{2}+\frac{3}{2}=2$ by (R2). So $\omega'(f)\geq -2+2=0$.

For $(3,3,2,3,3,2)$-face, we know these 3-vertices all are good 3-vertices by Lemmas 2.4 and 2.5. So $\omega'(f)=-2+\frac{1}{2}\times 4=0$ by (R2).

For $(3,3,2,3,4,2)$-face, we know v_1 and v_2 are good 3-vertices by Lemmas 2.4 and 2.5. So $\tau(v_1\to f)+\tau(v_2\to f)+\tau(v_5\to f)\geq \frac{1}{2}+\frac{1}{2}+\frac{3}{2}=\frac{5}{2}$ by (R2). So $\omega'(f)\geq -2+\frac{5}{2}=\frac{1}{2}$.

By Lemma 2.6, we know that there are no $(3,2,3,2,3,2)$-faces and $(4,2,3,2,3,2)$-faces in G. By Lemma 2.3, we know that there are no adjacent 2-vertices in graph G. So there is no case of $d_2(f)\geq 3$.

Let f be a 7-face. If f is incident with at least one 4^+-vertex, then $\omega'(f)\geq \frac{1}{2}\times 7-5+\frac{3}{2}=0$ by (R2). Therefore, we only need to consider the case that f is only incident with 2-vertices and 3-vertices.

Case 1. $d_2(f)=0$.

For $(3,3,3,3,3,3,3)$-face, we know $\omega'(f)\geq -\frac{3}{2}+\frac{1}{2}\times 7=2$ by (R2).

Case 2. $d_2(f)=1$.

For $(3,3,3,3,3,3,2)$-face, we know $\tau(v_2\to f)+\tau(v_3\to f)+\tau(v_4\to f)+\tau(v_5\to f)\geq \frac{1}{2}\times 4=2$ by (R2). So $\omega'(f)\geq -\frac{3}{2}+2=\frac{1}{2}$.

Case 3. $d_2(f)=2$.

For $(3,3,3,2,3,3,2)$-face, we know v_5 and v_6 are good 3-vertices by Lemmas 2.4 and 2.5. So $\tau(v_2\to f)+\tau(v_5\to f)+\tau(v_6\to f)\geq \frac{1}{2}\times 3=\frac{3}{2}$ by (R2). So $\omega'(f)\geq -\frac{3}{2}+\frac{3}{2}=0$.

For $(3,3,3,3,2,3,2)$-face, we know v_1 and v_4 are good 3-vertices by Lemma 2.6. So $\tau(v_1\to f)+\tau(v_2\to f)+\tau(v_3\to f)+\tau(v_4\to f)\geq \frac{1}{2}\times 4=2$ by (R2). So $\omega'(f)\geq -\frac{3}{2}+2=\frac{1}{2}$.

By Lemma 2.6, we know that there are no $(3,2,3,2,3,2,3)$-faces in G. By Lemma 2.3, we know that there are no adjacent 2-vertices in graph G. So there is no case of $d_2(f)\geq 3$.

Let f be a 8-face. If f is incident with at least one 4^+-vertex, then $\omega'(f)\geq \frac{1}{2}\times 8-5+\frac{3}{2}=\frac{1}{2}$ by (R2). Therefore, we only need to consider the case that f is only incident with 2-vertices and 3-vertices.

Case 1. $d_2(f)=0$.

For $(3,3,3,3,3,3,3,3)$-face, we know $\omega'(f)\geq -1+\frac{1}{2}\times 8=3$ by (R2).

Case 2. $d_2(f)=1$.

For $(3,3,3,3,3,3,3,2)$-face, we know $\tau(v_2\to f)+\tau(v_3\to f)\geq \frac{1}{2}\times 2=1$ by (R2). So $\omega'(f)\geq -1+1=0$.

Case 3. $d_2(f)=2$.

For $(3,3,3,2,3,3,3,2)$-face, we know $\tau(v_2 \to f) + \tau(v_6 \to f) \geq \frac{1}{2} \times 2 = 1$ by (R2). So $\omega'(f) \geq -1+1 = 0$.

For $(3,3,3,3,2,3,3,2)$-face, we know v_6 and v_7 are good 3-vertices by Lemmas 2.4 and 2.5. So $\omega'(f) \geq -1 + \frac{1}{2} \times 2 = 0$ by (R2).

For $(3,3,3,3,3,2,3,2)$-face, we know v_1 and v_5 are good 3-vertices by Lemma 2.6. So $\omega'(f) \geq -1 + \frac{1}{2} \times 2 = 0$ by (R2).

Case 4. $d_2(f) = 3$.

For $(3,2,3,3,2,3,3,2)$-face, we know v_6 and v_7 are good 3-vertices by Lemmas 2.4 and 2.5. So $\omega'(f) \geq -1 + \frac{1}{2} \times 2 = 0$ by (R2).

By Lemma 2.6, we know that there are no $(3,2,3,2,3,2,3,3)$-faces and $(3,2,3,2,3,2,3,2)$-faces in G. By Lemma 2.3, we know that there are no adjacent 2-vertices in graph G. So there is no case of $d_2(f) \geq 4$.

Let f be a 9-face. If f is incident with at least two 4^+-vertices, then $\omega'(f) \geq \frac{1}{2}\times 9-5+\frac{3}{8}\times 2 = \frac{1}{4}$ by (R3). If f is incident with a 5^+-vertex, then $\omega'(f) \geq \frac{1}{2} \times 9 - 5 + \frac{1}{2} = 0$ by (R3). Therefore, we only need to consider the case that there is at most one 4-vertex on the 9-face, and the rest are 2-vertices and 3-vertices.

Case 1. $d_2(f) = 0$.

For $(3,3,3,3,3,3,3,3,3)$-face and $(3,3,3,3,3,3,3,3,4)$-face, we know $\tau(v_1 \to f) + \tau(v_2 \to f) + \tau(v_3 \to f) \geq \frac{1}{6} \times 3 = \frac{1}{2}$ by (R3). So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$.

Case 2. $d_2(f) = 1$.

For $(3,3,3,3,3,3,3,3,2)$-face, $(3,3,3,3,4,3,3,3,2)$-face, $(3,3,3,3,3,4,3,3,2)$-face, $(3,3,3,3,3,3,4,3,2)$-face and $(3,3,3,3,3,3,3,4,2)$-face, we know that at least five of these 3-vertices are either good 3-vertices or best 3-vertices. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{6} \times 5 = \frac{1}{8}$ by (R3).

Case 3. $d_2(f) = 2$.

For $(3,3,3,3,3,3,2,3,2)$-face and $(3,3,3,3,3,2,3,3,2)$-face, we know $\tau(v_2 \to f) + \tau(v_3 \to f) + \tau(v_4 \to f) \geq \frac{1}{6} \times 3 = \frac{1}{2}$ by (R3). So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$.

For $(3,3,3,3,2,3,3,3,2)$-face, we know $\tau(v_2 \to f) + \tau(v_3 \to f) + \tau(v_7 \to f) \geq \frac{1}{6} \times 3 = \frac{1}{2}$ by (R3). So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$.

For $(3,3,3,3,3,3,2,4,2)$-face, $(3,3,3,3,3,4,2,3,2)$-face, $(3,3,3,3,4,3,2,3,2)$-face, $(3,3,3,4,3,3,2,3,2)$-face, $(3,3,3,3,3,2,3,4,2)$-face, $(3,3,3,3,4,2,3,3,2)$-face, $(3,3,3,4,3,2,3,3,2)$-face and $(3,3,4,3,3,2,3,3,2)$-face, $(3,3,3,3,2,3,3,4,2)$-face, $(3,3,3,3,2,3,4,3,2)$-face, $(3,3,3,4,2,3,3,3,2)$-face and $(3,3,4,3,2,3,3,3,2)$-face, we know 4-vertex sends charge at least $\frac{3}{8}$ to face and $\tau(v_2 \to f) \geq \frac{1}{6}$ by (R3). So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{6} + \frac{3}{8} = \frac{1}{24}$.

Case 4. $d_2(f) = 3$.

For $(3,3,3,2,3,3,2,3,2)$-face, we know v_1 is a good 3-vertex by Lemma 2.6. By Lemmas 2.4 and 2.5, we know v_5 and v_6 are good 3-vertices. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{6} \times 3 = 0$ by (R3).

By Lemma 2.6, we know that there are no $(3,3,3,3,2,3,2,3,2)$-faces, $(4,3,3,3,2,3,2,3,2)$-faces and $(3,4,3,3,2,3,2,3,2)$-faces in G.

For $(3,3,3,3,2,4,2,3,2)$-face, $(3,3,3,2,3,3,2,4,2)$-face, $(3,3,3,2,3,4,2,3,2)$-face, $(3,3,3,2,4,3,2,3,2)$-face and $(3,3,4,2,3,3,2,3,2)$-face, we know v_2 is either good 3-vertex or best 3-vertex.

So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{6} + \frac{3}{8} = \frac{1}{24}$ by (R3).

For $(3,4,3,2,3,3,2,3,2)$-face and $(4,3,3,2,3,3,2,3,2)$-face, we know v_5 and v_6 are good 3-vertices by Lemmas 2.4 and 2.5. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{3}{8} = \frac{5}{24}$ by (R3).

For $(3,3,2,3,3,2,3,3,2)$-face and $(3,3,2,3,3,2,3,4,2)$-face, we know v_1, v_2, v_4 and v_5 are good 3-vertices Lemmas 2.4 and 2.5. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{6} \times 4 = \frac{1}{6}$ by (R3).

Case 5. $d_2(f) = 4$.

By Lemma 2.6, we know that there are no $(3,2,3,2,3,2,3,2,3)$-faces, $(3,2,4,2,3,2,3,2,3)$-faces and $(4,2,3,2,3,2,3,2,3)$-faces in G.

For $(3,2,3,2,4,2,3,2,3)$-face, we know v_1 and v_9 are good 3-vertices by Lemmas 2.4 and 2.5. So $\omega'(f) \geq -\frac{1}{2} + \frac{3}{8} + \frac{1}{6} \times 2 = \frac{5}{24}$ by (R3).

By Lemma 2.3, we know that there are no adjacent 2-vertices in graph G. So there is no case of $d_2(f) \geq 5$.

Let f be a 10^+-face. We know that a 10^+-face is not involved in discharging rules, so $\omega'(f) = \omega(f) = \frac{1}{2}d(f) - 5 \geq \frac{1}{2} \times 10 - 5 = 0$. $\square$

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