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Right c-Group Inverses and Their Applications

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Abstract We study a new class of group inverses determined by right c-regular elements. The new concept of right c-group inverses is introduced and studied. It is shown that every right c-group invertible element is group invertible, and an example is given to show that group invertible elements need not be right c-group invertible. The conditions that right c-group invertible elements are precisely group invertible elements are investigated. We also study the strongly clean decompositions of right c-group invertible elements. As applications, we give some new characterizations of abelian rings and directly finite rings from the point of view of right c-group inverses.

Keywords right c-group inverse; group inverse; right c-regular elements; strongly clean decomposition

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1. Introduction

Throughout this paper, R is a unitary associative ring, the center of R is denoted by $C(R)$ and the group of units of the ring R is $U(R)$. An involution $*$: $R \to R$ is an anti-isomorphism which satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$, $(a+b)^* = a^* + b^*$ for all $a, b \in R$. For any $a \in R$, we use $\mathrm{lann}(a) = \{x \in R : xa = 0\}$ and $\mathrm{rann}(a) = \{x \in R : ax = 0\}$ to denote the left and right annihilator of a, respectively. Recall that an element $a \in R$ is Drazin invertible [1] if there is $x \in R$ such that $xax = x$, $ax = xa$, $a^k = a^{k+1}x$ for some $k \geq 0$. The least such k is called the index of a. The Drazin inverse is called the group inverse of a when $k = 1$. It is well known that an element a is group invertible if and only if a is strongly regular (that is, $a \in a^2R \cap Ra^2$). More results on group inverse of elements in various setting can be found in [2] and [3–5].

In [5], the Moore-Penrose inverse was introduced for a ring with involution. Also a detailed study of core inverses and dual core inverses in rings was undertaken in [3]. For any element $a \in R$, consider the following conditions:

(1) $axa = a$; (2) $xax = x$; (3) $xa = ax$; (4) $(ax)^* = ax$; (5) $(xa)^* = xa$; (6) $xa^2 = a$; (7) $ax^2 = x$.

Any element x satisfying (1) is called an inner inverse of a, and is denoted by a^-. If x satisfies (1)–(3), then x is called the group inverse of a, denoted by $a^\#$. If x satisfies (1), (2),

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(4) and (5), then x is called the Moore-Penrose inverse of a and is denoted by $a^{\dagger}$. The set of all group invertible elements and Moore-Penrose invertible elements are denoted by $R^{\#}$ and $R^{\dagger}$, respectively. It is well known that a is an EP element if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$. Moreover, x is the core inverse of a if it satisfies (1), (2), (4), (6) and (7), which is denoted by $a^{\oplus}$. And x is the dual core inverse of a if it satisfies (1), (2) and (5)–(7), which is denoted by $a_{\oplus}$. The set of all core invertible elements and dual core invertible elements are denoted by $R^{\oplus}$ and $R_{\oplus}$, respectively.

In 2012, Drazin defined a class of outer generalized inverses in [4]. Let $a, b, c, y \in R$. Then y is called the (b, c)-inverse of a if $y \in bRy \cap yRc, yab = b$ and $cay = c$. Later, Drazin shed a new light on (b, c)-inverse by introducing left and right (b, c)-inverses in [6]. Let $a, b, c, x \in R$. Recalled from [6] that x is a left (resp., right) (b, c)-inverse of a if it satisfies $xab = b, x \in Rc$ (resp., $cax = c, x \in bR$). According to [7], for $a, c \in R$, a is right (resp., left) c-regular if there exists $x \in R$ such that $a = axca$ (resp., $a = acxa$), and x is called a right (resp., left) c-regular inverse of a. It is clear that every right c-regular element is regular, but in general a regular element need not be right c-regular by [7, Example 2.1].

In this paper, we investigate a new class of group inverses in unitary associative rings. More precisely, we give an explicit description of group inverse determined by left and right c-regular elements. The concepts of right and left c-group inverses are defined and investigated. It is proved that if a is right c-group invertible, then a is group invertible. However, we shall give examples to show that group invertible elements need not be right c-group invertible, and right c-group invertible elements need not be left c-group invertible. We also study the strongly clean decompositions of right c-group invertible elements, and study the relationship between right c-group inverses and other generalized inverses including group inverses, Moore-Penrose inverses, core inverses, dual core inverses, one-sided (b, c)-inverses and (b, c)-inverses. As applications, we give some new characterizations of abelian rings, directly finite rings and EP elements by using right c-group inverses.

This paper is organized as follows:

In Section 2, we define and study right and left c-group inverses of an element in a ring R. We show that an element a is right c-group invertible if and only if a is group invertible and $Ra \subseteq Rc$ (Proposition 2.8). In Section 3, we further study the properties of right c-group invertible elements. Of particular interest are the new characterization of strongly clean decompositions of elements with respect to right c-group invertible elements (Theorem 3.4). Also we show that every right c-group invertible element of R has a unique right c-group inverse if and only if R is abelian (Proposition 3.7). Section 4 is devoted to study the relationships between right c-group inverse, Moore-Penrose inverse, core inverse and (b, c)-inverse. As applications, we give some new characterizations of EP elements and directly finite rings from the point of view of right c-group inverses (Proposition 4.4 and Theorem 4.14).

2. Right and left c-group inverses

This section is dedicated to the question of exploring the properties of group inverses determined by right c-regular elements. The new concepts of left and right c-group inverses are defined and discussed. An example is given to show that group invertible elements need not be right c-group invertible. We also study the condition under which right c-group invertibility coincides with group invertibility.

We begin with the following definition.

Definition 2.1 *Let $a, c \in R$. We say that a is right c-group invertible if there exists $x \in R$ such that $a = axca$, $x = xcax$, $axc = xca$. Any element x, which satisfies the above conditions, is called a right c-group inverse of a and is denoted as $a^{\#}_{c}$.*

Dually, a is said to be left c-group invertible if there is $y \in R$ such that $a = acya$, $y = yacy$, $cya = acy$. Any element y satisfying the above conditions is called a left c-group inverse of a and is defined as ${}_{c}a^{\#}$.

In what follows, we use $R^{\#}_{c}$ (resp., ${}_{c}R^{\#}$) to denote the set of all right (resp., left) c-group invertible elements of R. It is clear that if a is right (resp., left) c-group invertible, then a is group invertible. However, the next example shows that a group invertible element need not be right c-group invertible.

Example 2.2 Let $R = M_2(\mathbb{F})$ be the ring of all 2 by 2 matrices over a field $\mathbb{F}$. Let

$$a = x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R.$$

Then it can be easily checked that a is group invertible and x is the group inverse of a. However, it is clear

$$axca = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq a$$

for any element x since $ca = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Thus, a is not right c-group invertible.

The following proposition gives a characterization of right c-group inverse.

Proposition 2.3 *Let $a, x, c \in R$. Then the following statements are equivalent:*

(1) x is a right c-group inverse of a;

(2) $a = axca$, $Rxc = Ra$, $xR = aR$;

(3) $a = axca$, $\mathrm{rann}(xc) = \mathrm{rann}(a)$, $\mathrm{lann}(a) = \mathrm{lann}(x)$;

(4) $a = axca$, $Rxc \subseteq Ra$, $xR \subseteq aR$;

(5) $a = axca$, $\mathrm{rann}(a) \subseteq \mathrm{rann}(xc)$, $\mathrm{lann}(a) \subseteq \mathrm{lann}(x)$.

Proof (1) $\Rightarrow$ (2). Since x is a right c-group inverse of a, we have $a = axca = xca^2 \in xR$ and $x = xcax = axcx \in aR$. This implies that $aR = xR$. Also, we have $xc = xcaxc = (xc)^2a \in Ra$ and $a = axca = a^2xc \in Rxc$. This shows that $Rxc = Ra$.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are straightforward.

(3) $\Rightarrow$ (4). Since $a = axca$, we have $(1 - xca) \in \mathrm{rann}(a) = \mathrm{rann}(xc)$. It follows that $xc = (xc)^2 a \in Ra$. Therefore, we have $Rxc \subseteq Ra$. Similarly, since $(axc-1) \in \mathrm{lann}(a) = \mathrm{lann}(x)$, we get $x = axcx \in aR$, and hence $xR \subseteq aR$.

(5) $\Rightarrow$ (1). Since $a = axca$, we deduce that $(1 - xca) \in \mathrm{rann}(a) \subseteq \mathrm{rann}(xc)$. Then $xc = (xc)^2 a$. Similarly, since $(axc - 1) \in \mathrm{lann}(a) \subseteq \mathrm{lann}(x)$, we get $x = axcx$. Therefore, we have $axc = a(xc)^2 a = (axcx)ca = xca$. This implies that $x = axcx = xcax$, as desired. $\square$

In particular, if c is a central element, then we can give a description of right c-group invertible elements, which is closely related to the idempotents of R.

Theorem 2.4 *Let $a, c \in R$ and $c \in C(R)$. Then the following statements are equivalent:*

(1) $a \in R_c^{\#}$;

(2) There exists a unique idempotent element $p \in R$ such that $aR = caR = pR$, $Ra = Rca = Rp$;

(3) $ca \in R^-$ and there is a unique idempotent element $p \in R$ such that $\mathrm{lann}(a) = \mathrm{lann}(ca) = \mathrm{lann}(p), \mathrm{rann}(a) = \mathrm{rann}(ca) = \mathrm{rann}(p)$.

Proof (1) $\Rightarrow$ (2). Let $p = aa_c^{\#}c$. Then $p^2 = aa_c^{\#}caa_c^{\#}c = aa_c^{\#}c = p$. Since $a = aa_c^{\#}ca = pa \in pR$ and $p = aa_c^{\#}c \in aR$, we get $aR = pR$. Also since $c \in C(R)$, we have

$$ca = caa_c^{\#}ca = aa_c^{\#}cac \in pR, \quad p = aa_c^{\#}c = caa_c^{\#} \in caR,$$

thus $pR = caR$. Next, since

$$p = aa_c^{\#}c = a_c^{\#}ca \in Ra, \quad a = aa_c^{\#}ca = aaa_c^{\#}c = ap \in Rp,$$

we have $Rp = Ra$. Furthermore, since $ca = caa_c^{\#}ca \in Rp$ and $p = a_c^{\#}ca \in Rca$, we conclude that $Rp = Rca$.

(2) $\Rightarrow$ (3). Since $Rca = Rp$, there exist $s, t \in R$ such that $ca = tp$ and $p = sca$. It follows that $ca = cap = casca$ since p is an idempotent, and thus $ca \in R^-$. By [3, Lemma 2.5], we have

$$\mathrm{lann}(a) = \mathrm{lann}(ca) = \mathrm{lann}(p), \quad \mathrm{rann}(a) = \mathrm{rann}(ca) = \mathrm{rann}(p),$$

as desired.

(3) $\Rightarrow$ (1). By the assumption, it is clear that

$$(1 - p) \in \mathrm{rann}(p) = \mathrm{rann}(a), \quad [(ca)^- ca - 1] \in \mathrm{rann}(ca) = \mathrm{rann}(a) = \mathrm{rann}(p).$$

Then we conclude that $a = ap = a(ca)^- ca$ and $p = p(ca)^- ca$. Also, since

$$(p - 1) \in \mathrm{lann}(p) = \mathrm{lann}(a), \quad [1 - ca(ca)^-] \in \mathrm{lann}(ca) = \mathrm{lann}(a) = \mathrm{lann}(p),$$

we get $a = pa = ca(ca)^- a$ and $p = ca(ca)^- p$. Since $a = ap$ and $(p - 1) \in \mathrm{lann}(ca)$, we have $ca = cap$ and $pca = ca$. Let $x = p(ca)^- p$. Then we conclude that

$$axca = ap(ca)^- pca = a(ca)^- ca = a, \quad xcax = p(ca)^- pcap(ca)^- p = p(ca)^- p = x,$$

$$xca = p(ca)^- pca = p, \quad axc = ap(ca)^- pc = ca(ca)^- p = p.$$

It remains to show the uniqueness of p. In fact, if there are two idempotent elements $p_1, p_2 \in R$ such that $\mathrm{lann}(p_1) = \mathrm{lann}(a) = \mathrm{lann}(p_2)$, $\mathrm{rann}(p_1) = \mathrm{rann}(a) = \mathrm{rann}(p_2)$. Then it can be easily checked that

$$(1 - p_1) \in \mathrm{lann}(p_1) = \mathrm{lann}(p_2), \quad (p_2 - 1) \in \mathrm{rann}(p_2) = \mathrm{rann}(p_1),$$

which imply that $p_1 = p_1 p_2 = p_2$. $\square$

It is a well-known fact that the group inverse of a group invertible element is unique. Similarly, one may suspect that if $a \in R_c^{\#}$, then the right c-group inverse of a is also unique. However, the following example eliminates the possibility.

Example 2.5 Let $R = M_2(\mathbb{F})$ be the ring of all 2 by 2 matrices over a field $\mathbb{F}$. Take

$$a = c = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} m & n \\ s & t \end{pmatrix} \in R$$

for some $m, n, s, t \in \mathbb{F}$. If $xcax = x$, $a = axca$ and $xca = axc$, then

$$x = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}.$$

This shows that $\left(\begin{smallmatrix} 1 & n \\ 0 & 0 \end{smallmatrix}\right)$ is the right c-group inverse of a for some $n \in \mathbb{F}$. Therefore, the right c-group inverse of a is not unique.

The following proposition gives a more straightforward way to show the right c-group invertibility of an element.

Proposition 2.6 *Let $a, c \in R$. Then $a \in R_c^{\#}$ if and only if $a = a^2xc = yca^2$ for some $x, y \in R$. In this case, $ycax = axcx$ is a right c-group inverse of a.*

Proof If $a \in R_c^{\#}$ and x, y are two right c-group inverses of a, then we have

$$a = axca = ayca, \ \ xca = axc, \ \ yca = ayc.$$

This implies that $a = a^2xc = xca^2$. Analogously, we get $a = yca^2 = a^2yc$, that is, $a = yca^2 = a^2xc$. Conversely, if $a = a^2xc = yca^2$, then $yca = yca^2xc = axc$. Let $z = ycax$. Then we get

$$zcaz = (ycax)ca(yca)x = ycaxca^2xcx = yc(axc)ax = ycyca^2x = ycax = z,$$

$$azca = a(yca)xca = a^2xcxca = axca = yca^2 = a.$$

Moreover, since we have

$$zca = yc(axc)a = ycycaa = yca = axc, \quad azc = a(yca)xc = a^2xcxc = axc.$$

We conclude that $zca = azc$. Therefore, a is right c-group invertible with a right c-group inverse $z = ycax = axcx$. $\square$

Note that if the right c-group inverse of a is unique, then Proposition 2.6 can be rephrased as $a \in R_c^{\#}$ if and only if $a = a^2xc = xca^2$ for some $x \in R$. In this case, $a_c^{\#} = xcax = axcx$.

Similarly, we have the following proposition.

Proposition 2.7 *Let $a, c \in R$. Then $a \in {}_cR^{\#}$ if and only if $a = a^2cx = cya^2$ for some $x, y \in R$. In this case, $yacx = ycya$ is a left c-group inverse of a.*

The next proposition shows the condition under which right c-group invertibility coincides with group invertibility.

Proposition 2.8 *Let $a, c \in R$. Then $a \in R_c^{\#}$ if and only if $a \in R^{\#}$ and $Ra \subseteq Rc$.*

Proof Since $a \in R_c^{\#}$, there is $x \in R$ such that $xcax = x$. It is clear that xc is the group inverse of a. Since $xca = axc$, we have $a = axca = a^2xc \in Rc$. Thus $Ra \subseteq Rc$. Conversely, if $a \in R^{\#}$ and $Ra \subseteq Rc$, then there exist $y, t \in R$ such that $a = aya, ya = ay$ and $a = tc$. This implies that

$$a = ya^2 = y^2a^3 = y^2aa^2 = y^2tca^2 \in Rca^2,$$

$$a = ayaya = a^2y^2a = a^2y^2tc \in a^2Rc.$$

Therefore, $a \in R_c^{\#}$ by Proposition 2.6. □

The proof of the following proposition can be given similarly.

Proposition 2.9 *Let $a, c \in R$. Then $a \in {}_cR^{\#}$ if and only if $a \in R^{\#}$ and $aR \subseteq cR$.*

We next examine under what conditions the right (resp., left) c-group inverse of a right (resp., left) c-group invertible element is unique.

Theorem 2.10 *Let $a, c \in R$. If $a \in R_c^{\#} \cap {}_cR^{\#}$ such that $a_c^{\#} = {}_ca^{\#}$, then a has at most one right (resp., left) c-group inverse.*

Proof If $a_c^{\#} = {}_ca^{\#}$, then there is $x \in R$ such that $a = axca = acxa$ and $xcax = x = xacx$. If y is also a right c-group inverse of a with $x \neq y$. Then $y = ycay = yacy$. It follows that

$$y = ycay = ycaxcay = aycaxcy = axcy = xcay,$$

$$x = xacx = xacyacx = xcyaacx = xcyacxa = xcya = xacy.$$

Then we deduce that

$$ya = xcaya = axcya = axacy = ax,$$

$$y = yacy = yacxacy = yacx.$$

It follows that $y = yacx = axcx = xcax = x$. Therefore, a has at most one right c-group inverse. Similarly, we can show the uniqueness of left c-group inverse. □

Note that the condition in Theorem 2.10 is not superfluous. In fact, if a is right c-group invertible, then a need not be left c-group invertible by the following example.

Example 2.11 Let $R = M_2(\mathbb{F})$ be the ring of all 2 by 2 matrices over a field $\mathbb{F}$. Take

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R.$$

Then it is clear that

$$a = a^2 \begin{pmatrix} 0 & p \\ 1 & q \end{pmatrix} c = \begin{pmatrix} 0 & m \\ 1 & n \end{pmatrix} ca^2 \in a^2Rc \cap Rca^2$$

for $p, q, m, n \in \mathbb{F}$. Therefore, a is right c-group invertible by Proposition 2.6. However,

$$a^2c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $a \notin a^2cR \cap cRa^2$, that is, a is not left c-group invertible by Corollary 2.7.

Remark 2.12 In view of Example 2.5 and Theorem 2.10, we observe that in general the right and left c-group inverses of an element a are not unique. However, $a_c^{\#}ca$ and $ac_ca^{\#}$ are unique. In fact, if $x, y \in R$ are two right c-group inverses of a, then we have

$$xc = xcaxc = xcaycaxc = xcaaycxc = aycxc = ycaxc.$$

Therefore, $xca = ycaxca = yca$. Similarly, we can show that $ac_ca^{\#}$ is also unique.

We next discuss some further properties related to right c-group invertible elements.

Proposition 2.13 *Let $a, c \in R$. If $a_c^{\#} = {}_ca^{\#}$, then $(a_c^{\#})_c^{\#}$ and ${}_c({}_ca^{\#})^{\#}$ exist. In this case, a is both a left c-group inverse of ${}_ca^{\#}$ and a right c-group inverse of $a_c^{\#}$.*

Proof If $a_c^{\#} = {}_ca^{\#}$, then $a_c^{\#}$ is unique by Theorem 2.10. Let $x = a_c^{\#} = {}_ca^{\#}$. Then $a = axca = acxa$, $x = xcax = xacx$, $axc = xca$ and $cxa = acx$. Then we conclude that

$$xa = xcaxa = axcxa = ax,$$

$$axc = xca = xcacxa = axccxa = axcacx = acx.$$

Let $y = a$. Then we have

$$ycxy = acxa = a = y, \ xycx = xacx = x, \ xyc = xac = axc, \ ycx = acx.$$

Since $acx = axc$, we get $xyc = ycx$. Therefore, $(a_c^{\#})_c^{\#}$ exists and a is a right c-group inverse of $a_c^{\#}$. Similarly, we conclude that

$$yxcy = axca = a = y, \quad xcyx = xcax = x,$$

$$cyx = cax = cxa = acx, \quad xcy = xca = axc.$$

Since $acx = axc$, we get $xcy = cyx$. This implies that ${}_c({}_ca^{\#})^{\#}$ exists and a is a left c-group inverse of ${}_ca^{\#}$. □

Corollary 2.14 *Let $a, c \in R$. If $a_c^{\#} = {}_ca^{\#}$ such that $(a_c^{\#})_c^{\#} = {}_c({}_ca^{\#})^{\#}$, then $((a_c^{\#})_c^{\#})_c^{\#} = {}_c({}_c({}_ca^{\#})^{\#})^{\#} = {}_ca^{\#} = a_c^{\#}$.*

If R is a ring with an involution $*$, then we have the following lemma.

Lemma 2.15 *Let $a, c \in R$. If $a \in R_c^{\#}$, then $a^* \in {}_{c^*}R^{\#}$.*

The following theorem shows that if an element a is right c-group invertible, then a^* may be left c-group invertible under some mild conditions.

Theorem 2.16 *Let $a, c \in R$ such that $(ca^2)^* = ca^2$. If $a \in R_c^{\#}$, then $a^* \in {}_cR^{\#}$.*

Proof If $a \in R_c^{\#}$, then by Proposition 2.6, there exist $m, n \in R$ such that $a = a^2mc = nca^2$.

Next, it suffices to show $a^* \in (a^*)^2cR \cap cR(a^*)^2$ by Corollary 2.7. Since $(ca^2)^* = ca^2$ and $an^* = nca^2n^*$, an^* is symmetrical, thus $(an^*)^* = an^* = na^*$. Then we have

$$a = nca^2 = n(ca^2)^* = na^*a^*c^* = an^*a^*c^* = a(can)^*.$$

This implies that $a^* = cana^*$. Since $a^* = c^*m^*(a^*)^2$, we conclude that $a^* = cana^* = canc^*m^*(a^*)^2 \in cR(a^*)^2$. Furthermore, since $a = nca^2$, we have

$$\begin{aligned} a^* &= (a^*)^2c^*n^* = a^*a^*c^*n^* = (a^*)^2a^*c^*n^*c^*n^* \\ &= (a^*)^2canc^*m^*(a^*)^2c^*n^*c^*n^* \\ &= (a^*)^2canc^*m^*a^*c^*n^* \in (a^*)^2cR, \end{aligned}$$

proving a^* is left c-group invertible. □

3. Strongly clean decompositions for right c-group invertible elements

In this section, we study the strongly clean decompositions of right c-group invertible elements. A ring R is abelian if every idempotent element is central. An element a in a ring R is called clean [8] if $a = e + u$ where $e^2 = e$ and $u \in U(R)$, and an element a is strongly clean if $a = e + u$ where $e^2 = e, u \in U(R)$ and $eu = ue$. Note that an element a is strongly regular if and only if there is an idempotent $e \in R$ and $u \in U(R)$ such that $a = e + u, ae = ea$ and eae is zero.

Lemma 3.1 *An element $a \in R^{\#}$ if and only if $a = ue$, $ue = eu$ for some $u \in U(R)$ and idempotent $e = e^2$. In this case, $u = a - 1 + a^{\#}a$.*

Proof If $a \in R^{\#}$, then a is strong regular. By [2, Lemma 3.5], we have $a = ue$, $ue = eu$ and $u = a - 1 + a^{\#}a$. Conversely, since $ue = eu$ and $u \in U(R)$, we get

$$a = ue = ueueu^{-1} = a^2u^{-1} \in a^2R, \quad a = u^{-1}eueu = u^{-1}a^2 \in Ra^2.$$

This implies that $a \in a^2R \cap Ra^2$. Therefore, $a \in R^{\#}$. □

The following theorem shows that an element is group invertible if and only if it is both left c-group invertible and right c-group invertible.

Theorem 3.2 *Let $a, c \in R$. Then the following statements are equivalent:*

(1) $a \in R_c^{\#} \cap {}_cR^{\#}$;

(2) $a \in R^{\#}$;

(3) There exist $c \in U(R)$ and $e = e^2$ such that $a = ce$ and $ce = ec$.

In this case, $c = a - 1 + a^{\#}a$ is unique.

Proof (1) ⇒ (2). If $a \in R_c^{\#} \cap {}_cR^{\#}$, then $a \in Rca^2 \subseteq Ra^2$ and $a \in a^2cR \subseteq a^2R$ by Propositon 2.6 and Corollary 2.7. It follows that $a \in Ra^2 \cap a^2R$.

(2) ⇔ (3) is clear by Lemma 3.1. Since $a^{\#}$ is unique, c is unique.

(3) ⇒ (1). Since $a = ce = ec$, we have $Ra \subseteq Rc$ and $aR \subseteq cR$. Combining with $a \in R^{\#}$, we get $a \in R_c^{\#} \cap {}_cR^{\#}$ by Propositon 2.8 and Corollary 2.9. □

By Proposition 4.4, Lemma 2.15 and Theorem 3.2, we can give the following corollary immediately which shows the equivalence of right c-group invertible elements and group invertible elements.

Corollary 3.3 *If R is a ring with involution and $a^* = a$, then $a \in R^{\#}_{a-1+a^{\#}a}$ if and only if $a \in R^{\#}$.*

The next result shows the relationship between right c-group invertible elements and strongly clean elements.

Theorem 3.4 *Let $a, c \in R$. Then the following statements are equivalent:*

(1) $a \in R^{\#}_c$;

(2) $a \in R^{\#}$ and there exist $c \in U(R)$ and $f = f^2$ such that $a = c + f$ is a strongly clean element.

In this case, $c = a - 1 + a^{\#}_c ca$.

Proof (1) $\Rightarrow$ (2). Since $a \in R^{\#}_c$, it is clear that $a \in R^{\#}$ and there is $x \in R$ such that $xcax = x$. It follows that $xcaxca = xca$. Let $e = xca$. Then $e^2 = e$ is an idempotent element. Since $a = axca = xca^2$, we have $a = ae = ea$. Let $c = a - 1 + xca$. Then c is a unit since

$$(a - 1 + xca)(xc - 1 + xca) = (xc - 1 + xca)(a - 1 + xca) = 1.$$

Therefore, $a = c + 1 - xca = c + 1 - e$. Let $f = 1 - e$. Then $f^2 = f = 1 - e$ is an idempotent element. This implies that $a = c + f$ is a clean element. Since $a = ae = ea$, we have

$$af = a(1 - e) = a - ae = a - ea = (1 - e)a = fa.$$

It follows that $cf = (a - f)f = af - f = fa - f = f(a - f) = fc$. Therefore, $a = c + f$ is a strongly clean element.

(2) $\Rightarrow$ (1). Since $a = c + f$ is a strongly clean element, we get $a^2 = a(c + f) = ac + af$. Also since $a \in R^{\#}$, there is $y \in R$ such that $a = ya^2$. This implies that

$$a = y(ac + af) = yac + yaf = yac + yafc^{-1}c = (ya + yafc^{-1})c \in Rc.$$

By Proposition 2.8, we get $a \in R^{\#}_c$. $\square$

Corollary 3.5 *Let $a, c \in R$. Then the following statements are equivalent:*

(1) $a \in {}_cR^{\#}$;

(2) $a \in R^{\#}$ and there exist $c \in U(R)$ and $e = e^2$ such that $a = c + e$ is a strongly clean element.

In this case, $c = a - 1 + ac_ca^{\#}$ and $c^{-1} = c_ca^{\#} - 1 + ac_ca^{\#}$.

An element $a \in R^{\#}$ if and only if there is $e^2 = e \in R$ and $u \in U(R)$ such that $a = e + u$, $ae = ea$ and $eae = 0$. Accordingly, we have the following theorem for $a \in R^{\#}_c$.

Theorem 3.6 *Let $a, c \in R$. Then the following statements are equivalent:*

(1) $a \in R^{\#}_c$;

(2) There exist $c \in U(R)$ and $e = e^2$ such that $a = e + c$, $ec = ce$ and $aR \cap eR = \{0\}$;

(3) There exist $c \in U(R)$ and $e = e^2$ such that $a = e + c$, $ec = ce$ and $ae = ea = 0$.

Proof (1) $\Rightarrow$ (3). Let $c = a - 1 + a_c^{\#}ca$ and $e = 1 - a_c^{\#}ca$. By Theorem 3.4, we have $ae = a - aa_c^{\#}ca = a - a_c^{\#}ca^2 = ea = 0$, $a = e + c$, $ec = ce$.

(3) $\Rightarrow$ (1). Since there exist $c \in U(R)$ and $e = e^2$ such that $a = e + c$ and $ec = ce$, a is strongly clean. Also since $ea = ae = 0$, we get $ae = e + ce = e + ec = 0$. It follows that

$$a^2 = (e + c)^2 = e + 2ec + c^2 = ec + c^2 = (e + c)c = ac.$$

Thus, $a = a^2c^{-1} \in a^2R$. Analogously, we get $a = c^{-1}a^2 \in Ra^2$. Then $a \in R^{\#}$. Therefore, $a \in R_c^{\#}$ by Theorem 3.4 again.

(2) $\Rightarrow$ (3). Since $ae = e + ce = e + ec = ea$, we have $ae \in aR \cap eR = \{0\}$. Thus, $ae = 0 = ea$.

(3) $\Rightarrow$ (2). Let $x \in aR \cap eR$. Then there exist $s, t \in R$ such that $x = as = et$. It follows that $ex = eas = 0$ since $ea = 0$. Thus, $x = et = ex = 0$. Then $aR \cap eR = \{0\}$. □

We conclude this section by showing a particularly nice behaviour of the uniqueness of right c-group inverse on abelian rings.

Proposition 3.7 *Let $c \in R$. Then every right c-group invertible element of R has a unique right c-group inverse if and only if R is abelian.*

Proof If R is abelian and $a \in R$ is right c-group invertible, then there exist $x, y \in R$ such that $a = axca = ayca$, $xcax = x$ and $yca = ayc$ with $x \neq y$. Since cax is an idempotent element, it follows that

$$x = xcax = xcaycax = xcaaycx = aycx = ycax = caxy.$$

Moreover, because axc, cax and yca are idempotent elements, we also have

$$\begin{aligned} y &= ycaxcay = (ayc)axcy = (axc)yayc = yaycaxc \\ &= caxyayc = caxy(yca) = caxycay = caxy = x. \end{aligned}$$

Therefore, every right c-group invertible element has a unique right c-group inverse in an abelian ring.

Conversely, suppose that every right c-group invertible element of R has a unique right c-group inverse. If R is not abelian, then there is $e^2 = e \in R$ such that e is not central. Then $ex \neq xe$ for some $x \in R$, and thus $ex(1 - e) \neq 0$. This implies that $e \neq e + ex(1 - e)$. Let $c = e$. Then e is right e-group invertible with a right e-group inverse e. Moreover, we have

$$(e + ex(1 - e))e(e + ex(1 - e)) = e + ex(1 - e),$$

$$e(e + ex(1 - e))e = e = (e + ex(1 - e))e.$$

This shows that $e + ex(1 - e)$ is also a right e-group inverse of e, a contradiction. □

4. Relationships of various generalized inverses

In this section, we investigate the relationships between right c-group inverses and other various generalized inverses including group inverses, Moore-Penrose inverses, core inverses, dual core inverses, one-sided (b,c)-inverses and (b,c)-inverses. Some work has already been done in this topic (for example, see [3]). We start with the following result which shows that $a \in R^{\#} \cap R^{\dagger}$ implies $a \in R^{\#}_{aa^{\#}} \cap {}_{aa^{\#}}R^{\#}$ under some conditions.

Theorem 4.1 *Let $a \in R$. Then the following statements are equivalent:*

(1) $a \in R^{\#} \cap R^{\dagger}$;

(2) There exist $x, y \in R$ such that x is a right $aa^{\#}$-group inverse of a, y is a left $aa^{\#}$-group inverse of a and ax, ya are projections.

Proof (1) $\Rightarrow$ (2). If $a \in R^{\#} \cap R^{\dagger}$, then $a^{\#}$ and $a^{\dagger}$ exist. Let $x = a^{\#}aa^{\dagger}$. Then we have

$$xaa^{\#}ax = a^{\#}aa^{\dagger}aa^{\#}aa^{\#}aa^{\dagger} = a^{\#}aa^{\dagger} = x,$$

$$axaa^{\#}a = aa^{\#}aa^{\dagger}aa^{\#}a = a, \quad xaa^{\#}a = a^{\#}aa^{\dagger}aa^{\#}a = a^{\#}a,$$

$$axaa^{\#} = aa^{\#}aa^{\dagger}aa^{\#} = aa^{\#} = a^{\#}a.$$

This implies that x is a right $aa^{\#}$-group inverse of a. Since $ax = aa^{\#}aa^{\dagger} = aa^{\dagger}$ and $axax = aa^{\dagger}aa^{\dagger} = aa^{\dagger} = ax$, it follows that ax is a projection. Similarly, if we let $y = a^{\dagger}aa^{\#}$, then

$$aaa^{\#}ya = aaa^{\#}a^{\dagger}aa^{\#}a = a, \ yaaa^{\#}y = y, \ aa^{\#}ya = aaa^{\#}y.$$

This shows that y is a left $aa^{\#}$-group inverse of a. Since $ya = a^{\dagger}aa^{\#}a = a^{\dagger}a$ and $yaya = ya$, ya is also a projection.

(2) $\Rightarrow$ (1). If x is a right $aa^{\#}$-group inverse of a, then $a = axaa^{\#}a = axa$. Also, if y is a left $aa^{\#}$-group inverse of a, then $a = aaa^{\#}ya = aya$. Combining with $(ax)^* = ax$ and $(ya)^* = ya$, we have $a \in R^{\dagger}$ by [9, Lemma 2.18]. Therefore, $a \in R^{\#} \cap R^{\dagger}$. □

Proposition 4.2 *If $a \in R^{\dagger}$, then the following statements are equivalent:*

(1) $a \in R^{EP}$;

(2) There is $x \in R$ such that x is a right $aa^{\dagger}$-group inverse of a with $xa = ax$.

Proof (1) $\Rightarrow$ (2). Since $a \in R^{EP}$, $aa^{\dagger} = a^{\dagger}a$. Let $x = a^{\dagger}$. Then we have $xa = ax$ and

$$xaa^{\dagger}ax = a^{\dagger}aa^{\dagger} = a^{\dagger} = x, \quad axaa^{\dagger}a = a,$$

$$xaa^{\dagger}a = xa = a^{\dagger}a = aa^{\dagger} = axaa^{\dagger}.$$

Therefore, we deduce that x is a right $aa^{\dagger}$-group inverse of a.

(2) $\Rightarrow$ (1). If x is a right $aa^{\dagger}$-group inverse of a such that $ax = xa$, then we have

$$axaa^{\dagger} = xaa^{\dagger}a = xa, \ a = ax(aa^{\dagger})a = axa, \ xaa^{\dagger}ax = xax = x.$$

This implies that x is the group inverse of a and $xa = aa^{\dagger}$. Therefore, $a^{\#}a = aa^{\dagger}$, that is, $a \in R^{EP}$. □

Corollary 4.3 *Let $a \in R$. If $a \in R^{\#}_{aa^{\dagger}} \cap {}_{aa^{\dagger}}R^{\#}$ such that $a^{\#}_{aa^{\dagger}} = {}_{aa^{\dagger}}a^{\#}$, then a is an EP element.*

Proof Since $a^{\#}_{aa^{\dagger}} = {}_{aa^{\dagger}}a^{\#}$, there exists $x \in R$ such that $x = a^{\#}_{aa^{\dagger}} = {}_{aa^{\dagger}}a^{\#}$. Then we have

$$a = axaa^{\dagger}a = axa, \quad x = xaa^{\dagger}ax = xax,$$

$$axaa^\dagger = xaa^\dagger a = xa, \quad aa^\dagger xa = aaa^\dagger x = a^2a^\dagger x.$$

It follows that $xa = axaa^\dagger = aa^\dagger$, thus $a^2a^\dagger = axa = a$ and $aa^\dagger xa = aa^\dagger$. Then $a^2a^\dagger x = ax = aa^\dagger$. Hence, $ax = aa^\dagger = xa$. It follows that a is an EP element by Proposition 4.2. □

If $a^* = a$, then the next proposition shows that a being right c-group invertible implies the EP property of a.

Proposition 4.4 *If $a = a^*$, then a is EP if and only if there is $c \in R$ such that a is right c-group invertible. In this case, $c = aa^\# = aa^\dagger = aa^{\oplus} = aa_{\oplus}$.*

Proof If a is EP, then a is right $aa^\dagger$-group invertible by Proposition 4.2. Let $c = aa^\# = aa^\dagger = aa^{\oplus} = aa_{\oplus} \in R$. Then a is right c-group invertible. Conversely, if a is right c-group invertible, then there is $x \in R$ such that $axc = xca$ and $xcax = x$. Since $a^* = a$, we have $(axc)^* = (xc)^*a = (xca)^* = a(xc)^*$. It follows that $[a(xc)^*xc]^* = a(xc)^*xc = (xc)^*xca$. Since $a = axca = xca^2$, we have $a(xc)^*xca = (xc)^*xca^2 = (xc)^*a$. Then $[(xc)^*a]^* = (xc)^*a = axc = xca$. Therefore, we have $(xc)^*a(xc) = xcaxc = xc$. Thus $(xc)^* = xc$. Let $z = xc$. Then we have

$$(za)^* = (xca)^* = axc = xca = za, \quad (az)^* = (axc)^* = xca = axc = az,$$

$$aza = a, \quad zaz = z.$$

Therefore, $z = a^\dagger$. Moreover, it is clear that xc is the group inverse of a. Then $a \in R^\# \cap R^\dagger$ and $xc = a^\# = a^\dagger$, that is, a is EP. □

Corollary 4.5 *Let $a \in R$ such that $a = a^*$. Then a is EP if and only if there is $c \in R$ such that a is left c-group invertible. In this case, $c = aa^\# = aa^\dagger = aa^{\oplus} = aa_{\oplus}$.*

Proposition 4.4 together with Corollary 4.5 implies the following corollary valid in the rings with an involution.

Corollary 4.6 *Let $a \in R$ such that $a = a^*$. Then there is $c \in R$ such that $a \in R_c^\#$ if and only if $a \in {}_cR^\#$.*

When $a^* = a$ and $c = aa^\dagger$, we next show that $a \in R^{EP}$ is equivalent to $a \in R_c^\# \cap {}_cR^\#$.

Proposition 4.7 *If $a^* = a$, then a is an EP element if and only if $a \in R^\#_{aa^\dagger} \cap {}_{aa^\dagger}R^\#$. In this case, $a^\dagger$ is both a left $aa^\dagger$-group inverse of a and a right $aa^\dagger$-group inverse of a.*

Proof If $a \in R^{EP}$, then a is right $aa^\dagger$-group invertible by Proposition 4.2. Let x be a right $aa^\dagger$-group inverse of a. Then we have

$$axaa^\dagger a = axa = a, \quad xaa^\dagger ax = xax = x,$$

$$xaa^\dagger a = xa = axaa^\dagger.$$

Hence $xa = aa^\dagger$ and $a = xa^2$. Since $a^* = a$ and $a \in R^\#_{aa^\dagger}$, we conclude that a is left $aa^\dagger$-group invertible by Lemma 2.15. Suppose that $y \in R$ is a left $aa^\dagger$-group inverse of a. Then $a = aaa^\dagger ya$ and $aa^\dagger ya = aaa^\dagger y = ay$ since a is an EP element. Thus $a = a^2y$. This implies that $a = a^2y = a^2(aa^\dagger)y = (aa^\dagger)xa^2$. By Corollary 2.7, we get $xay = xaa^\dagger xa = aa^\dagger a^\dagger aa^\dagger = a^\dagger$.

Therefore, $a^\dagger$ is a left $aa^\dagger$-group inverse of a. Also since

$$a = a^2y = a^2y(aa^\dagger) = xa^2 = x(aa^\dagger)a^2,$$

we have xay is also a right $aa^\dagger$-group inverse of a by Proposition 2.6. This implies that $a \in R^{\#}_{aa^\dagger} \cap {}_{aa^\dagger}R^{\#}$ and $a^\dagger$ is both a left $aa^\dagger$-group inverse of a and a right $aa^\dagger$-group inverse of a. Conversely, since $a \in R^{\#}_{aa^\dagger} \cap {}_{aa^\dagger}R^{\#}$, a is an EP element by Proposition 4.4 and Corollary 4.5. □

We need the following lemma, which is closely related to EP elements and has been investigated in [3, Theorem 3.1].

Lemma 4.8 *Let $a \in R^{\#} \cap R^\dagger$. Then:*

(1) a is EP if and only if $a^{\#} = a^\dagger = a^{\oplus} = a_{\oplus}$;

(2) a is EP if and only if $aa^{\#} = aa^\dagger = aa^{\oplus} = aa_{\oplus}$.

If $c = aa^\dagger$, then the following theorem not only gives a new characterization of EP elements, but also reveals the relations between $a^{\#}$, $a^\dagger$, $a^{\oplus}$, $a_{\oplus}$, $a^{\#}_c$, ${}_ca^{\#}$ and EP elements.

Theorem 4.9 *Let $a \in R^\dagger$ and $a^* = a$. Then the following statements are equivalent:*

(1) $a \in R^{EP}$;

(2) $a \in R^{\#} \cap {}_{aa^\dagger}R^{\#}$ and $a^{\#}$ is a left $aa^\dagger$-group inverse of a;

(3) $a \in R^{\#} \cap R^{\#}_{aa^\dagger}$ and $a^{\#}$ is a right $aa^\dagger$-group inverse of a;

(4) $a \in R^{\oplus} \cap {}_{aa^\dagger}R^{\#}$ and $a^{\oplus}$ is a left $aa^\dagger$-group inverse of a;

(5) $a \in R_{\oplus} \cap R^{\#}_{aa^\dagger}$ and $a_{\oplus}$ is a right $aa^\dagger$-group inverse of a;

(6) $a \in R^{\#} \cap R^\dagger$ and $a^\dagger$ is a left $aa^\dagger$-group inverse of a;

(7) $a \in R^{\#} \cap R^\dagger$ and $a^\dagger$ is a right $aa^\dagger$-group inverse of a;

(8) $a \in R^{\#} \cap R^\dagger$ and $a \in R^{\#}_{aa^\dagger} \cap {}_{aa^\dagger}R^{\#}$.

Proof (1) ⇔ (8) is clear by Proposition 4.7.

(1) ⇒ (2)–(7). Since $a \in R^{EP}$, $a^\dagger$ is both a left $aa^\dagger$-group inverse of a and a right $aa^\dagger$-group inverse of a by Proposition 4.7. By Lemma 4.8, we get $a^\dagger = a^{\oplus} = a_{\oplus} = a^{\#}$. The other implications are clear by Proposition 4.7 and [3, Theorem 3.1].

(7) ⇒ (1). Since $a^\dagger$ is a right $aa^\dagger$-group inverse of a, $a^\dagger aa^\dagger = a^\dagger$ is the group inverse of a, we deduce that $a^\dagger = a^{\#}$. Hence $a \in R^{EP}$.

(6) ⇒ (1). Since $a^\dagger$ is a left $aa^\dagger$-group inverse of a, $aa^\dagger a^\dagger$ is the group inverse of a. Thus $aa^\dagger a^\dagger = a^{\#}$. It follows that $(a^{\#})^* = (a^\dagger)^*aa^\dagger$, thus $(a^{\#})^*a = (a^\dagger)^*a$. Since $a^* = a$, we get $aa^{\#} = aa^\dagger$. By Lemma 4.8, we get $a \in R^{EP}$.

(5) ⇒ (1). Since $a \in R^{\#}_{aa^\dagger}$, a is group invertible. Combining with $a \in R^\dagger$, we have $a \in R^{\#} \cap R^\dagger$. If $a_{\oplus}$ is a right $aa^\dagger$-group inverse of a, then $a_{\oplus}aa^\dagger = a^{\#}$. Therefore, $aa^\dagger = aa^{\#}$, that is, $a \in R^{EP}$.

(4) ⇒ (1). If $a^{\oplus}$ is a left $aa^\dagger$-group inverse of a, then $aa^\dagger a^{\oplus} = a^{\#}$. Since $a^* = a$, we deduce that $(aa^{\#})^* = [a(aa^\dagger)a^{\oplus}]^* = (a^{\oplus})^*aa^\dagger a = (a^{\oplus})^*a = (aa^{\oplus})^*$. Thus $aa^{\#} = aa^{\oplus}$, that is, $a \in R^{EP}$ by Lemma 4.8.

(3) $\Rightarrow$ (1). If $a^{\#}$ is a right $aa^{\dagger}$-group inverse of a, then $a^{\#}aa^{\dagger} = a^{\#}$. It follows that $aa^{\#} = aa^{\dagger}$, thus $a \in R^{EP}$.

(2) $\Rightarrow$ (1). If $a^{\#}$ is a left $aa^{\dagger}$-group inverse of a, then $aa^{\dagger}a^{\#} = a^{\#} = (aa^{\dagger})^* a^{\#} = (a^{\dagger})^* aa^{\#}$. Therefore, $aa^{\#} = a^{\#}a = (a^{\dagger})^* a = aa^{\dagger}$ since $a^* = a$, that is, $a \in R^{EP}$. $\square$

Now, we study the relationship between right c-group inverses and one-sided (b, c)-inverses.

Proposition 4.10 *Let $a, c \in R$ such that $a = a^*$. Then a is right c-group invertible if and only if a is left (a, c)-invertible and $Ra \subseteq Rc$.*

Proof Since a is right c-group invertible, there is $y \in R$ such that $a = a^2yc \in Rc$ and $a = yca^2 \in Rca^2$. This implies that a is left (a, c)-invertible. Conversely, if a is left (a, c)-invertible, then there is $x \in R$ such that $a = xca^2 \in Ra^2$. It follows that $a = a^2(xc)^* \in a^2R$ since $a^* = a$. Therefore, $a \in Ra^2 \cap a^2R$, that is, $a \in R^{\#}$. Since $Ra \subseteq Rc$, a is right c-group invertible by Proposition 2.8. $\square$

Corollary 4.11 *Let $a, c \in R$ such that $a = a^*$. Then a is left c-group invertible if and only if a is right (c, a)-invertible and $aR \subseteq cR$.*

In particular, we have the following corollary which is related to (b, c)-inverses.

Corollary 4.12 *If $a \in R$ such that $a = a^*$, then the following statements are equivalent:*

(1) a is right (resp., left) a-group invertible;

(2) a is (a, a)-invertible;

(3) $a \in R^{\#}$.

Over a directly finite ring R, the following proposition shows that 1 may be the unique right c-group inverse of a for some $a, c \in R$.

Proposition 4.13 *Let $a, c \in R$. Then the following statements are equivalent:*

(1) R is a directly finite ring;

(2) If $ac = 1$, then 1 is the unique right c-group inverse of a;

Proof (1) $\Rightarrow$ (2). If $ac = 1$, then $ca = 1$ since R is directly finite. Therefore, we have

$$a = a(ca)ca, \ ca = cacaca, \ caca = acac = 1.$$

This implies that $ca = 1$ is a right c-group inverse of a. Let any $x \in R$ such that x is a right c-group inverse of a. Then we have $a = axca = ax$. Since $ac = ca = 1$, we have $a \in U(R)$. It follows that $x = 1$, and thus 1 is the unique right c-group inverse of a.

(2) $\Rightarrow$ (1). It suffices to show $ca = 1$. In fact, since 1 is the unique right c-group inverse of a, we have $ca = (1c)a = a(1c) = ac = 1$, as desired. $\square$

We conclude this section by giving a new characterization of directly finite rings.

Theorem 4.14 *Let $a, c \in R$. Then the following statements are equivalent:*

(1) R is a directly finite ring;

(2) If $ac = 1$, then $a_c^{\#} = c_a^{\#}$.

Proof (1) $\Rightarrow$ (2). If R is a directly finite ring and $ac = 1$, then 1 is the unique right c-group inverse of a by Proposition 4.13. Moreover, since $ca = ac = 1$, 1 is also the unique right a-group inverse of c again by Proposition 4.13. Therefore, we have $a_c^{\#} = c_a^{\#} = 1$.

(2) $\Rightarrow$ (1). Let $x \in R$ be a right c-group inverse of a. Since $a_c^{\#} = c_a^{\#}$ and $ac = 1$, we have $axc = xca$, $cxa = x$. Therefore, we get $cxac = cx = xc$. Then $axca = acxa = xa = a$, and hence $xac = ac = 1$. This implies that $ca = cxa = xac = 1$. □

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