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The Edge-Balanced Properties of Product Graphs of Paths

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Abstract In 2009, Kwong and Lee considered a new labeling problem of graph theory–the edge-balance index set of graph. In this paper, we investigated the edge-balanced properties of product of paths by using a method known as embedding labeling graph method.

Keywords vertex labeling; edge labeling; embedding graph; boundary vertex; inner vertex; $P_m \times P_n$

MR(2020) Subject Classification 05C78; 05C25

1. Introduction

In this paper, all graphs are simple graphs without isolated vertices. In [1], Kwong and the third author considered a new labeling problem of graph theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and $\mathbb{Z}_2 = \{0, 1\}$. An edge labeling f induces a partial vertex labeling f^+: $V(G) \to \mathbb{Z}_2$ defined by $f^+(v) = 0$ if the number of the edges labeled by 0 incident on v is more than the number of edges labeled by 1 incident on v and $f^+(v) = 1$ if the number of the edges labeled by 1 incident to v is more than the number of edges labeled by 0 incident to v. $f^+(v)$ is not defined if the number of the edges labeled by 0 is equal to the number the edges labeled by 1 incident on v. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f(e) = i\}|$. An edge labeling f is said to be edge-friendly if $|e_f(1) - e_f(0)| \le 1$. With these notations, we now introduce the definition of an edge-balanced graph.

Definition 1.1 ([1]) *A graph G is said to be an edge-balance graph if there is an edge-friendly labeling f of G satisfying $|v_f(1) - v_f(0)| \le 1$.*

Definition 1.2 ([1]) *The edge-balance index set of the graph G, denoted by $EBI(G)$, is defined as $\{|v_f(1) - v_f(0)| : f$ is edge-friendly$\}$.*

Kwong and Lee investigated the edge-balance index sets of generalized theta graphs [1] and flower graphs [2]. Lee, Su and Wang [3] investigated the edge-balance index sets of $(p, p+1)$-graphs. Chung and Lee [4] investigated the edge-balance index sets of the envelope graphs of

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stars, paths, and cycles. In [5,6], the edge-balance index sets of L-product of cycles with stars are investigated. Bouchard, Clark and Su [7] gave the exact values of the edge-balance index sets of L-product of cycles with cycles. Chopra, Lee and Su [8] investigated the edge-balance index sets of the fan $P_n + K_1$. Lee, Su and Todt [9] investigated the edge-balance index sets of broken wheels. Lee, Lee and Su [10] present a technique that determines the balance index sets of a graph from its degree sequence.

One can see that if $\{0,1\} \subseteq EBI(G)$, then the graph is edge-balanced. Hence, the notion of edge-balance indices generalizes that of edge-balanced labeling in the sense that if the edge-balance index set for a graph G is known, then the edge-balances-ness of G is determined.

Notation 1.3 *For graph $P_m \times P_n$ $(m, n \geq 2)$, the vertex set $V = \{u_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$, the edge set $E = \{(u_{i,j}, u_{i,j+1}) : 1 \leq i \leq m,\ 1 \leq j \leq n-1\} \cup \{(u_{s,t}, u_{s+1,t}) : 1 \leq s \leq m-1,$ $1 \leq t \leq n\}$.*

$P_4 \times P_6$ is shown in Figure 1.

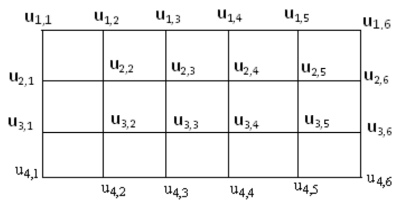

Figure 1 $P_4 \times P_6$

In $P_m \times P_n$, $|V| = mn$, $|E| = 2mn - m - n$.

2. Preliminaries

Before we discuss the edge-balanced properties of Product of Paths, we present a result and some notations which will be used to obtain our main results.

Notation 2.1 *In the following discussions, for the vertices and edges on $P_m \times P_n$, the vertices $u_{i,j}$ $(i = 1, m, 1 \leq j \leq n$ and $2 \leq i \leq m-1,\ j = 1,\ n)$ are said to be boundary vertices, the others are said to be interior vertices. Similarly, the edges $(u_{i,j}, u_{i,j+1})$ $(i = 1, m, 1 \leq j \leq n-1)$ and $(u_{s,t}, u_{s,t+1})$ $(1 \leq s \leq m-1, t = 1, n)$ are said to be boundary edges, the others are said to be interior edges.*

The degrees of the boundary vertices are 2 or 3, and the degrees of all interior vertices are 4.

Notation 2.2 *For any friendly labeling f of $P_m \times P_n$, the maximum value of all $|v_f(1) - v_f(0)|$ is denoted by $M(m,n)$.*

Without losing generality, assume $m \leq n$ on $P_m \times P_n$. An edge e is called a k-edge if $f(e) = k$, $k \in \{0,1\}$, a vertex v is called a k-vertex if $f^+(v) = k$, $k \in \{0,1\}$, a vertex v is called a $*$-vertex if $f^+(v)$ is not defined. We will use $v(0)$, $v(1)$, $v(*)$, $e(0)$, $e(1)$, instead of $v_{f^+}(0)$, $v_{f^+}(1)$, $v_{f^+}(*)$, $e_f(0)$, $e_f(1)$, provided there is no ambiguity.

Theorem 2.3 *For any edge-friendly labeling f of graph G, $v(1) - v(0) = 2v(1) + v(*) - |V|$.*

Proof Let f be an edge-friendly labeling of graph G. Since $v(1)+v(0)+v(*) = |V|$, $v(1)-v(0) = v(1) - (|V| - v(1) - v(*)) = 2v(1) + v(*) - |V|$. □

Notation 2.4 *For a friendly labeling f of $P_m \times P_n$ with index a, when $a = M(m,n)$, then the number of 1-vertices in the interior vertices is denoted by $A_{m,n}$, the number of $*$-vertices in the interior vertices is denoted by $B_{m,n}$.*

Labeled graph A graph G with an edge labeling is called a labeled graph of G.

Notation 2.5 *Embedding labeled graph method 1. Given a labeling of $P_m \times P_n$, for some fixed j with $1 \le j < n$, let the label of the edge $(u_{i,j}, u_{i,j+1})$ be $k_{i,j}$, where $\le i \le m$. Let the vertices of a labeled path P_m be $v_1, v_2, \dots, v_m$ and the label of the edge (v_i, v_{i+1}) be l_i, where $1 \le i < m$. In the type 1 embedding labeled graph method, each edge $(u_{i,j}, u_{i,j+1})$ is subdivided into two edges $(u_{i,j}, v_i)$ and $(v_i, u_{i,j+1})$, both of them are labeled $k_{i,j}$. The newly inserted vertices v_is are connected to form a path P_m, and the edges are labeled a_i, b_i and c_i, respectively. The result is a labeling of $P_m \times P_{n+1}$.*

Embedding labeled graph method 2. Let the vertices of a labeled $P_m \times P_2$ be $v_{i,j}$, where $1 \le i \le m$ and $j = 1, 2$, and the labels of the edges $(v_{i,1}, v_{i,2})$, $(v_{i,1}, v_{i+1,1})$ and $(v_{i,2}, v_{i+1,2})$ be a_i, b_i and c_i, respectively. In a type 2 embedding labeled graph method, each edge $(u_{i,j}, u_{i,j+1})$ is subdivided into three edges $(u_{i,j}, v_{i,1})$, $(v_{i,1}, v_{i,2})$ and $(v_{i,2}, u_{i,j+1})$, with both edges $(u_{i,j}, v_{i,1})$ and $(v_{i,2}, u_{i,j+1})$ labeled $k_{i,j}$. The vertices $v_{i,j}$ are connected to form a $P_m \times P_2$, and the edges are labeled a_i, b_i and c_i, respectively. The result is a labeling of $P_m \times P_{n+2}$.

In general, given a labeled $P_m \times P_k$, in a type k embedding labeled graph method, each edge $(u_{i,j}, u_{i,j+1})$ is split into two (while keeping the same label $k_{i,j}$) and attached to the left-most and right-most column, respectively, of the given labeled $P_m \times P_k$ to form a labeling of $P_m \times P_{n+k}$.

For example, embedding a labeled graph of P_2 on $P_2 \times P_2$ resulting in a labeled graph of $P_2 \times P_3$, is shown in Figure 2.

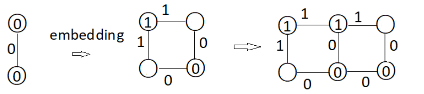

Figure 2 Embedding P_2 on $P_2 \times P_2$

3. EBI($P_2 \times P_n$)

When $n = 2$, the graph $P_2 \times P_2$ is cycle C_4. The fact that EBI(C_4) = {0} was obtained in [1]. So, in this section, we investigate the edge-balanced properties of $P_2 \times P_n$ ($n > 2$).

Theorem 3.1 *For any edge-friendly labeling f of $P_2 \times P_n$,* $M(2,n) = \begin{cases} n-2, & \text{if } n \text{ is even,} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$

Proof First, in $P_2 \times P_n$, $|E| = 3n - 2$, all vertices are the boundary vertices. Next, all cycles are even. Finally, for any edge-friendly labeling f, if the label of any vertex $u_{i,j}$ is 1, then there are at least two 1-edges incident on $u_{i,j}$. Thereby, choose an even cycle that contains the edge $(u_{1,1}, u_{2,1})$, and its length is: $[\frac{3n+1}{2}]$ for $n \equiv 0$ or 1 (mod 4); $[\frac{3n-1}{2}]$ for $n \equiv 2$ or 3 (mod 4).

When $n \equiv 0$ or 1 (mod 4), define the labels of the edges on the above cycle as 1 (except edge $(u_{1,1}, u_{2,1})$), the labels of the remaining edges as 0, then $e(1) = e(0)$ for $n \equiv 0 \pmod 4$; $e(1) = e(0) + 1$ for $n \equiv 1 \pmod 4$.

When $n \equiv 2$ or 3 (mod 4), define the labels of the edges on the cycle as 1, the labels of the remaining edges as 0, then $e(1) = e(0)$ for $n \equiv 2 \pmod 4$; $e(1) = e(0) + 1$ for $n \equiv 3 \pmod 4$.

(1) $n \equiv 0 \pmod 4$.

Since $e(1) = e(0) = \frac{3n}{2} - 1$, $\frac{3n}{2} - 1$ is odd, the maximum length of a cycle formed by 1-edges is $\frac{3n}{2} - 2$, two $*$-vertices can be obtained by one 1-edge and two 0-edges. Hence, the maximum value of $v(1)$ is $\frac{3n}{2} - 2$, at this time, the maximum value of $v(*)$ is 2. $M(2,n) = 2v(1) + v(*) - |V| = 2 \times \frac{3n}{2} - 4 + 2 - 2n = n - 2$.

(2) $n \equiv 2 \pmod 4$.

$v(1) = \frac{3n-2}{2}$, $\frac{3n-2}{2}$ is even, the maximum length of a cycle formed by 1-edges is $\frac{3n-2}{2}$. Hence, the maximum value of $v(1)$ is $\frac{3n-2}{2}$, at this time $v(*) = 0$. $M(2,n) = 2v(1) + v(*) - |V| = 2 \times \frac{3n-2}{2} - 2n = n - 2$.

(3) $n \equiv 1 \pmod 4$.

Using the manner of the discussion in Case 1, the maximum value of $v(1)$ is $\frac{3n+1}{2} - 2$, at this time, the maximum value of $v(*)$ is 2, $M(2,n) = 2v(1) + v(*) - |V| = 2 \times \frac{3n+1}{2} - 4 + 2 - 2n = n - 1$.

(4) $n \equiv 3 \pmod 4$.

Using the manner of the discussion in Case 2, the maximum value of $v(1)$ is $\frac{3n-1}{2}$, at this time $v(*) = 0$, $M(2,n) = 2v(1) + v(*) - |V| = 2 \times \frac{3n-1}{2} - 2n = n - 1$. □

Theorem 3.2 *For any odd integer $n \geq 3$,*

$$\mathrm{EBI}(P_2 \times P_n) = \begin{cases} \{0, 1, \ldots, n-2\}, & \text{if } n \text{ is even,} \\ \{0, 1, \ldots, n-1\}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof The construction in Theorem 3.1 produces a labeling f with index $M(2,n)$, where $M(2,n) = n - 2$ for even n and $M(2,n) = n - 1$ for odd n. Starting with f with index $M(2,n)$, we can construct another labeling g by exchanging the labels of the two edges $(u_{2,1}, u_{2,2})$ and $(u_{1,n}, u_{2,n})$, then

(1) when $n \equiv 0$ or 1 (mod 4), $v(1)$ is decreased by 1, $v(*)$ is increased by 1, the labeling g with index $M(2,n) - 1$ is obtained.

(2) when $n \equiv 2$ or 3 (mod 4), $v(1)$ is decreased by 2, $v(0)$ is decreased by 1, the labeling g with index $M(2,n) - 1$ is obtained.

In the labeling f and g, there exist the edge labeling forms as illustrated in Figure 3.

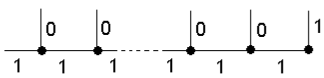

Figure 3 Edge labeling form

Starting with f and g, if exchanging 1-edge and 0-edge successively respectively, as illustrated in Figure 4,

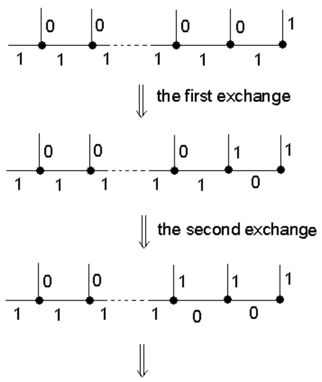

Figure 4 Exchange the edge labels

then in each exchange, $v(1)$ is decreased by 1, $v(*)$ is increased by 1.

When $n > 3$, in the labeling f, there are at least $\frac{M(2,n)}{2}$ exchanges as illustrated in Figure 4, in the labeling g, there are at least $\frac{M(2,n)-2}{2}$ exchanges as illustrated in Figure 4. Once the $\frac{M(2,n)}{2}$ exchanges in the labeling f and the $\frac{M(2,n)-2}{2}$ exchanges in the labeling g are completed, then the labelings with indices $n-4, n-5, \ldots, 0$ (since n is even) can be obtained; the labelings with indices $n-3, n-4, \ldots, 0$ (since n is odd) can be obtained.

When $n = 3$, starting with the labeling with index 2, exchange the labels of $(u_{1,2}, u_{2,2})$ and $(u_{1,3}, u_{2,3})$, then the labeling with index 0 is obtained. The conclusion holds. □

4. EBI($P_m \times P_n$)

When $n > 2$, there exist interior vertices on $P_m \times P_n$. Since the degrees of the interior vertices are 4, the degrees of the boundary vertices are 2 or 3. If index $a = M(m,n)$, then the labels of all boundary vertices must be 1, thus, when the number of 1-vertices on the interior vertices is the greatest too, $2v(1) + v(*)$ must be the greatest. We can see the contribution of two $*$-vertices is the same as that of one 1-vertices for $2v(1) + v(*)$, and when index $a = M(m,n)$, 0-vertices must be adjacent to each other, most of these 0-vertices associated with 0-edges, i.e., as illustrated in Figure 5.

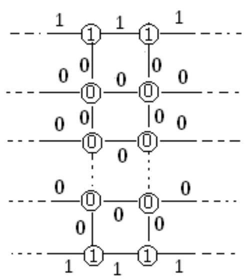

Figure 5 0-vertices in interior vertices

Now, we find the labeling with index $M(m,n)$ by embedding a labeled graph method.

Theorem 4.1 *For $P_m \times P_n$ and $P_m \times P_{n+\frac{6(m-2)}{a}}$ $(m > 2)$, has $M(m, n+\frac{6(m-2)}{a}) = M(m,n) + \frac{2m^2-8}{a}$, where a is the greatest common factor of $2m-5$ and $2m-2$.*

Proof Define a graph with (0,1)-edge labeling of $P_m \times P_2$ and a graph with 0-edge labeling of P_m as illustrated in Figure 6.

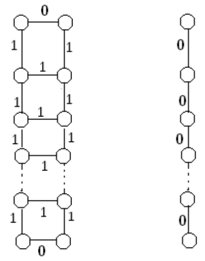

Figure 6 A graph with (0,1)-edge labeling of $P_m \times P_2$ and a graph with 0-edge labeling of P_m

Assume the labeling with index $M(m,n)$ is obtained, $e(1) = b$, $e(0) = c$ ($b = c$ or $|b-c| = 1$). First, embedding a graph with (0,1)-edge labeling of $P_m \times P_2$ as illustrated in Figure 7, then the number of 1-edges is increased by $3m-2$, the number of 0-edges is increased by m.

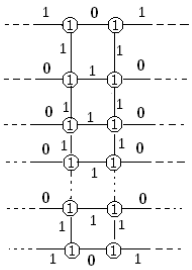

Figure 7 Embedding a graph with (0,1)-edge labeling of $P_m \times P_2$

Since at most two $*$-vertices can be obtained by three interior 1-edges, at most one 1-vertices and one $*$-vertices can be obtained by four interior 1-edges, at most two 1-vertices can be obtained by five interior 1-edges, at most two 1-vertices and two $*$-vertices can be obtained by six interior 1-edges, $\ldots$, at most $2(m-2)$ 1-vertices can be obtained by $3m-2$ 1-edges on interior 1-edges, thereby, embedding a labeled graph with an (0,1)-edge labeling of $P_m \times P_2$ with $2m-2$ interior 1-vertices as shown in Figure 7, does not alter the 1-vertices and 0-vertices that have already be obtained.

When embedding a labeled graph with 0-edge labeling of P_m on the label labeled graph with index k as shown in Figure 5, this is equal to adding a row in interior of the label graph with index k. The number of 1-edges is increased by 2, the number of 0-edges is increased by $2m-3$.

For $e(1) = e(0)$ or $|e(1) - e(0)| = 1$ are satisfied, embedding some graphs with (0,1)-edge labeling of $P_m \times P_2$ and some graphs with 0-edge labeling of P_m, the number of 1-edges increased is equal to the number of 0-edges increased. Embedding a graph with (0,1)-edge labeling of $P_m \times P_2$, the number of 1-edges is $2m-2$ more than the number of 0-edges; embedding a graph with 0-edge labeling of P_m, the number of 0-edges is $2m-5$ more than the number of 1-edges. Hence, let a be the greatest common factor of $2m-5$ and $2m-2$, then when embedding $\frac{2m-5}{a}$

graphs with (0,1)-edge labeling of $P_m \times P_2$ and $\frac{2m-2}{a}$ graphs with 0-edge labeling of P_m, the number of 1-edges increased is equal to the number of 0-edges increased, the number of the rows in interior of the labeled graph with index k is equal to $2\times\frac{2m-5}{a}+\frac{2m-2}{a}=\frac{6(m-2)}{a}$. Thus, we have $M(m,n+\frac{6(m-2)}{a}) = M(m,n)+2m\times\frac{2m-5}{a}+2\times\frac{2m-2}{a}-\frac{2m-2}{a}\times(m-2) = M(m,n)+\frac{2m^2-8}{a}$.□

By Theorem 4.1, we obtain the following labeling with index $M(m,n)$.

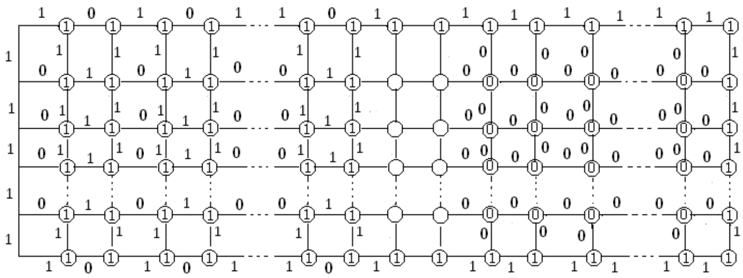

Figure 8 The labeling form in $M(m,n)$

The labels of the edges and the vertices that are not yet labeled will be determined by the concrete values of m and n.

By Theorem 4.1, if the labeling with index $M(m,n)$ is obtained, then the labeling with index $M(m,n+\frac{6(m-2)}{a})$ is also obtained. Thus, when the labelings with indices $M(m,m), M(m,m+1),\ldots,M(m,m+\frac{6(m-2)}{a}-1)$ are obtained, then the labelings with indices $M(m,m+\frac{6k(m-2)}{a})$, $M(m,m+1+\frac{6k(m-2)}{a}),\ldots,M(m,m+\frac{6k(m-2)}{a}-1)$ $(k=1,2,\ldots,)$ are also obtained. By exhaust algorithm, when m is determined, the labelings with indices $M(m,m), M(m,m+1),\ldots,M(m,m+\frac{6(m-2)}{a}-1)$ can be obtained.

When $m>4$, starting with the labeling with index $M(m,m+t)$ $(t=0,1,\ldots,\frac{6(m-2)}{a}-1)$, exchange the labels of edge $(u_{1,1},u_{2,1})$ and $(u_{2,1},u_{2,2})$, then $v(1)$ is decreased by 1, $v(*)$ is increased by 1, the labeling with index $M(m,m+t)-1$ is obtained.

In the labeling with index $M(m,m+t)$ and the labeling with index $M(m,n+t)-1$, there are some 1-edges and 0-edges associated with $u_{1,j}$, $u_{m,s}$ and $u_{t,n}$ as shown in Figure 3. These 1-edges and 0-edges can be exchanged, and these two exchanges do not influence each other, refer to the following discussions of $P_3\times P_n$, $P_4\times P_n$, the edge-balance index sets of $P_m\times P_n$ $(m>4)$ can be obtained.

$$\mathrm{EBI}(P_m\times P_n)=\{0,1,\ldots,M(m,n)\}.$$

When $m=3$ and 4, the 1-edges and 0-edges associated with $u_{1,j}$, $u_{m,s}$ cannot be exchanged at the same time, in the following sections, the edge-balance index sets of $P_3\times P_n$, $P_4\times P_n$ are obtained, respectively.

5. EBI($P_3\times P_n$)

On $P_3\times P_n$, $|V|=3n$, $|E|=5n-3$. Since $m=3$, $2m-5=1$, $2m-2=4$, by Theorem 4.1, $a=1$, $M(3,n+6)=M(3,n)+10$. Hence, assume $M(3,n)=C$, then $M(3,n+6t)=C+10t$. When $n>5$, the number of the boundary vertices is more than or equals to $[\frac{5n-3}{2}]$, thereby, we investigate $M(3,n)$ for $n=6,7,\dots,11$. We obtain the following results.

(1) $n=6$.

Define the labels of the boundary edges as 1 and the labels of the remaining edges as 0, then $e(1)=14=e(0)+1$, $v(1)=14$, $v(0)=4$, $M(3,6)=10$.

(2) $n=7$.

Define the labels of the boundary edges as 1 and the labels of the remaining edges as 0, then $e(1)=16=e(0)$, $v(1)=16$, $v(0)=5$, $M(3,7)=11$.

(3) $n=8$.

Define the labels of the boundary edges and $(u_{1,2},u_{2,2})$ as 1 and the labels of the remaining edges as 0, then $e(1)=19=e(0)+1$, $v(1)=18$, $v(0)=6$, $M(3,8)=12$.

(4) $n=9$.

Define the labels of the boundary edges and $(u_{1,2},u_{2,2})$ as 1 and the labels of the remaining edges as 0, then $e(1)=21=e(0)$, $v(1)=20$, $v(0)=7$, $M(3,9)=13$.

(5) $n=10$.

Define the labels of the boundary edges (except $(u_{1,2},u_{1,3})$), $(u_{1,2},u_{2,2})$, $(u_{1,3},u_{2,3})$ and $(u_{2,2},u_{2,3})$ as 1 and the labels of the remaining edges as 0, then $e(1)=24=e(0)+1$, $v(1)=22$, $v(*)=2$, $v(0)=6$, $M(3,10)=16$.

(6) $n=11$.

Define the labels of the boundary edges (except $(u_{1,2},u_{1,3})$), $(u_{1,2},u_{2,2})$, $(u_{1,3},u_{2,3})$ and $(u_{2,2},u_{2,3})$ as 1 and the labels of the remaining edges as 0, then $e(1)=26=e(0)$, $v(1)=24$, $v(*)=2$, $v(0)=7$, $M(3,11)=17$.

By the above results and Theorem 4.1, when $n>5$,

$$M(3,n)=10+10\times\frac{n-6}{6}=\frac{5n}{3}\ \text{for } n\equiv 0 \pmod 6;$$
$$M(3,n)=12+10\times\frac{n-8}{6}=\frac{5n-4}{3}\ \text{for } n\equiv 2 \pmod 6;$$
$$M(3,n)=16+10\times\frac{n-10}{6}=\frac{5n-2}{3}\ \text{for } n\equiv 4 \pmod 6;$$
$$M(3,n)=11+10\times\frac{n-7}{6}=\frac{5n-2}{3}\ \text{for } n\equiv 1 \pmod 6;$$
$$M(3,n)=13+10\times\frac{n-9}{6}=\frac{5n-6}{3}\ \text{for } n\equiv 3 \pmod 6;$$
$$M(3,n)=17+10\times\frac{n-11}{6}=\frac{5n-4}{3}\ \text{for } n\equiv 5 \pmod 6.$$

Theorem 5.1 *For* $n\equiv 0 \pmod 6$, $\mathrm{EBI}(P_3\times P_n)=\{0,1,\dots,\frac{5n}{3}\}$.

Proof (1) $n>6$.

Step 1. Define the labels of the boundary edges (except $(u_{1,j},u_{1,j+1})$, $(u_{3,j},u_{3,j+1})$ ($j=2,4,\dots,\frac{n}{3}-2$)), the interior edges $(u_{2,j},u_{2,j+1})$, $(u_{1,j},u_{2,j})$, $(u_{2,j},u_{3,j})$, $(u_{1,j+1},u_{2,j+1})$, $(u_{2,j+1},u_{3,j+1})$ ($j=2,4,\dots,\frac{n}{3}-2$) as 1, the labels of the remaining edges as 0, then $e(1)=2n+2-(\frac{n}{3}-2)+(\frac{n}{6}-1)+2(\frac{n}{3}-2)=\frac{5n}{2}-1$, $e(0)=(\frac{5n}{3}-2)$. $v(1)=2n+\frac{n}{3}=\frac{7n}{3}$, $v(*)=0$, $M(3,n)=\frac{14n}{3}-3n=\frac{5n}{3}$, the labeling with index $M(3,n)$ is obtained.

Step 2. Start with the labeling with index $M(3,n)$, exchange the labels $(u_{1,\frac{n}{3}-2}, u_{1,\frac{n}{3}-1})$ and $(u_{1,\frac{n}{3}-1}, u_{2,\frac{n}{3}-1})$, then $v(1)$ is decreased by 1, $v(0)$ is not changed and $v(*)$ is increased by 1, the labeling with index $\frac{5n}{3}-1$ is obtained.

Step 3. Start with the labelings with indices $\frac{5n}{3}$, $\frac{5n}{3}-1$, successively exchange the labels of $(u_{2,j}, u_{3,j})$ and $(u_{3,j-1}, u_{3,j})$ $(\frac{n}{3} \le j \le n-1)$, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1, there are $(\frac{2n}{3})$ exchanges. Once all exchanges are completed, the labelings with indices $\frac{5n}{3}-2, \frac{5n}{3}-3, \ldots, \frac{n}{3}$ can be obtained.

Step 4. Start with the labelings with indices $\frac{n}{3}+1$, $\frac{n}{3}$, exchange the labels of $(u_{2,n-1}, u_{2,n})$ and $(u_{2,n-1}, u_{3,n-1})$, then the labelings with indices $\frac{n}{3}-1$, $\frac{n}{3}-2$ are obtained.

Step 5. Start with the labelings with indices $\frac{n}{3}-1$, $\frac{n}{3}-2$, successively exchange the labels of $(u_{2,t-1}, u_{2,t})$ and $(u_{2,t}, u_{3,t})$ $(t = 2, 4, \ldots, \frac{n-6}{3})$, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1, there are $(\frac{n-6}{6})$ exchanges. Once all exchanges are completed, the labelings with indices $\frac{n}{3}-3, \frac{n}{3}-4, \ldots, 0$ can be obtained.

(2) $n = 6$.

Define the labels of the boundary edges as 1, the labels of the remaining edges as 0, then the labeling with index 10 is obtained.

Start with the labeling with index 10, exchange the labels of $(u_{1,6}, u_{2,6})$ and $(u_{2,5}, u_{2,6})$, the labeling with index 9 is obtained. Start with the labelings with indices 10 and 9, successively exchange the labels of $(u_{2,j}, u_{3,j})$ and $(u_{3,j}, u_{3,j+1})$ $(j = 2, 3, 4)$, then the labelings with indices 8, 7, 6, 5, 4, 3 are obtained.

Start with the labeling with index 4, successively exchange the labels of $(u_{2,2}, u_{3,2})$ and $(u_{2,5}, u_{2,6})$, $(u_{1,2}, u_{2,2})$ and $(u_{1,2}, u_{1,3})$, then the labelings with indices 2, 0 are obtained.

Start with the labeling with index 3, exchange the labels of $(u_{1,5}, u_{1,6})$ and $(u_{1,2}, u_{2,2})$, then the labeling with index 1 is obtained. □

Theorem 5.2 *For* $n \equiv 2 \pmod 6$, $\mathrm{EBI}(P_3 \times P_n) = \{0, 1, \ldots, \frac{5n-4}{3}\}$.

Proof (1) $n > 8$.

Step 1. Define the labels of the boundary edges (except $(u_{1,j}, u_{1,j+1})$, $(u_{3,j}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-8}{3})$), the interior edges $(u_{2,j}, u_{2,j+1})$, $(u_{1,j}, u_{2,j})$, $(u_{2,j}, u_{3,j})$, $(u_{1,j+1}, u_{2,j+1})$, $(u_{2,j+1}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-8}{3})$ as 1, the labels of the remaining edges as 0, then $e(1) = 2n + 2 - (\frac{n-2}{3} - 2) + (\frac{n-2}{6} - 1) + 2(\frac{n-2}{3} - 2) + 1 = \frac{5n}{2} - 2$, $e(0) = (\frac{5n}{3} - 1)$. $v(1) = 2n + \frac{n-2}{3}$, $v(*) = 0$, $M(3,n) = 4n + \frac{2n-4}{3} - 3n = \frac{5n-4}{3}$, the labeling with index $\frac{5n-4}{3}$ is obtained.

Step 2. Start with the labeling with index $\frac{5n-4}{3}$, exchange the labels $(u_{1,\frac{n-8}{3}}, u_{1,\frac{n-5}{3}})$ and $(u_{1,\frac{n-5}{3}}, u_{2,\frac{n-5}{3}})$, then $v(*)$ is increased by 1, $v(1)$ is decreased by 1 and $v(0)$ is not changed, the labeling with index $\frac{5n-7}{3}$ is obtained.

Step 3. Start with the labelings with indices $\frac{5n-4}{3}$, $\frac{5n-7}{3}$, successively exchange the labels of $(u_{2,j}, u_{3,j})$ and $(u_{3,j-1}, u_{3,j})$ $(\frac{n-2}{3} \le j \le n-1)$, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1, there are $(\frac{2n+2}{3})$ exchanges. Once all exchanges are completed, the labelings with indices $\frac{5n-10}{3}, \frac{5n-13}{3}, \ldots, \frac{n-8}{3}$ can be obtained.

Step 4. Start with the labelings with indices $\frac{n-5}{3}$, $\frac{n-8}{3}$, exchange the labels of $(u_{2,n-1}, u_{2,n})$

and $(u_{1,n-1}, u_{2,n-1})$, the labelings with indices $\frac{n-11}{3}$, $\frac{n-14}{3}$ are obtained.

Step 5. Start with the labelings with indices $\frac{n-11}{3}$, $\frac{n-14}{3}$, successively exchange the labels of $(u_{2,t-1}, u_{2,t})$ and $(u_{2,t}, u_{3,t})$ $(t = 2, 4, \ldots, \frac{n-14}{3})$, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1, there are $(\frac{n-14}{6})$ exchanges. Once all exchanges are completed, the labelings with indices $\frac{n-17}{3}, \frac{n-20}{3}, \ldots, 0$ can be obtained.

(2) $n = 8$.

Define the labels of the boundary edges as 1, the labels of the remaining edges as 0, then the labeling with index 12 is obtained.

Start with the labeling with index 12, exchange the labels of $(u_{1,8}, u_{2,8})$ and $(u_{2,7}, u_{2,8})$, the labeling with index 11 is obtained.

Start with the labelings with indices 12, 11, successively exchange the labels of $(u_{2,j}, u_{3,j})$ and $(u_{3,j-1}, u_{3,j})$ $(3 \le j \le 7)$, the labelings with indices $10, 9, \ldots, 1$ are obtained.

Start with the labeling with index 2, exchange the labels of $(u_{1,6}, u_{1,7})$ and $(u_{1,7}, u_{2,7})$, the labeling with index 0 is obtained. □

Theorem 5.3 *For* $n \equiv 4 \pmod 6$, $\mathrm{EBI}(P_3 \times P_n) = \{0, 1, \ldots, \frac{5n-2}{3}\}$.

Proof Step 1. Define the labels of the boundary edges (except $(u_{1,j}, u_{1,j+1})$, $(u_{3,j}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-10}{3})$, $(u_{1,\frac{n-4}{3}}, u_{1,\frac{n-1}{3}})$), the interior edges $(u_{2,j}, u_{2,j+1})$, $(u_{1,j}, u_{2,j})$, $(u_{2,j}, u_{3,j})$, $(u_{1,j+1}, u_{2,j+1})$, $(u_{2,j+1}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-10}{3})$, $(u_{2,\frac{n-4}{3}}, u_{2,\frac{n-1}{3}})$, $(u_{1,\frac{n-4}{3}}, u_{2,\frac{n-4}{3}})$, $(u_{2,\frac{n-1}{3}}, u_{3,\frac{n-1}{3}})$ as 1, the labels of the remaining edges as 0, then

$$e(1) = 2n + 2 - \frac{n-4}{3} + 1 + \frac{n-4}{6} + 2(\frac{n-4}{3} - 1) = \frac{5n}{2} - 1,\ e(0) = (\frac{5n}{3} - 2),$$

$$v(1) = 2n + 2 + \frac{n-10}{3},\ v(*) =,\ |v(1) - v(0)| = 4n + 4 + \frac{2n-20}{3} + 2 - 3n = \frac{5n-2}{3},$$

the labeling with index $M(3,n)$ is obtained.

Step 2. Start with the labeling with index $M(3,n)$, exchange the labels of $(u_{1,\frac{n-4}{3}}, u_{1,\frac{n-1}{3}})$ and $(u_{1,\frac{n-1}{3}}, u_{2,\frac{n-1}{3}})$, $v(*)$ is decreased by 1, $v(0)$ is increased by 1, $v(1)$ is not changed, the labeling with index $\frac{5n-2}{3} - 1$ is obtained.

Step 3. Start with the labelings with indices $\frac{5n-2}{3}$, $\frac{5n-2}{3} - 1$, successively exchange the labels of $(u_{2,j}, u_{3,j})$ and $(u_{3,j-1}, u_{3,j})$ $(\frac{n+2}{3} \le j \le n-1)$, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1, there are $(\frac{2n-2}{3})$ exchanges. Once all exchanges are completed, the labelings with indices $\frac{5n-8}{3} - 2, \frac{5n-11}{3}, \ldots, \frac{n-1}{3}$ can be obtained.

Step 4. Start with the labelings with indices $\frac{n+2}{3}$, $\frac{n-1}{3}$, exchange the labels of $(u_{2,n-1}, u_{3,n-1})$ and $(u_{2,n-1}, u_{2,n})$, then the labelings with indices $\frac{n-4}{3}$, $\frac{n-7}{3}$ are obtained

Step 5. Start with the labelings with indices $\frac{n-4}{3}$, $\frac{n-7}{3}$, successively the labels of $(u_{2,t}, u_{3,t})$ and $(u_{2,t-1}, u_{2,t})$ $(s = 2, 4, \ldots, \frac{n-4}{3})$, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1, there are $\frac{n-4}{6}$ exchanges. Once all exchanges are completed, the labelings with indices $\frac{n-10}{3}, \frac{n-13}{3}, \ldots, 0$, can be obtained. □

Theorem 5.4 *For* $n \equiv 1 \pmod 6$, $\mathrm{EBI}(P_3 \times P_n) = \{0, 1, \ldots, \frac{5n-5}{3}\}$.

Proof $n > 7$.

Step 1. Define the labels of the boundary edges (except $(u_{1,j}, u_{1,j+1})$, $(u_{3,j}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-7}{3})$), the interior edges $(u_{2,j}, u_{2,j+1})$, $(u_{1,j}, u_{2,j})$, $(u_{2,j}, u_{3,j})$, $(u_{1,j+1}, u_{2,j+1})$, $(u_{2,j+1}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-7}{3})$ as 1, the labels of the remaining edges as 0, then $e(1) = 2n + 2 - (\frac{n-1}{3} - 2) + (\frac{n-1}{6} - 1) + 2(\frac{n-1}{3} - 2) = \frac{5n-3}{2}$, $e(0) = (\frac{5n-3}{3})$, $v(1) = 2n + \frac{n-1}{3}$, $v(*) = 0$, $M(3, n) = 4n + \frac{2n-2}{3} - 3n = \frac{5n-2}{3}$, the labeling with index $M(3, n)$ is obtained.

The steps 2–5 are similar to those in Theorem 5.1, we can know that the conclusion holds.

For $n = 7$, the discussions are similar to those about $n = 6$. □

Theorem 5.5 *For* $n \equiv 3 \pmod 6$, $\mathrm{EBI}(P_3 \times P_n) = \{0, 1, \ldots, \frac{5n-6}{3}\}$.

Proof $n > 9$.

Step 1. Define the labels of the boundary edges (except $(u_{1,j}, u_{1,j+1})$, $(u_{3,j}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-9}{3})$), the interior edges $(u_{2,j}, u_{2,j+1})$, $(u_{1,j}, u_{2,j})$, $(u_{2,j}, u_{3,j})$, $(u_{1,j+1}, u_{2,j+1})$, $(u_{2,j+1}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-9}{3})$ as 1, the labels of the remaining edges as 0, then $e(1) = 2n + 2 - (\frac{n-3}{3} - 2) + (\frac{n-3}{6} - 1) + 2(\frac{n-3}{3} - 2) + 1 = \frac{5n-3}{2} = e(0)$, $v(1) = 2n + \frac{n-3}{3}$, $v(*) = 0$, $M(3, n) = 4n + \frac{2n-6}{3} - 3n = \frac{5n-6}{3}$, the labeling with index $M(3, n)$ is obtained.

The steps 2–5 are similar to those in Theorem 5.2, we can know that the conclusion holds.

For $n = 9$, the discussions are similar to those about $n = 8$. □

Theorem 5.6 *For* $n \equiv 5 \pmod 6$, $\mathrm{EBI}(P_3 \times P_n) = \{0, 1, \ldots, \frac{5n-4}{3}\}$.

Proof Define the labels of the boundary edges (except $(u_{1,j}, u_{1,j+1})$, $(u_{3,j}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-11}{3})$, $(u_{1,\frac{n-5}{3}}, u_{1,\frac{n-2}{3}})$), the interior edges $(u_{2,j}, u_{2,j+1})$, $(u_{1,j}, u_{2,j})$, $(u_{2,j}, u_{3,j})$, $(u_{1,j+1}, u_{2,j+1})$, $(u_{2,j+1}, u_{3,j+1})$ $(j = 2, 4, \ldots, \frac{n-11}{3})$, $(u_{2,\frac{n-5}{3}}, u_{2,\frac{n-2}{3}})$, $(u_{1,\frac{n-5}{3}}, u_{2,\frac{n-5}{3}})$, $(u_{2,\frac{n-2}{3}}, u_{3,\frac{n-2}{3}})$ as 1, the labels of the remaining edges as 0, then $e(1) = 2n + 2 - \frac{n-5}{3} + 1 + \frac{n-5}{6} + 2(\frac{n-5}{3} - 1) = \frac{5n-3}{2} = e(0)$, $v(1) = 2n + 2 + \frac{n-11}{3}$, $v(*) = 2$, $M(3, n) = 4n + 4 + \frac{2n-22}{3} + 2 - 3n = \frac{5n-4}{3}$, the labeling with index $M(3, n)$ is obtained.

The steps 2–5 are similar to those in Theorem 5.3, and we can know that the conclusion holds. □

Theorem 5.7 $\mathrm{EBI}(P_3 \times P_3) = \{0, 1, 2, 3\}$; $\mathrm{EBI}(P_3 \times P_4) = \{0, 1, \ldots, 5\}$; $\mathrm{EBI}(P_3 \times P_5) = \{0, 1, \ldots, 6\}$.

Proof (1) $n = 3$.

Define the labels of the boundary edges (except $(u_{1,1}, u_{2,1})$, $(u_{2,1}, u_{3,1})$) as 1, the labels of the remaining edges as 0, then $M(3, 3) = 3$, the labeling with index 3 is obtained.

Start with the labeling with index 3, exchange the labels of $(u_{1,3}, u_{2,3})$ and $(u_{2,2}, u_{2,3})$, the labeling with index 2 is obtained. Start with the labelings with indices 3 and 2, exchange the labels of $(u_{2,1}, u_{3,1})$ and $(u_{3,1}, u_{3,2})$, the labelings with indices 1 and 0 are obtained.

(2) $n = 4$.

Define the labels of the boundary edges (except $(u_{1,1}, u_{2,1})$) as 1, the labels of the remaining edges as 0, then the labeling with index 5 is obtained. Start with the labeling with index 5, exchange the labels of $(u_{1,3}, u_{1,4})$ and $(u_{2,3}, u_{2,4})$, the labeling with index 4 is obtained. Start with the labelings with indexes 5 and 4, exchange the labels of $(u_{3,2}, u_{3,3})$ and $(u_{2,2}, u_{3,2})$, the labelings with indices 3 and 2 are obtained. Start with the labeling with index 3, exchange the labels of $(u_{1,2}, u_{1,3})$ and $(u_{1,3}, u_{2,3})$, the labeling with index 1 is obtained. Start with the labeling with index 2, exchange the labels of $(u_{1,3}, u_{1,4})$ and $(u_{1,2}, u_{2,2})$, the labeling with index 0 is obtained.

(3) $n = 5$.

Define the labels of the boundary edges (except $(u_{1,1}, u_{2,1})$) as 1, the labels of the remaining edges as 0, then the labeling with index 6 is obtained. Start with the labeling with index 6, exchange the labels of $(u_{2,1}, u_{3,1})$ and $(u_{2,4}, u_{2,5})$, the labeling with index 5 is obtained. Start with the labelings with indices 6 and 5, successively exchange the labels of $(u_{3,j}, u_{3,j+1})$ and $(u_{2,j}, u_{3,j})$ $(j = 2, 3)$, then the labelings with indices 4, 3, 2, 1 are obtained. Start with the labeling with index 2, exchange the labels of $(u_{1,3}, u_{1,4})$ and $(u_{1,4}, u_{2,4})$, the labeling with index 0 is obtained. □

6. EBI($P_4 \times P_n$)

On $P_4 \times P_n$, $|V| = 4n$, $|E| = 7n - 4$.

Since $m = 4$, $2m - 5 = 3$, $2m - 2 = 6$, It follows from Theorem 4.1, $a = 3$, $M(3, n + 4) = M(3, n) + 8$. Hence, assume $M(4, n) = C$, then $M(4, n + 6t) = C + 8t$. Thereby, we investigate $M(4, n)$ for $n = 4$, 5, 6 and 7. By exhaust algorithm, then

(1) $n = 4$.

Define the labels of all boundary edges as 1, the labels of the remaining edges as 0, then $e(1) = 12 = e(0)$, $v(1) = 12$, $v(0) = 4$, $M(4, 4) = 8$.

(2) $n = 5$.

Define the labels of the boundary edges (except $(u_{1,2}, u_{1,3})$), the interior edges $(u_{2,2}, u_{2,3})$ and $(u_{1,j+1}, u_{2,j+1})$ $(j = 1, 2)$ as 1, the labels of the remaining edges as 0, then $e(1) = 16 = e(0) + 1$, $v(1) = 10$, $v(0) = 4$, $M(4, 5) = 10$.

(3) $n = 6$.

Define the labels of the boundary edges (except $(u_{1,2}, u_{1,3})$), the interior edges $(u_{2,2}, u_{2,3})$ and $(u_{1,j+1}, u_{2,j+1})$, $(u_{2,j+1}, u_{3,j+1})$ $(j = 1, 2)$, $(u_{2,2}, u_{3,2})$ as 1, the labels of the remaining edges as 0, then $e(1) = 19 = e(0)$, $v(1) = 17$, $v(0) = 6$, $M(4, 6) = 11$.

(4) $n = 7$.

Define the labels of the boundary edges (except $(u_{1,2}, u_{1,3})$), the interior edges $(u_{i,2}, u_{i,3})$ $(i = 2, 3)$ and $(u_{s,t}, u_{s+1,t})$ $(s = 1, 2, t = 2, 3)$ as 1, the labels of the remaining edges as 0, then $e(1) = 23 = e(0) + 1$, $v(1) = 20$, $v(0) = 6$, $M(4, 7) = 14$.

By the above results and Theorem 4.1, then

$$M(4, n) = 8 + 8 \times \frac{n - 4}{4} = 2n \text{ for } n \equiv 0 \pmod 4;$$

$$M(4,n)=10+8\times\frac{n-5}{4}=2n \text{ for } n\equiv 1 \pmod 4;$$
$$M(4,n)=11+8\times\frac{n-6}{4}=2n-1 \text{ for } n\equiv 2 \pmod 4;$$
$$M(4,n)=14+8\times\frac{n-7}{4}=2n \text{ for } n\equiv 3 \pmod 4.$$

Theorem 6.1 *For even* n, $\mathrm{EBI}(P_4\times P_n)=\begin{cases}\{0,1,\ldots,2n\}, & \text{if } n\equiv 0 \pmod 4,\\ \{0,1,\ldots,2n-1\}, & \text{if } n\equiv 2 \pmod 4.\end{cases}$

Proof (1) $n\equiv 0 \pmod 4$ and $n>4$.

Step 1. Define the labels of the boundary edges (except $(u_{i,2j},u_{i,2j+1})$ $(i=1,4,\ j=1,2,\ldots,\frac{n}{4}-1)$), the interior edges $(u_{i,k},u_{i+1,k})$ $(i=1,2,3,k=2,3,\ldots,\frac{n-2}{2})$, $(u_{s,2j},u_{s,2j+1})$ $(s=2,3,j=1,2,\ldots,\frac{n-4}{4})$ as 1, the labels of the remaining edges as 0, then $e(1)=2n+4-2\times\frac{n-4}{4}+6\times\frac{n-4}{4}+2\times\frac{n-4}{4}=\frac{7n-4}{2}=e(0)$. $v(1)=2n+4+n-4=3n$, $v(*)=0$, $v(0)=n$, $M(4,n)=2n$, the labeling with index $M(4,n)$ is obtained.

Step 2. Start with the labeling with index $M(4,n)$, exchange the labels of $(u_{1,1},u_{2,1})$ and $(u_{2,1},u_{2,2})$, $v(1)$ is decreased by 1, $v(*)$ is increased by 1, $v(0)$ is not changed, then the labeling with index $2n-1$ is obtained.

Step 3. Start with the labelings with indices $2n$, $2n-1$, successively exchange the labels of $(u_{1,t},u_{1,t+1})$ and $(u_{1,t+1},u_{2,t+1})$, $(u_{4,t},u_{4,t+1})$ and $(u_{3,t+1},u_{4,t+1})$ $(t=\frac{n-2}{2},\frac{n}{2},\ldots,n-2)$, there are n exchanges, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1. Once all exchanges are completed, the labelings with indices $2n-2,2n-3,\ldots,0$ can be obtained.

(2) $n=4$.

Define the labels of all boundary edges as 1,the labels of the remaining edges as 0, then the labeling with index 8 is obtained. Start with the labeling with index 8, exchange the labels of $(u_{1,1},u_{2,1})$ and $(u_{2,1},u_{2,2})$, the labeling with index 7 is obtained.

Start with the labelings with indices 8, 7, successively exchange the labels of $(u_{1,2},u_{1,3})$ and $(u_{1,3},u_{2,3})$, $(u_{2,1},u_{3,1})$ and $(u_{3,1},u_{3,2})$, $(u_{4,2},u_{4,3})$ and $(u_{3,3},u_{4,3})$, $(u_{3,3},u_{4,3})$ and $(u_{3,3},u_{3,4})$, the labelings with indices $6,5,\ldots,0$ can be obtained.

(3) $n\equiv 2 \pmod 4$ and $n>6$.

Step 1. Define the labels of the boundary edges (except $(u_{i,2j},u_{i,2j+1})$, $i=1,4,j=1,2,\ldots,\frac{n-2}{4}-1)$, $(u_{1,\frac{n-2}{4}},u_{1,\frac{n+2}{4}})$), the interior edges $(u_{i,k},u_{i+1,k})$ $(i=1,2,3,k=2,3,\ldots,\frac{n-4}{2})$, $(u_{s,2j},u_{s,2j+1})$ $(s=2,3,j=1,2,\ldots,\frac{n-6}{4})$, $(u_{2,\frac{n-4}{2}},u_{2,\frac{n-2}{2}})$ and $(u_{2,\frac{n-2}{2}},u_{2,\frac{n}{2}})$ as 1, the labels of the remaining edges as 0, then $e(1)=2n+4-2\times\frac{n-6}{4}-1+6\times\frac{n-6}{4}+2\times\frac{n-6}{4}+4=\frac{7n-4}{2}=e(0)$. $v(1)=2n+4+n-6+1=3n-1$, $v(*)=1$, $v(0)=n$, $M(4,n)=2n-1$, the labeling with index $M(4,n)$ is obtained.

Step 2. Start with the labeling with index $M(4,n)$, exchange the labels of $(u_{2,1},u_{2,2})$ and $(u_{2,2},u_{2,3})$, $v(1)$ is decreased by 1, $v(*)$ is increased by 1, $v(0)$ is not changed, then the labeling with index $2n-2$ is obtained.

Step 3. Start with the labelings with indices $2n-1$, $2n-2$, successively exchange the labels of $(u_{4,\frac{n-2}{2}},u_{4,\frac{n}{2}})$ and $(u_{3,\frac{n}{2}},u_{4,\frac{n}{2}})$, $(u_{1,t},u_{1,t+1})$ and $(u_{1,t+1},u_{2,t+1})$, $(u_{4,t},u_{4,t+1})$ and

$(u_{3,t+1}, u_{4,t+1})$ $(t = \frac{n}{2}, \frac{n+2}{2}, \ldots, n-2)$, there are $n-1$ exchanges, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1. Once all exchanges are completed, the labelings with indices $2n-3, 2n-4, \ldots, 0$ can be obtained.

(4) $n = 6$.

Define the labels of the boundary edges (except $(u_{1,2}, u_{1,3})$), the interior edges $(u_{i,k}, u_{i+1,k})$ $(i = 1, 2, 3, k = 2, 3)$, $(u_{2,1}, u_{2,2})$ and $(u_{2,2}, u_{2,3})$ as 1, the labels of the remaining edges as 0, then the labeling with index 11 is obtained. Start with the labeling with 11, exchange the labels of $(u_{3,1}, u_{4,1})$ and $(u_{3,1}, u_{3,2})$, the labeling with index 10 is obtained.

Start with the labelings with indices 11, 10, successively exchange the labels of $(u_{4,2}, u_{4,3})$ and $(u_{3,3}, u_{4,3})$, $(u_{1,t}, u_{1,t+1})$ and $(u_{1,t+1}, u_{2,t+1})$, $(u_{4,t}, u_{4,t+1})$ and $(u_{3,t+1}, u_{4,t+1})$ $(t = 3, 4)$, the labelings with indices $9, 8, \ldots, 0$ can be obtained. □

Theorem 6.2 *If* $n > 3$ *is odd, then* $\mathrm{EBI}(P_4 \times P_n) = \{0, 1, \ldots, 2n\}$.

Proof (1) $n \equiv 1 \pmod 4$ and $n > 5$.

Step 1. Define the labels of the boundary edges (except $(u_{i,2j}, u_{i,2j+1})$ $(i = 1, 4, j = 1, 2, \ldots, \frac{n-5}{4})$, $(u_{1,\frac{n+3}{4}}, u_{1,\frac{n+7}{4}})$), the interior edges $(u_{i,k}, u_{i+1,k})$ $(i = 1, 2, 3, k = 2, 3, \ldots, \frac{n-5}{2})$, $(u_{1,\frac{n-3}{2}}, u_{2,\frac{n-3}{2}})$, $(u_{1,\frac{n-1}{2}}, u_{2,\frac{n-1}{2}})$, $(u_{s,2j}, u_{s,2j+1})$ $(s = 2, 3, j = 1, 2, \ldots, \frac{n-5}{4})$, $(u_{2,\frac{n-1}{2}}, u_{2,\frac{n+1}{2}})$ as 1, the labels of the remaining edges as 0, then $e(1) = 2n+4-2\times\frac{n-5}{4}-1+6\times\frac{n-5}{4}+2\times\frac{n-5}{4}+3 = \frac{7n-3}{2} = e(0)+1$. $v(1) = 2n+4+n-5 = 3n-1$, $v(*) = 2$, $v(0) = n-1$, $M(4, n) = 2n$, the labeling with index $M(4, n)$ is obtained.

Step 2. Start with the labeling with index $M(4, n)$, exchange the labels of $(u_{1,1}, u_{2,1})$ and $(u_{2,1}, u_{2,2})$, $v(1)$ is decreased by 1, $v(*)$ is increased by 1, $v(0)$ is not changed, then the labeling with index $2n-1$ is obtained.

Step 3. Start with the labelings with indices $2n$, $2n-1$, successively exchange the labels of $(u_{4,\frac{n-3}{2}}, u_{4,\frac{n-1}{2}})$ and $(u_{3,\frac{n-1}{2}}, u_{4,\frac{n-1}{2}})$, $(u_{4,\frac{n-1}{2}}, u_{4,\frac{n+1}{2}})$ and $(u_{3,\frac{n+1}{2}}, u_{4,\frac{n+1}{2}})$, $(u_{1,t}, u_{1,t+1})$ and $(u_{1,t+1}, u_{2,t+1})$, $(u_{4,t}, u_{4,t+1})$ and $(u_{3,t+1}, u_{4,t+1})$ $(t = \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-2)$, $(u_{3,n-1}, u_{4,n-1})$ and $(u_{3,n-1}, u_{3,n})$, there are n exchanges, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1. Once all exchanges are completed, the labelings with indices $2n-2, 2n-3, \ldots, 0$ can be obtained.

(2) $n = 5$.

Define the labels of the boundary edges (except $(u_{1,2}, u_{1,3})$), the interior edges $(u_{i,k}, u_{i+1,k})$ $(i = 1, 2, 3, k = 2, 3)$, $(u_{2,2}, u_{2,3})$ as 1, the labels of the remaining edges as 0, then $e(1) = 16 = e(0) + 1$. $v(1) = 14$, $v(*) = 2$, $v(0) = 4$, $M(4, 5) = 10$, the labeling with index 10 is obtained. Start with the labeling with index 10, exchange the labels of $(u_{1,1}, u_{2,1})$ and $(u_{2,1}, u_{2,2})$, the labeling with index 9 is obtained.

Start with the labelings with indices 10, 9, exchange the labels of $(u_{2,1}, u_{3,1})$ and $(u_{3,1}, u_{3,2})$, $(u_{4,2}, u_{4,3})$ and $(u_{3,3}, u_{4,3})$, $(u_{1,3}, u_{1,4})$ and $(u_{1,4}, u_{2,4})$, $(u_{4,3}, u_{4,4})$ and $(u_{3,4}, u_{4,4})$, $(u_{3,4}, u_{4,5})$ and $(u_{3,4}, u_{3,5})$ successively, the labelings with indices $8, 7, \ldots, 0$ can be obtained.

(3) $n \equiv 3 \pmod 4$ and $n > 7$.

Step 1. Define the labels of the boundary edges (except $(u_{i,2j}, u_{i,2j+1})$ $(i = 1, 4,\ j =$

$1, 2, \ldots, \frac{n-7}{4}$), $(u_{1,\frac{n-3}{2}}, u_{1,\frac{n+1}{2}})$), the interior edges $(u_{i,k}, u_{i+1,k})$ ($i = 1, 2, 3, k = 2, 3, \ldots, \frac{n-1}{2}$, except $(u_{3,\frac{n-3}{2}}, u_{4,\frac{n-3}{2}})$, $(u_{3,\frac{n-1}{2}}, u_{4,\frac{n-1}{2}})$), $(u_{s,2j}, u_{s,2j+1})$ ($s = 2, 3, j = 1, 2, \ldots, \frac{n-3}{4}$) as 1, the labels of the remaining edges as 0, then $e(1) = \frac{7n-3}{2} = e(0) + 1$. $v(1) = 3n - 1$, $v(*) = 2$, $v(0) = n - 1$, $M(4, n) = 2n$, the labeling with index $M(4, n)$ is obtained.

Step 2. Start with the labeling with index $M(4, n)$, exchange the labels of $(u_{1,1}, u_{2,1})$ and $(u_{2,1}, u_{2,2})$, $v(1)$ is decreased by 1, $v(*)$ is increased by 1, $v(0)$ is not changed, then the labeling with index $2n - 1$ is obtained.

Step 3. Start with the labelings with indices $2n$, $2n - 1$, successively exchange the labels of $(u_{1,t}, u_{1,t+1})$ and $(u_{1,t+1}, u_{2,t+1})$, $(u_{4,t}, u_{4,t+1})$ and $(u_{3,t+1}, u_{4,t+1})$ ($t = \frac{n-1}{2}, \frac{n+1}{2}, \ldots, n - 2$), $(u_{3,n-1}, u_{4,n-1})$ and $(u_{3,n-1}, u_{3,n})$, there are n exchanges, in each exchange such that $v(1)$ is decreased by 1, $v(0)$ is increased by 1. Once all exchanges are completed, the labelings with indices $2n - 2, 2n - 3, \ldots, 0$ can be obtained.

(2) $n = 7$.

Define the labels of the boundary edges (except $(u_{1,2}, u_{1,3})$), the interior edges $(u_{i,k}, u_{i+1,k})$ ($i = 1, 2, k = 2, 3$), $(u_{s,2j}, u_{s,2j+1})$ ($s = 2, 3, j = 1$) as 1, the labels of the remaining edges as 0, then $M(4, 7) = 14$, the labeling with index 14 is obtained. Start with the labeling with index 14, exchange the labels of $(u_{1,1}, u_{2,1})$ and $(u_{2,1}, u_{2,2})$, the labeling with index 13 is obtained.

Start with the labelings with indices 14, 13, successively exchange the labels of $(u_{1,t}, u_{1,t+1})$ and $(u_{1,t+1}, u_{2,t+1})$, $(u_{4,t}, u_{4,t+1})$ and $(u_{3,t+1}, u_{4,t+1})$ ($t = 3, 4, 5$), $(u_{3,6}, u_{4,6})$ and $(u_{3,6}, u_{3,7})$, the labelings with indices 12, 11, …, 0 can be obtained. □

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