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Some Remarks on Locally Almost Perfect Domains

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Abstract An integral domain R is called a locally almost perfect domain provided that $R_{\mathfrak{m}}$ is an almost perfect domain for any maximal ideal $\mathfrak{m}$ of R. In this paper, we give several characterizations of locally almost perfect domains in terms of locally perfect rings, almost projective modules, weak-injective modules, almost strongly flat modules and strongly Matlis cotorsion modules.

Keywords locally almost perfect domain; almost projective module; almost strongly flat module; strongly Matlis cotorsion module

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1. Introduction

Throughout this paper, R is always a commutative integral domain with identity and $Q := Q(R)$ is the quotient field of R. Recall from [1] that an integral domain R is called an almost perfect domain if all its proper homomorphic images are perfect. It first emerged in the investigation of the existence of strongly flat covers of modules over integral domains. Actually, Bazzoni [2] showed that an integral domain R is almost perfect if and only if any R-module has a strongly flat cover, if and only if any flat R-module is strongly flat. In 2009, Fuchs and Lee [3] obtained that an integral domain R is almost perfect if and only if any divisible R-module is weak-injective, if and only if homomorphic images of weak-injective R-modules are weak-injective, if and only if the class $\mathcal{F}_1$ of R-modules with weak dimension at most 1 is equal to the class $\mathcal{P}_1$ of R-modules with projective dimension at most 1. In 2010, Bazzoni [4] characterized almost perfect domains R as integral domains over which any R-module has an $\mathcal{P}_1$-cover, or any R-module has a divisible envelope, or any direct sum of weak-injective R-modules is weak-injective. Recently, Hrbek [5] showed that an integral domain R is almost perfect if and only if the class of all divisible R-modules is closed under flat covers.

It is an important approach to study integral domains via their localizations at all maximal ideals. For example, almost Dedekind domains are defined to be integral domains whose localizations at all maximal ideals are Dedekind domains. Certainly, any Dedekind domain is an almost Dedekind domain. However, an almost Dedekind domain is a Dedekind domain exactly

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when it is also of finite character [6, Corollary 3.4.8]. In 2019, Zhou et. al [7] introduced almost global dimensions of commutative rings using almost projective modules and then characterized almost Dedekind domains as integral domains with almost global dimension at most 1. Very recently, Zhou et. al [8] introduced locally perfect rings whose localization at any maximal ideal are perfect rings. They characterized locally perfect rings as commutative rings over which the classes of all almost projective modules have covers, or all flat modules are almost projective. Recall from [1] that an integral domain R is called a locally almost perfect domain (LAPD for short) provided that $R_{\mathfrak{m}}$ is an almost perfect domain for any maximal ideal $\mathfrak{m}$ of R. Certainly, every almost perfect domain is an LAPD. However, an LAPD is an almost perfect domain if and only if it is h-local [1, Theorem 3.6]. In this paper, we give some new characterizations of LAPDs in terms of locally perfect rings, the class $\mathcal{AP}_1$ of all R-modules with almost projective dimensions at most 1, the class $\mathcal{F}_1$ of all R-modules with flat dimensions at most 1, the class $\mathcal{WI}$ of all weak-injective modules, the class $\mathcal{ASF}$ of all almost strongly flat modules and the class $\mathcal{SMC}$ of all strongly Matlis cotorsion modules. In conclusion, we mainly prove the following result.

Theorem 1.1 *Let R be a non-field integral domain. Then the following statements are equivalent for R:*

(1) R *is an* LAPD*;*

(2) $R/\langle u\rangle$ *is locally perfect for any nonzero non-unit element* u *in* R*;*

(3) R/I *is locally perfect for any nonzero proper ideal* I *of* R*;*

(4) $\mathcal{AP}_1 = \mathcal{F}_1$*;*

(5) $\mathcal{AP}_1^{\perp} = \mathcal{WI}$*;*

(6) $(\mathcal{AP}_1, \mathcal{AP}_1^{\perp})$ *is a perfect cotorsion pair;*

(7) any R*-module has an* $\mathcal{AP}_1$*-cover;*

(8) any R*-module has an* $\mathcal{AP}_1^{\perp}$*-envelope;*

(9) any flat R*-module is almost strongly flat;*

(10) any direct limit of almost strongly flat R*-modules is almost strongly flat;*

(11) any direct limit of projective R*-modules is almost strongly flat;*

(12) any R*-module has an* $\mathcal{ASF}$*-cover;*

(13) any $R_{\mathfrak{m}}$*-module viewed as an* R*-module has an almost strongly flat cover for any maximal ideal* $\mathfrak{m}$ *of* R*;*

(14) $(\mathcal{ASF}, \mathcal{SMC})$ *is a perfect cotorsion pair;*

(15) any strongly Matlis cotorsion module is cotorsion.

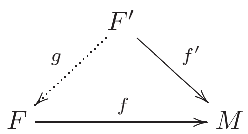
2. Almost strongly flat modules

Recall from [9] that an R-module M satisfying $\mathrm{Ext}_R^1(Q, M) = 0$ is said to be a Matlis cotorsion module (denoted by $M \in \mathcal{MC}$), and an R-module F is said to be *strongly flat*, denoted by $F \in \mathcal{SF}$, if $\mathrm{Ext}_R^1(F, M) = 0$ for any $M \in \mathcal{MC}$. Obviously, any strongly flat module is flat. Recall from [7] that an R-module M is said to be *almost projective* provided that $\mathrm{Ext}_R^1(M, N) = 0$

for any $R_{\mathfrak{m}}$-module N, where $\mathfrak{m} \in \mathrm{Max}(R)$. By [7, Theorem 2.3], an R-module M is almost projective if and only if $M_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$, if and only if $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$. Now we introduce almost strongly flat modules which can be seen as generations of both almost projective modules and strongly flat modules.

Definition 2.1 *Let M be an R-module. Then M is said to be* almost strongly flat *if* $\mathrm{Ext}_R^1(M, N) = 0$ *for any Matlis cotorsion $R_{\mathfrak{m}}$-module N and any $\mathfrak{m} \in \mathrm{Max}(R)$. Denote by $\mathcal{ASF}$ the class of all almost strongly flat modules.*

First we recall some basic notions on (pre)covers and (pre)envelopes. Given a class $\mathcal{F}$ of R-modules, denote by ${}^{\perp}\mathcal{F}$ (resp., $\mathcal{F}^{\perp}$) the class of R-modules N such that $\mathrm{Ext}_R^1(N, F) = 0$ (resp., $\mathrm{Ext}_R^1(F, N) = 0$) for all $F \in \mathcal{F}$. Let M be an R-module. An R-homomorphism $f : F \to M$ with $F \in \mathcal{F}$ is an $\mathcal{F}$-precover of M, provided that the natural homomorphism $\mathrm{Hom}_R(F', f) : \mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M)$ is surjective for any $F' \in \mathcal{F}$. That is, for any R-homomorphism $f' : F' \to M$ there exists an R-homomorphism $g : F' \to F$ such that $f' = gf$:



An $\mathcal{F}$-precover $f : F \to M$ is said to be special provided that f is an epimorphism and $\mathrm{Ker}(f) \in \mathcal{F}^{\perp}$. An $\mathcal{F}$-precover $f : F \to M$ is said to be an $\mathcal{F}$-cover if f is left minimal, that is, provided $f = gf$ implies g is an automorphism for each $g \in \mathrm{Hom}_R(F, F)$. The definitions of (special) preenvelopes and envelopes can be given dually.

Lemma 2.2 ([10, Corollary 7.2]) *Let R be an integral domain. Then each R-module has an $\mathcal{MC}$-envelope and a special $\mathcal{SF}$-precover.*

Lemma 2.3 ([10, Corollary 7.51]) *An R-module is strongly flat if and only if it is a direct summand of a module which is an extension of a free module by a divisible torsionfree module. Moreover, if M is a strongly flat R-module, then $M_{\mathfrak{p}}$ is a strongly flat $R_{\mathfrak{p}}$-module for any $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Proposition 2.4 *Let R be an integral domain and M be an R-module. Then the following statements are equivalent:*

(1) M is almost strongly flat over R;

(2) $M_{\mathfrak{m}}$ is strongly flat over $R_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$;

(3) $M_{\mathfrak{p}}$ is strongly flat over $R_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$.

Consequently, any strongly flat module is almost strongly flat, and any almost strongly flat module is flat.

Proof (1) $\Rightarrow$ (2). Suppose M is almost strongly flat over R. Let $\mathfrak{m} \in \mathrm{Max}(R)$ and $0 \to C_{\mathfrak{m}} \to F_{\mathfrak{m}} \to M_{\mathfrak{m}} \to 0$ be a special strongly flat precover of $M_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$ (see Lemma 2.2). Then $C_{\mathfrak{m}}$ is Matlis cotorsion over $R_{\mathfrak{m}}$. Since M is almost strongly flat, $\mathrm{Ext}_R^1(M, C_{\mathfrak{m}}) \cong \mathrm{Ext}_{R_{\mathfrak{m}}}^1(M_{\mathfrak{m}}, C_{\mathfrak{m}}) = 0$

for any $\mathfrak{m} \in \mathrm{Max}(R)$ (see [11, Chapter VI, Proposition 4.1.3]). Thus the exact sequence $0 \to C_\mathfrak{m} \to F_\mathfrak{m} \to M_\mathfrak{m} \to 0$ splits. Consequently, $M_\mathfrak{m}$ is strongly flat over $R_\mathfrak{m}$.

(2)$\Rightarrow$ (3). For any $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. Thus $M_\mathfrak{p} = (M_\mathfrak{m})_\mathfrak{p}$ is strongly flat over $R_\mathfrak{p}$ by Lemma 2.3.

(3)$\Rightarrow$ (2). Trivial.

(2)$\Rightarrow$ (1). Let $\mathfrak{m} \in \mathrm{Max}(R)$ and N a Matlis cotorsion $R_\mathfrak{m}$-module. Then by [11, Chapter VI Proposition 4.1.3] again, $\mathrm{Ext}^1_R(M, N) \cong \mathrm{Ext}^1_{R_\mathfrak{m}}(M_\mathfrak{m}, N) = 0$. Thus M is almost strongly flat over R.

For the consequence, we first suppose M is a strongly flat R-module. Then $M_\mathfrak{p}$ is strongly flat over $R_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$ by Lemma 2.3. Thus M is almost strongly flat over R by (3) $\Rightarrow$ (1). Then we suppose N is an almost strongly flat R-module. It follows that $N_\mathfrak{p}$ is strongly flat over $R_\mathfrak{p}$ by (3) $\Rightarrow$ (1). Thus $N_\mathfrak{p}$ is flat over $R_\mathfrak{p}$. So N is flat over R. □

Definition 2.5 *Let R be an integral domain. An R-module is said to be strongly Matlis cotorsion if* $\mathrm{Ext}^1_R(M, N) = 0$ *for any $M \in \mathcal{ASF}$. Denote by $\mathcal{SMC}$ the class of all strongly Matlis cotorsion modules.*

By Proposition 2.4, any cotorsion module is strongly Matlis cotorsion and any strongly Matlis cotorsion is Matlis cotorsion.

Next, we recall some basic notions on cotorsion pairs. A pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules is called a cotorsion pair, if $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. If each R-module has a special $\mathcal{A}$-precover, then $(\mathcal{A}, \mathcal{B})$ is called complete. If each R-module has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope, then $(\mathcal{A}, \mathcal{B})$ is called perfect. If there exists a class $\mathcal{C}$ of R-modules such that $\mathcal{A} = {}^\perp\mathcal{C}$ (resp., $\mathcal{B} = \mathcal{C}^\perp$), then $(\mathcal{A}, \mathcal{B})$ is said to be cogenerated (resp., generated) by $\mathcal{C}$. By [10, Theorem 6.11], if a cotorsion pair is generated by a set, then it is complete. It is well-known that $(\mathcal{SF}, \mathcal{MC})$ is a cotorsion pair generated by Q, so it is a complete cotorsion pair.

Lemma 2.6 *Let $\mathcal{M} := \{M \in R\text{-Mod} \mid M$ is a Matlis cotorsion $R_\mathfrak{m}$-module for some $\mathfrak{m} \in \mathrm{Max}(R)\}$. Then $(\mathcal{ASF}, \mathcal{SMC})$ is a cotorsion pair cogenerated by $\mathcal{M}$.*

Proof By Proposition 2.4, we have $\mathcal{ASF} = {}^\perp\mathcal{M}$. Then $\mathcal{SMC} = \mathcal{ASF}^\perp = ({}^\perp\mathcal{M})^\perp$. Thus $\mathcal{M} \subseteq \mathcal{SMC}$, and so $\mathcal{ASF} = {}^\perp\mathcal{M} \supseteq {}^\perp\mathcal{SMC}$. Since $\mathcal{ASF} \subseteq {}^\perp(\mathcal{ASF}^\perp) = {}^\perp\mathcal{SMC}$, we have $\mathcal{ASF} = {}^\perp\mathcal{SMC}$. □

Proposition 2.7 *Let R be an integral domain. Then the following statements are equivalent for R:*

(1) R is a field;

(2) any almost strongly flat module is almost projective;

(3) any R-module is almost strongly flat;

(4) any R-module is strongly Matlis cotorsion.

Proof (1)$\Rightarrow$ (2), (1)$\Rightarrow$ (3) and (1)$\Rightarrow$ (4). Trivial.

(2)$\Rightarrow$ (1). Let $\mathfrak{m}$ be a maximal ideal of R. Since Q is a strongly flat R-module, Q is almost projective by (2). Then, for any $\mathfrak{m} \in \mathrm{Max}(R)$, we have $Q = Q(R_\mathfrak{m})$ is projective, and thus free over $R_\mathfrak{m}$. Hence $R_\mathfrak{m}$ is a field for any $\mathfrak{m} \in \mathrm{Max}(R)$. So R is a von Neumann regular ring. Since R is an integral domain, R is a field.

(3)⇒ (1). By (3), any R-module is flat. So R is a von Neumann regular ring. Since R is an integral domain, R is a field.

(4)⇒ (2). Since any R-module is strongly Matlis cotorsion by (4), any almost strongly flat module is projective by Lemma 2.6, thus almost projective. □

Recall from [12] that an integral domain R is called an almost Dedekind domain if $R_\mathfrak{m}$ is a Dedekind domain for any maximal ideal $\mathfrak{m}$ of R.

Proposition 2.8 *Let R be an integral domain. Then the following statements are equivalent for R:*

(1) R is an almost Dedekind *domain;*

(2) any ideal of R is almost projective;

(3) any submodule of a projective module is almost projective;

(4) any ideal of R is almost strongly flat;

(5) any submodule of a projective module is almost strongly flat.

Proof (1)⇔ (2)⇔ (3). Refer to [7, Theorem 4.4].

(3)⇒ (5)⇒ (4). Trivial.

(4)⇒ (2). By (4), any ideal of R is flat. So R is a Prüfer domain. If R is a field, then (2) trivially holds. Now, suppose R is not a field. Let I be a nonzero ideal R and $\mathfrak{m}$ be a maximal ideal of R. Then $I_\mathfrak{m}$ is strongly flat over $R_\mathfrak{m}$. Since $R_\mathfrak{m}$ is a valuation domain, by [10, Theorem 7.64] there is an $R_\mathfrak{m}$-exact sequence $0 \to R_\mathfrak{m}^{(\kappa_1)} \to I_\mathfrak{m} \to Q^{(\kappa_2)} \to 0$, where $R \neq Q = Q(R_\mathfrak{m})$. Since I is a nonzero ideal R, the rank of $I_\mathfrak{m}$ over $R_\mathfrak{m}$ is 1. Thus $\kappa_1 = 1$ and hence $I_\mathfrak{m} \cong R_\mathfrak{m}$. So I is almost projective. □

3. Locally almost perfect domains

Recall from [13] that a commutative ring T is said to be *locally perfect* provided that $T_\mathfrak{m}$ is perfect for any maximal ideal $\mathfrak{m}$ of T. The locally perfect rings can be characterized as commutative rings T over which any T-module has a maximal submodule [13, Theorem A]. It is certain that a commutative ring T is perfect if and only if T is semi-perfect and locally perfect. Recall from [1] that an integral domain R is said to be an *almost perfect domain* (APD for short) if all its proper homomorphic images are perfect. In 2011, Salce [1] introduced the notion of locally almost perfect domains.

Definition 3.1 ([1, P. 369]) *An integral domain R is called* a locally almost perfect domain *(*LAPD *for short) provided that $R_\mathfrak{m}$ is an almost perfect domain for any maximal ideal $\mathfrak{m}$ of R.*

Recall from [9, P. 131] that an integral domain R is said to be h-local if it satisfies the following two conditions:

(1) R is of finite character;

(2) any nonzero prime ideal is contained in only one maximal ideal.

Theorem 3.2 ([1, Theorem 3.6]) *An integral domain R is an* APD *if and only if it is h-local and an* LAPD.

Note that Salce [1, Example 3.7] showed there exist LAPDs which fail to be h-local, and thus provided examples of LAPDs are not almost perfect.

Proposition 3.3 *Let R be a non-field integral domain. Then the following statements are equivalent for R:*

(1) R is an LAPD*;*

(2) $R/\langle u\rangle$ is locally perfect for any nonzero non-unit element u in R;

(3) R/I is locally perfect for any nonzero proper ideal I of R.

Proof (1)$\Rightarrow$ (2). Let u be a nonzero non-unit in R and $\mathfrak{m}$ a maximal ideal of R. Then $\langle u\rangle_{\mathfrak{m}}$ is a nonzero ideal over $R_{\mathfrak{m}}$ as R is an integral domain. Thus $(R/\langle u\rangle)_{\mathfrak{m}}\cong R_{\mathfrak{m}}/\langle u\rangle_{\mathfrak{m}}$ is perfect, and so $R/\langle u\rangle$ is locally perfect.

(2)$\Rightarrow$ (3). Let I be a proper nonzero ideal I of R. Then there exist a nonzero non-unit u in I and a natural epimorphism $R/\langle u\rangle \twoheadrightarrow R/I$. Localizing at any maximal ideal $\mathfrak{m}$, there is an epimorphism $(R/\langle u\rangle)_{\mathfrak{m}} \twoheadrightarrow (R/I)_{\mathfrak{m}}$. Since $(R/\langle u\rangle)_{\mathfrak{m}}$ is perfect by (2), $(R/I)_{\mathfrak{m}}$ is also perfect by [14, Corollary 3.10.23(2)]. Hence R/I is locally perfect.

(3)$\Rightarrow$ (1). Let $\mathfrak{m}$ be a maximal ideal of R and $I_{\mathfrak{m}}$ a nonzero proper ideal of $R_{\mathfrak{m}}$. Then obviously I is also a nonzero ideal of R. Then $R_{\mathfrak{m}}/I_{\mathfrak{m}}\cong (R/I)_{\mathfrak{m}}$ is perfect. Thus $R_{\mathfrak{m}}$ is an almost perfect domain. □

Recall from [7] that an R-module M has almost projective dimension $A.\mathrm{pd}_R(M)\leq n$ if there is an exact sequence of R-modules

$$0\to P_n\to P_{n-1}\to\cdots\to P_1\to P_0\to M\to 0, \qquad (\diamondsuit)$$

where each P_i is almost projective. The exact sequence $(\diamondsuit)$ is said to be an almost projective resolution of length n of M. Define $A.\mathrm{pd}_R(M)=n$ if n is the length of the shortest almost projective resolution of M. If no such finite resolution exists, then $A.\mathrm{pd}_R(M)=\infty$. Since every projective module is locally projective, $A.\mathrm{pd}_R(M)\leq \mathrm{pd}_R(M)$ for any R-module M. We denote by $\mathcal{AP}_1$ the class of R-modules M with $A.\mathrm{pd}_R(M)\leq 1$. By [7, Theorem 3.3], we have

$$\mathcal{AP}_1=\{M\in R\text{-Mod}|\ \mathrm{pd}_{R_{\mathfrak{m}}}M_{\mathfrak{m}}\leq 1 \text{ for any } \mathfrak{m}\in \mathrm{Max(R)}\}.$$

Denote by $\mathcal{P}_1$ the class of R-modules with projective dimensions at most 1 and by $\mathcal{DI}$ the class of all divisible modules.

Lemma 3.4 ([10, Theorem 8.6(a), Theorem 9.1(a)]) *Let R be an integral domain and M an R-module. Then M has projective dimension ≤ 1 if and only if $\mathrm{Ext}_R^1(M,D)=0$ for each divisible module D. Hence, $(\mathcal{P}_1,\mathcal{DI})$ is a complete cotorsion pair.*

Proposition 3.5 *Let R be an integral domain and $\mathcal{D}:=\{N\in R\text{-Mod}|\ N$ is an divisible $R_{\mathfrak{m}}$-*

module for some $\mathfrak{m} \in \mathrm{Max}(R)\}$. *Then* $(\mathcal{AP}_1, \mathcal{AP}_1^{\perp})$ *is a cotorsion pair cogenerated by* $\mathcal{D}$.

Proof Let M be an R-module in $\mathcal{AP}_1$. Then $\mathrm{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \le 1$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. Let $\mathfrak{m}$ be a maximal ideal of R and N a divisible $R_\mathfrak{m}$-module. Then $\mathrm{Ext}^1_R(M, N) \cong \mathrm{Ext}^1_{R_\mathfrak{m}}(M_\mathfrak{m}, N) = 0$ by Lemma 3.4. Thus $\mathcal{AP}_1 \subseteq^{\perp} \mathcal{D}$. Let N be a divisible $R_\mathfrak{m}$-module for some $\mathfrak{m} \in \mathrm{Max}(R)$ and M an R-module such that $\mathrm{Ext}^1_R(M, N) = 0$. Then $\mathrm{Ext}^1_{R_\mathfrak{m}}(M_\mathfrak{m}, N) = 0$, and thus $M_\mathfrak{m}$ is an $R_\mathfrak{m}$-module with projective dimension at most 1 by Lemma 3.4 again. It follows that $\mathcal{AP}_1 =^{\perp} \mathcal{D}$. Consequently, $(\mathcal{AP}_1, \mathcal{AP}_1^{\perp})$ is a cotorsion pair cogenerated by $\mathcal{D}$. □

Lemma 3.6 ([4, Lemma 4.2]) *Let* R *be an integral domain and* S *a multiplicative subset of* R. *Denote by* $\mathcal{P}_1(R_S)$ *the class of* R_S*-modules with projective dimensions at most* 1. *If* M *is an* R_S*-module and* $0 \to D \to A \to M \to 0$ *is a* $\mathcal{P}_1$*-cover of* M *viewed as an* R*-module, then* $0 \to D \to A \to M \to 0$ *is a* $\mathcal{P}_1(R_S)$*-cover of* M *viewed as an* R_S*-module.*

Lemma 3.7 ([4, Lemma 3.7]) *Let* R *be an integral domain and* S *a multiplicative subset of* R. *Denote by* $\mathcal{DI}(R_S)$ *the class of divisible* R_S*-modules. If* M *is an* R_S*-module and* $0 \to M \to D \to D/M \to 0$ *is a* $\mathcal{DI}$*-envelope of* M *viewed as an* R*-module, then* $0 \to M \to D \to D/M \to 0$ *is a* $\mathcal{DI}(R_S)$*-envelope of* M *viewed as an* R_S*-module.*

Let R be an integral domain. Denote by $\mathcal{F}_1$ the class of R-modules with flat dimensions at most 1 and by $\mathcal{WI} := \mathcal{F}_1^{\perp}$ the class of all weak-injective modules. Then $(\mathcal{F}_1, \mathcal{WI})$ is a perfect cotorsion pair by [10, Theorem 8.3].

Theorem 3.8 *Let* R *be an integral domain. Then the following statements are equivalent:*

(1) R *is an* LAPD*;*

(2) $\mathcal{AP}_1 = \mathcal{F}_1$*;*

(3) $\mathcal{AP}_1^{\perp} = \mathcal{WI}$*;*

(4) $(\mathcal{AP}_1, \mathcal{AP}_1^{\perp})$ *is a perfect cotorsion pair;*

(5) any R*-module has an* $\mathcal{AP}_1$*-cover;*

(6) any R*-module has an* $\mathcal{AP}_1^{\perp}$*-envelope.*

Proof (1)⇒ (2). By [7, Theorem 3.3], $\mathcal{AP}_1 \subseteq \mathcal{F}_1$. On the other hand, let M be an R-module with $\mathrm{fd}_R(M) \le 1$. Then $\mathrm{fd}_{R_\mathfrak{m}} M_\mathfrak{m} \le 1$ for each $\mathfrak{m} \in \mathrm{Max}(R)$. Since $R_\mathfrak{m}$ is an APD, $\mathrm{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \le 1$ for each $\mathfrak{m} \in \mathrm{Max}(R)$ (see [4, Proposition 3.5]). Hence, M has almost projective dimension at most 1 by [7, Theorem 3.3] again.

(2)⇔ (3). It follows from that $(\mathcal{AP}_1, \mathcal{AP}_1^{\perp})$ and $(\mathcal{F}_1, \mathcal{WI})$ are both cotorsion pairs (see Proposition 3.5 and [10, Theorem 8.3]).

(2)⇒(4). It follows from that $(\mathcal{F}_1, \mathcal{F}_1^{\perp})$ is a perfect cotorsion pair [10, Theorem 8.3].

(4)⇒(5) and (4)⇒(6). Trivial.

(5)⇒(1). Let $\mathfrak{m}$ be a maximal ideal of R and M an $R_\mathfrak{m}$-module. Let $f : A \to M$ be an $\mathcal{AP}_1$-cover over R. Then $f : A \to M$ is a $\mathcal{P}_1$-cover over $R_\mathfrak{m}$ by Lemma 3.6. It follows that $R_\mathfrak{m}$ is an APD by [4, Theorem 4.3].

(6)⇒(1). Let $\mathfrak{m}$ be a maximal ideal of R and M an $R_\mathfrak{m}$-module. Let $f : M \to D$ be a

$\mathcal{DI}$-envelope over R. Then $f : M \to D$ is a $\mathcal{DI}(R_\mathfrak{m})$-envelope over $R_\mathfrak{m}$ by Lemma 3.7. It follows that $R_\mathfrak{m}$ is an APD by [4, Theorem 3.6]. □

Similarly to [4, Lemma 3.7], we have the following result.

Lemma 3.9 *Let R be an integral domain and S a multiplicative subset of R. Denote by $\mathcal{SF}(R_S)$ the class of strongly flat R_S-modules. If M is an R_S-module and $0 \to K \to F \to M \to 0$ is an $\mathcal{SF}$-cover of M viewed as an R-module, then $0 \to K \to F \to M \to 0$ is an $\mathcal{SF}(R_S)$-cover of M viewed as an R_S-module.*

Proof Let M be an R_S-module and

$$0 \to K \to F \to M \to 0 \qquad (*)$$

an $\mathcal{SF}$-cover of M viewed as an R-module. Then by [4, Proposition 4.1], F is an R_S-module, and thus K is also an R_S-module. Since for any R_S-module N, we have $\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_{R_S}(M,N)$. So it is easy to verify $(*)$ is also an $\mathcal{SF}(R_S)$-cover of M viewed as an R_S-module. □

Theorem 3.10 *Let R be an integral domain. Then the following statements are equivalent:*

(1) R is an LAPD*;*

(2) any flat R-module is almost strongly flat;

(3) any direct limit of almost strongly flat R-modules is almost strongly flat;

(4) any direct limit of projective R-modules is almost strongly flat;

(5) any R-module has an $\mathcal{ASF}$-cover;

(6) any $R_\mathfrak{m}$-module viewed as an R-module has an almost strongly flat cover for any maximal ideal $\mathfrak{m}$ of R;

(7) $(\mathcal{ASF}, \mathcal{SMC})$ is a perfect cotorsion pair;

(8) any strongly Matlis cotorsion module is cotorsion.

Proof (1)⇒ (2). Let M be a flat R-module. By (1), $R_\mathfrak{m}$ is an APD for any $\mathfrak{m} \in \mathrm{Max}(R)$. Thus $M_\mathfrak{m}$ is a strongly flat $R_\mathfrak{m}$-module. Thus M is almost strongly flat by Proposition 2.4.

(2)⇒ (1). Let $\mathfrak{m}$ be a maximal ideal of R. Let M be a flat $R_\mathfrak{m}$-module. Then $M \cong M_\mathfrak{m}$ is a strongly flat $R_\mathfrak{m}$-module by (2). Thus $R_\mathfrak{m}$ is an APD. So R is an LAPD.

(2)⇒ (3). Since any almost strongly flat R-module is flat and the class of all flat modules is closed under direct limits, we have any direct limit of almost strongly flat R-modules is flat.

(3)⇒ (4), (5)⇒ (6) and (7)⇒ (5). Trivial.

(4)⇒ (2). Let M be a flat R-module. Then there exists a direct system $\{F_i\}_{i\in\Gamma}$ of finitely generated projective R-modules such that $M = \lim_{\longrightarrow} F_i$. Thus M is almost strongly flat by (4).

(2)⇒ (5). Since any almost strongly flat R-module is flat, the class of almost strongly flat R-modules coincides with that of flat R-modules. Thus any R-module has an almost strongly flat cover by [15, Theorem 3].

(5)⇒ (1). Let M be an $R_\mathfrak{m}$-module and $f : A \to M$ be an almost strongly flat cover over R. Then $f : A \to M$ is a strongly flat cover over $R_\mathfrak{m}$ by Lemma 3.9. Thus the class of strongly flat

$R_{\mathfrak{m}}$-modules is covering. It follows that $R_{\mathfrak{m}}$ is an APD for any $\mathfrak{m} \in \mathrm{Max}(R)$ by [2, Theorem 4.5]. So R is an LAPD.

(2)$\Leftrightarrow$ (8). It follows that $(\mathcal{ASF}, \mathcal{SMC})$ and $(\mathcal{F}, \mathcal{C})$ are both cotorsion pair.

(2)$\Rightarrow$ (7). We just note that $(\mathcal{F}, \mathcal{C})$ is a perfect cotorsion pair [15, Theorem 3]. □

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