*Journal of Mathematical Research with Applications*
Jan., 2023, Vol. 43, No. 1, pp. 74–82
DOI:10.3770/j.issn:2095-2651.2023.01.008
Http://jmre.dlut.edu.cn

# Finite Groups with Some $c_p$-Supplemented Subgroups

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**Abstract** Let $H$ be a subgroup of a finite group $G$ and $p$ a prime divisor dividing the order of $G$. We say $H$ is $c_p$-supplemented in $G$ if there exists a supplement $T$ to $H$ in $G$ containing $H_G$ such that $H \cap T/H_G$ is a $p'$-group, where $H_G$ is the core of $H$ in $G$. A $CS_p$-group is a group in which every $p$-subgroup is $c_p$-supplemented. In this paper, we characterize the $p$-solvability and $p$-supersolvability of groups $G$ with some certain $p$-subgroups being $c_p$-supplemented. Furthermore, we give some equivalent conditions of $CS_p$-group in $p$-solvable universe. Finally, we give some criteria of $CS_p$-groups for the direct product of two $CS_p$-groups. Our results extend some recent conclusions.

**Keywords** $c$-supplemented subgroup; $c_p$-normal subgroup; $c_p$-supplemented subgroup; $CS_p$-group; $p$-supersolvable group

**MR(2020) Subject Classification** 20D15; 20D20; 20D40

## 1. Introduction

In this paper, all groups are assumed to be finite. The symbol $G$ always means a group. For convenience, we denote by $|G|$ the order of group $G$, $\pi(G)$ the set of prime divisors dividing $|G|$ and $\mathrm{Exp}(G)$ the exponent of $G$. For some fixed $p \in \pi(G)$, $G_p$ stands for the Sylow $p$-subgroup of $G$ meanwhile $G_{p'}$ stands for the Hall $p'$-subgroup of $G$. Further, we denote by $\mathrm{Syl}_p(G)$ the set of all Sylow $p$-subgroups of $G$. Other unspecified notions and natation are standard as in [1, 2].

As a generalization of normality, a subgroup $H$ of $G$ is called a $c$-normal subgroup of $G$ if $G = HT$ and $H \cap T \leq H_G$ (see [3]), where $T$ is a normal subgroup of $G$ and $H_G$ is the core of $H$ in $G$. There are many extensions of $c$-normality, a subgroup $H$ is called $c$-supplemented in $G$ if $G = HT$ and $H \cap T \leq H_G$ for which $T$ is a subgroup of $G$ and $H_G$ is the core of $H$ in $G$ (see [4]). Following Li et al.[5], $G$ is called a $CN$-group if all of whose subgroups are $c$-normal. In [4], $G$ is called a $c$-supplemented group if all of whose subgroups are $c$-supplemented. Recently, many results involving the structure of a $CN$-group or $c$-supplemented group are investigated [5–7].

On the other hand, the authors in [8] extended the $c$-normality from a quantitative aspect: a subgroup $H$ is said to be $c_p$-normal in $G$ if there exists a normal supplement $T$ to $H$ in $G$ such that $H_G \leq T$ and $H \cap T/H_G$ is a $p'$-group, where $p \in \pi(G)$. $G$ is called a $CN_p$-group

Received February 24, 2022; Accepted June 26, 2022
Supported by the National Natural Science Foundation of China (Grant No. 12071092) and the Science and Technology Program of Guangzhou Municipality (Grant No. 201804010088).
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if all of whose $p$-subgroups are $c_p$-normal. Following this idea, we can extend the concepts of $c$-supplemented subgroup and $CN_p$-group:

**Definition 1.1** *Let $p \in \pi(G)$ be a fixed prime and let $H$ be a subgroup of $G$.*

*(1) $H$ is said to be $c_p$-supplemented in $G$ if $G = HT$ and $H \cap T/H_G$ is a $p'$-group, where $H_G \leq T \leq G$. We also call $T$ the $c_p$-supplement to $H$ in $G$;*

*(2) $G$ is said to be a $CS_p$-group if all of whose $p$-subgroups are $c_p$-supplemented in $G$.*

Obviously, a $c$-supplemented subgroup or a $c_p$-normal subgroup is $c_p$-supplemented and a $CN_p$-group or a $c$-supplemented group is a $CS_p$-group, but the converse is not true in general, see Examples 1.2–1.4.

**Example 1.2** Let $G = A_5 = \langle (1,2,3), (1,2,4), (1,2,5) \rangle$ and $H = \langle (1,2)(3,4), (2,3)(4,5) \rangle \cong D_{10}$. Then $H_G = 1$ and $H$ has proper supplemented subgroups $T_i$ $(i = 1,2,3,4,5)$ in $G$, where $T_1 = \langle (3,4,5), (2,4)(3,5) \rangle$, $T_2 = \langle (1,4,5), (1,4)(3,5) \rangle$, $T_3 = \langle (2,3,4), (1,3)(2,4) \rangle$, $T_4 = \langle (1,2,3), (1,3)(2,5) \rangle$, $T_5 = \langle (1,2,5), (1,4)(2,5) \rangle$. Note that $H \cap T_i \cong C_2$ $(i = 1,2,3,4,5)$, so $H$ is $c_5$-supplemented in $G$ but $H$ is neither $c$-supplemented nor $c_5$-normal in $G$.

**Example 1.3** Let $G = S_4$, the symmetric group of degree 4. It is well known that every element in $\mathrm{Syl}_2(G)$ and $\mathrm{Syl}_3(G)$ is not normal in $G$. Let $p = 3$. Clearly, $G$ is a $CS_3$-group. Let $H \in \mathrm{Syl}_3(G)$. Assume that $H$ is $c$-normal in $G$. Then there is a $T \in \mathrm{Syl}_2(G)$ and $T \trianglelefteq G$ such that $G = HT$ and $H \cap T = 1$. The contradiction indicates that $G$ is not a $CN_3$-group.

**Example 1.4** Let $G = A_4$, the alternating group of degree 4. Let $p = 3$. Then every 3-subgroup of $G$ is $c$-supplemented, so $G$ is a $CS_3$-group. Let $C_2 \cong H \leq G$. Then $H$ is not $c$-supplemented in $G$ since $G$ has no subgroup of order 6. So $G$ is not a $c$-supplemented group.

**Remark 1.5** In [8], the authors obtained some results about $p$-solvable groups under the assumption of some maximal subgroups being $c_p$-normal, for example, a group $G$ is $p$-solvable if and only if in which every maximal subgroup of $G$ is $c_p$-normal in $G$ (see [8, Corollary 3.3]). But the following two simple groups indicate we cannot weaken the condition to $c_p$-supplemented subgroup.

**Example 1.6** (1) Let $G = A_5$ be the alternating group of degree 5. Clearly, every subgroup of $G$ is $c_5$-supplemented in $G$.

(2) Let $G = \mathrm{PSL}(3,2)$ be the simple group of order 168. Note that all maximal subgroups of $G$ are complemented. Then they are $c_2$-supplemented in $G$.

In Section 2, we first give some elementary properties of $c_p$-supplemented subgroup and then some preliminary results we need. In Section 3, we first discuss the structure of $G$ with some minimal subgroups $c_p$-supplemented (or, $c$-supplemented), then characterize the structure of $CS_p$-groups and give some criteria for a group to be a $CS_p$-group. Many of our results may be regarded as the generalizations of results in [5, 7, 8] in $p$-solvable universe.

## 2. Preliminaries

In this section, we give some basic results which are essential in the sequel.

**Lemma 2.1** *Let $G$ be a group and $p \in \pi(G)$ be a fixed prime. Let $N, H, K$ be subgroups of $G$.*

*(1) If $H \leq K$ and $H$ is $c_p$-supplemented in $G$, then $H$ is $c_p$-supplemented in $K$.*

*(2) Assume that $N$ is normal in $G$ and $N \leq H$. Then $H$ is $c_p$-supplemented in $G$ if and only if $H/N$ is $c_p$-supplemented in $G/N$.*

*(3) If $H$ is a c-supplemented subgroup of $G$, then $H$ is $c_q$-supplemented in $G$ for every $q \in \pi(G)$.*

*(4) If $H$ is a $p$-group, then $H$ is c-supplemented in $G$ if and only if $H$ is $c_p$-supplemented in $G$.*

*(5) Assume that $N$ is a normal $p'$-subgroup of $G$ and $H$ is a $c_p$-supplemented $p$-subgroup of $G$. Then $HN/N$ is $c_p$-supplemented in $G/N$.*

*(6) Let $P \in Syl_p(G)$. Then $P$ is complemented in $G$ if and only if $P$ is $c_p$-supplemented in $G$.*

*(7) Let $M \leq G$ and $H \leq \Phi(M)$. If $H$ is a $c_p$-supplemented $p$-subgroup of $G$, then $H \trianglelefteq G$ and $H \leq \Phi(G)$.*

**Proof** (1) Assume that $H$ is $c_p$-supplemented in $G$ and $T$ is a $c_p$-supplement to $H$ in $G$. Then $H \cap T/H_G$ is a $p'$-group. Note that $K = H(T \cap K)$. Now we have

$$K = HH_K(T \cap K) = H(H_KT \cap K) = H(T_0 \cap K),$$

where $T_0 = H_KT$. Furthermore, $H_K \leq T_0 \cap K$ and $H \cap (T_0 \cap K)/H_K = (H_KT \cap H)/H_K = (H \cap T)H_K/H_K \cong (H \cap T)/(H_K \cap T)$. Since $H_G \leq K$ and $H_G \cap T \leq H_K \cap T$, it follows that $H_G = H_G \cap T \trianglelefteq H_K \cap T$. So $(H \cap T)/(H_K \cap T) \leq (H \cap T)/(H_G \cap T) = (H \cap T)/H_G$ is a $p'$-group. Therefore, $H \cap (K \cap T_0)/H_K$ is a $p'$-group, that is, $H$ is $c_p$-supplemented in $K$.

(2) Assume first that $H$ is $c_p$-supplemented in $G$ and $T$ is a $c_p$-supplement to $H$ in $G$. Then $H \cap T/H_G$ is a $p'$-group. Note that $G/N = H/N \cdot T/N$ and $(H/N \cap T/N)/(H/N)_{(G/N)} = ((H \cap T)/N)/(H_G/N) \cong (H \cap T)/H_G$ is a $p'$-group. Therefore, $H/N$ is $c_p$-supplemented in $G/N$.

Conversely, let $(H/N)_{G/N} \leq T/N \leq G/N$ such that $G/N = H/N \cdot T/N$ and $((H/N) \cap (T/N))/(H/N)_{(G/N)}$ is a $p'$-group. Then $G = HT$ and $H_G \leq T$. Since

$$((H/N) \cap (T/N))/(H/N)_{G/N} = ((H \cap T)/N)/(H_G/N) \cong H \cap T/H_G,$$

$H \cap T/H_G$ is a $p'$-group. Hence $H$ is $c_p$-supplemented in $G$.

(3) Assume that $H$ is $c$-supplemented in $G$. There then exists a supplement $T$ to $H$ in $G$ such that $H \cap T \leq H_G$. Denote $T_0 = TH_G$, then $G = HT_0$ and $H_G \leq T_0$. Note that

$$H \cap T_0/H_G = H \cap H_GT/H_G = (H \cap T)H_G/H_G = 1,$$

so $H$ is $c_q$-supplemented in $G$ for any $q \in \pi(G)$.

(4) If $H$ is $c$-supplemented in $G$, then $H$ is $c_p$-supplemented in $G$ by (3). Now assume that $H$ is $c_p$-supplemented in $G$. Then there exists a subgroup $H_G \leq T$ such that $G = HT$ and

$H \cap T/H_G$ is a $p'$-group, so $H \cap T = H_G$ since $H$ is a $p$-group. We have $H$ is $c$-supplemented in $G$.

(5) It follows from (4) and [4, Lemma 2.1(3)].

(6) If $P$ is complemented in $G$, then $P$ is $c$-supplemented in $G$. So $P$ is $c_p$-supplemented in $G$ by (4). Now if $P$ is $c_p$-supplemented in $G$, then we have a subgroup $P_G \leq T$ of $G$ such that $G = PT$ and $P \cap T/P_G$ is a $p'$-group. So $P \cap T = P_G$. Note that

$$|G : P| = |PT : P| = |T : P \cap T| = |T : P_G|,$$

thus $P_G \in \mathrm{Syl}_p(T)$. Now by Schur-Zassenhaus Theorem, $T = P_G T_{p'}$. So $G = PT = PP_G T_{p'} = PT_{p'}$, $P$ is complemented in $G$.

(7) It follows from [4, Lemma 2.1(4)] and (4). □

**Lemma 2.2** ([9, Lemma 2.6], [10, Lemma 2.9]) *Let $G$ be a group and let $P \in \mathrm{Syl}_p(G)$, where $p = \min \pi(G)$. Then $G$ is $p$-nilpotent if every cyclic subgroup of $P$ of order $p$ and 4 (if $p = 2$) is $c$-supplemented in $G$.*

**Lemma 2.3** ([1]) *Let $p \in \pi(G)$. Then $G$ is $p$-nilpotent if each element of $G$ of order $p$ lies in $Z(G)$ and in addition, each element of $G$ of order 4 still lies in $Z(G)$ when $p = 2$.*

As a special case of [11, Lemma 2.4], we have

**Lemma 2.4** *Let $N$ be a minimal normal $p$-subgroup of a group $G$. Then $|N| = p$ if $N$ has a subgroup $H$ such that $|H| = p$ and $H$ is $c$-supplemented in $G$.*

**Lemma 2.5** ([7, Corollary 3.1]) *Assume that $P$ is a $c$-supplemented 2-group. Then $|P : C_P(\Phi(P))| \leq 2$.*

**Lemma 2.6** ([7, Theorem 3.5]) *Let $G$ be a $c$-supplemented $p$-group and $a \in G$. If $p > 2$, then $g^a = g$ for any $g \in \Phi(G)$. If $p = 2$, then either $g^a = g$ for any $g \in \Phi(G)$ or $g^a = g^{-1}$ for any $g \in \Phi(G)$.*

Recall that $G$ is a $p$-complemented group if every $p$-subgroup of $G$ is complemented [8].

**Lemma 2.7** ([8, Lemma 2.8]) *Assume that $G$ is a $p$-solvable group for some prime $p \in \pi(G)$. Then $G$ is a $p$-complemented group if and only if $G$ is $p$-supersolvable and every Sylow $p$-subgroup of $G$ is elementary abelian.*

**Lemma 2.8** *Let $G$ be a $p$-solvable group. Then $G$ has a Hall $p'$-subgroup, say $G_{p'}$. In addition, if $G_{p'}$ is solvable, then $G$ is solvable.*

**Proof** Since $G$ is $p$-solvable, then $G$ is $p$-separable. Hence $G$ is $p'$-separable. By [2, Theorem 3.5], $G$ possesses a Hall $p'$-subgroup $G_{p'}$.

Note that every chief factor of $G$ is either an elementary abelian $p$-group or a $p'$-group. If $G_{p'}$ is solvable, then every non-$p$-chief factor of $G$ is an elementary abelian $q$-group for some $q \in \pi(G)$. So $G$ is solvable. □

## 3. Main results

In the literature, people usually assume that $p = \min \pi(G)$ to obtain the $p$-nilpotence or $p$-solvability of $G$, here we extend this discussion to the second minimal prime in $\pi(G)$.

**Theorem 3.1** *Let $G$ be a group and $\pi(G) = \{p_1, p_2, \dots, p_n\}$ with $p_n > \cdots > p_2 > p_1$. If every subgroup of $G$ of order $p$ and 4 (if $p = 2$) is $c_p$-supplemented (c-supplemented) in $G$, then $G$ is $p$-solvable, where $p \in \{p_1, p_2\}$.*

**Proof** If $p = p_1$, by Lemma 2.2, obviously, $G$ is $p$-solvable. Henceforth we may assume that $p = p_2$ is odd. Let $K < G$. If $p_1 \notin \pi(K)$ or $p_1 \in \pi(K)$ and $p \notin \pi(K)$, obviously, $K$ is $p$-solvable. If $\{p_1, p\} \subseteq \pi(K)$, then by Lemma 2.1, $K$ satisfies our hypothesis. Hence $K$ is $p$-solvable by induction. Therefore, we may suppose that $G$ is a non-$p$-solvable group all of whose proper subgroups are $p$-solvable. Pick $H \leq G$ and $|H| = p$. By hypothesis, $H$ is either normal in $G$ or complemented in $G$. Assume that $G$ has a complemented subgroup $H$ with order $p$. Then $G = HT$ and $H \cap T = 1$ for some $T \leq G$. So $|G : T| = p$ and $G/T_G \lesssim S_p$. Thus $G/T_G$ is solvable since $|G/T_G| = p_1^{\alpha} p_2^{\beta}$ for which $\alpha, \beta$ are nonnegative integers. If $T_G > 1$, then $T_G$ is $p$-solvable. It follows that $G$ is $p$-solvable, a contradiction. If $T_G = 1$, obviously, $G$ is $p$-solvable, a contradiction again. This implies that every subgroup of $G$ of order $p$ is normal in $G$. Now assume that $G$ has a subgroup $H$ of order prime $p$ such that $H \nleq Z(G)$. Then $C_G(H) < G$ and $G/C_G(H)$ is solvable. It follows that $G$ is $p$-solvable, a contradiction. Therefore, we may assume that $Z(G)$ contains every subgroup of $G$ of order $p$. By Lemma 2.3, $G$ is $p$-nilpotent, contrary to the choice of $G$. □

**Theorem 3.2** *Let $G$ be a group and $\pi(G) = \{p_1, p_2, \dots, p_n\}$ with $p_n > \cdots > p_2 > p_1$. Assume that $G$ is a $CS_p$-group for which $p = \min(\pi(G) \setminus \{r\})$. Then $G$ is $p$-supersolvable and $\Phi(G)_p = \Phi(G_p)$.*

**Proof** Clearly, $G$ is a $p$-solvable group by Theorem 3.1. Let $N$ be a minimal normal subgroup of $G$. We have $N$ is either an elementary abelian $p$-group or a $p'$-group. It follows from Lemma 2.1 that $G/N$ inherits our conditions, therefore, $G/N$ is $p$-supersolvable. Hence we may assume $N$ is unique and $N$ is an elementary abelian $p$-group. Let $x \in N$ and $|x| = p$. By hypothesis, $H = \langle x \rangle$ is $c$-supplemented in $G$. Then $|N| = p$ by Lemma 2.4. Thus $G$ is $p$-supersolvable.

We now prove that $\Phi(G)_p = \Phi(G_p)$. Since $G$ is $p$-solvable, we can write $G = G_p G_{p'}$ by Lemma 2.8. We first claim that $\Phi(G)_p \leq \Phi(G_p)$. If $\Phi(G)_p \nleq \Phi(G_p)$, note that $\Phi(G)_p \leq G_p$, there then exists a subgroup $T \lessdot G_p$ such that $G_p = \Phi(G)_p T$. Hence $G = G_p G_{p'} = \Phi(G)_p T G_{p'} = T G_{p'} < G$, a contradiction. We now prove that $\Phi(G_p) \leq \Phi(G)_p$. Let $M = G_p$ and $H = \Phi(G_p)$ in Lemma 2.1. Then $\Phi(G_p) \leq \Phi(G)$ since $G$ is a $CS_p$-group. Hence, $\Phi(G_p) \leq \Phi(G)_p$. □

**Corollary 3.3** *Let $G$ be a group and $\pi(G) = \{p_1, p_2, \dots, p_n\}$ with $p_n > \cdots > p_2 > p_1$. Assume that there exists a subgroup $M \lessdot G$ such that $|G : M| = p$ and every $p$-subgroup of $M$ is $c$-supplemented in $G$, where $p \in \{p_1, p_2\}$. Then $G$ is $p$-supersolvable.*

**Proof** Clearly, $M$ is a $CS_p$-group by Lemma 2.1. By Theorem 3.2, $M$ is $p$-supersolvable. So $G$ is $p$-solvable since $G/M_G \lesssim S_p$, where $p \in \{p_1, p_2\}$. Let $N$ be a minimal normal subgroup of $G$. Then $N$ is either an elementary abelian $p$-group or a $p'$-group. Assume that $N \leq M$. Then $G/N$ inherits our conditions by Lemma 2.1. Hence $G/N$ is $p$-supersolvable by induction on the order of $G$. If $p \notin \pi(N)$, obviously, $G$ is $p$-supersolvable. If $p \in \pi(N)$, then $N$ is an elementary abelian $p$-group. By Lemma 2.4, we have that $|N| = p$. So $G$ is $p$-supersolvable. Assume now that $N \nleq M$. Then $G = MN$ and $|N| = p$ since $|G : M| = p$. So $G$ is $p$-supersolvable since $G/N \cong M$ is $p$-supersolvable. □

Now we give some characterizations of $CS_p$-groups, the following theorem is a local vision of [4, Theorem 3.3].

**Theorem 3.4** *The following statements are pairwise equivalent for a $p$-solvable group $G$.*

*(1) $G$ is a $CS_p$-group;*

*(2) Every $p$-subgroup of $G$ is c-supplemented in $G$;*

*(3) $G$ is $p$-supersolvable. Let $M \leq G$ and $L \leq \Phi(M)$ be a $p$-subgroup of $G$. Then $L \trianglelefteq G$ and $L \leq \Phi(G)$;*

*(4) $G$ is $p$-supersolvable, every element in $\mathrm{Syl}_p(G/\Phi(G))$ is elementary abelian and every $p$-subgroup of $\Phi(G)$ is normal in $G$;*

*(5) $G/\Phi(G)$ is a $p$-complemented group in which every $p$-subgroup of $\Phi(G)$ is normal in $G$.*

**Proof** (1)⇔(2). It is obvious.

(2)⇒(3). The proof of $p$-supersolvability is similar to Theorem 3.2. Furthermore, if $L \leq \Phi(M) \leq G$, then by Lemma 2.1, $L \trianglelefteq G$ and $L \leq \Phi(G)$.

(3)⇒(4). By hypotheses every $p$-subgroup of $\Phi(G)$ is normal in $G$. Since $\Phi(P) \leq \Phi(G)$, we have $P\Phi(G)/\Phi(G)$ is elementary abelian.

(4)⇒(5). It follows from Lemma 2.7.

(5)⇒(1). Let $H \leq G$ be a $p$-subgroup. Then $G/\Phi(G) = H\Phi(G)/\Phi(G) \cdot T/\Phi(G)$ and $H\Phi(G)/\Phi(G) \cap T/\Phi(G) = \bar{1}$ since $G/\Phi(G)$ is $p$-complemented. So $G = HT$ and $H \cap T \leq \Phi(G)$. By hypothesis, $H \cap T \leq H_G$. Denote $K = H_G T$, then $G = HK$ and $H \cap K = H_G$. Hence, $G$ is a $CS_p$-group. □

**Corollary 3.5** *Let $G$ be a group. Then the following statements are pairwise equivalent.*

*(1) $G$ is a $CS_p$-group for any $p \in \pi(G)$;*

*(2) $G$ is a group whose every $p$-subgroup is c-supplemented for every $p \in \pi(G)$;*

*(3) $G$ is a group whose every cyclic $p$-subgroup is c-supplemented in $G$ for every $p \in \pi(G)$;*

*(4) $G$ is supersolvable. Let $L \leq \Phi(G) \leq G$. Then $L \trianglelefteq G$;*

*(5) $G$ is supersolvable, every subgroup of $\Phi(G)$ is normal in $G$ and every element of $\mathrm{Syl}_p(G/\Phi(G))$ is elementary abelian for every $p \in \pi(G)$;*

*(6) $G/\Phi(G)$ is $p$-complemented in which every subgroup of $\Phi(G)$ is normal in $G$ for every $p \in \pi(G)$.*

**Proof** The proofs of (4) ⇒ (5) ⇒ (6) ⇒ (1) ⇒ (2) ⇒ (3) follow from Theorem 3.4. Applying the similar arguments as in [11, Theorem 2.8], we have (3) $\Rightarrow$ (4). $\square$

Denote by $G^{\mathcal{E}_\pi}$ the smallest normal subgroup $N$ of $G$ such that every Sylow $p$-subgroup of $G/N$ is elementary abelian, where $p \in \pi$. In particular, if $\pi = \{p\}$ for some $p \in \pi(G)$, we denote $G^{\mathcal{E}_{\{p\}}} = G^{\mathcal{E}_p}$.

**Theorem 3.6** *The following statements are pairwise equivalent for a nilpotent group $G$.*

*(1) $G$ is a $CS_p$-group;*

*(2) $G_p$ is a c-supplemented group;*

*(3) c-supplemented property is a transitive relationship in $G_p$;*

*(4) Every subgroup of $\Phi(G_p)$ is normal in $G_p$;*

*(5) Every subgroup of $G_p^{\mathcal{E}_p}$ is normal in $G_p$.*

**Proof** (1) $\Leftrightarrow$ (2). Assume that $G$ is a $CS_p$-group. Clearly, by Lemma 2.1, $G_p$ is a $c$-supplemented group. Conversely, assume that $G_p$ is a $c$-supplemented group. Let $H$ be any $p$-subgroup of $G$. Then $H \leq G_p$. By hypothesis, there exists a subgroup $H_{G_p} \leq T \leq G_p$ such that $G_p = HT$ and $H \cap T = H_{G_p}$. Denote $K = G_{p'}T$, then $G = G_pG_{p'} = HK$ and $H \cap K = H \cap TG_{p'} = H \cap T = H_{G_p}$. Note that

$$H_G = \bigcap_{g \in G} H^g = \bigcap_{x \in G \setminus G_p, y \in G_p} H^{xy} = \bigcap_{y \in G_p} H^y = H_{G_p},$$

then $H$ is $c$-supplemented in $G$. Hence, $G$ is a $CS_p$-group.

(2) $\Leftrightarrow$ (3). It follows from [7, Theorem 3.3].

(2) $\Leftrightarrow$ (4). Since $G_p/\Phi(G_p)$ is an elementary abelian $p$-group, then $G_p/\Phi(G_p)$ is a complemented group. So the assertion follows from Theorem 3.4.

(4) $\Leftrightarrow$ (5). We only need to prove that $\Phi(G_p) = G_p^{\mathcal{E}_p}$. By Theorem 3.4, $G_p^{\mathcal{E}_p} \leq \Phi(G_p)$. On the other hand, since $\Phi(G_p/G_p^{\mathcal{E}_p}) = 1$, we have $\Phi(G_p) \leq G_p^{\mathcal{E}_p}$. $\square$

Let $A, B$ be two $CS_p$-groups. Now we give some criteria of the direct product $A \times B$ to be a $CS_p$-group. First, we give a local vision of [7, Theorem 3.7] in the $p$-solvable universe.

**Theorem 3.7** *Let $A, B$ be two $p$-solvable $CS_p$-groups. Then $A \times B$ is a $CS_p$-group if and only if every $p$-subgroup of $\Phi(A) \times \Phi(B)$ is normal in $A \times B$.*

**Proof** By Theorem 3.4, $A \times B$ is a $CS_p$-group if and only if $A \times B/\Phi(A \times B)$ is a $p$-complemented group and every $p$-subgroup of $\Phi(A \times B)$ is normal in $A \times B$. Since $\Phi(A \times B) = \Phi(A) \times \Phi(B)$, then $A \times B/\Phi(A \times B) \cong A/\Phi(A) \times B/\Phi(B)$. Hence $A \times B$ is a $CS_p$-group if and only if every $p$-subgroup of $\Phi(A) \times \Phi(B)$ is normal in $A \times B$. $\square$

Applying the similar arguments as in [8, Theorem 4.5], we have

**Theorem 3.8** *Assume that $A \times B$ is a $p$-solvable $CS_p$-group, where $p \in \pi(\Phi(A)) \cap \pi(\Phi(B))$ and $p$ is odd. Denote $p^e = \mathrm{Exp}(\Phi(A)_p)$, $p^f = \mathrm{Exp}(\Phi(B)_p)$. Then one of the following statements holds:*

*(1) $\Phi(A)_p \times \Phi(B)_p \leq Z(A \times B)$;*

*(2) $\Phi(A)_p \leq Z(A)$ and $\Phi(B)_p$ centralizes $B_{p'}$ and $e < f$;*

*(3)* $\Phi(B)_p \le Z(B)$ *and* $\Phi(A)_p$ *centralizes* $A_{p'}$ *and* $f < e$.

**Theorem 3.9** *Assume that* $A \times B$ *is a* $CS_2$*-group, where* $2 \in \pi(\Phi(A)) \cap \pi(\Phi(B))$. *Then one of the following statements holds:*

*(1)* $\Phi(A)_2 \times \Phi(B)_2 \le Z(A \times B)$;

*(2)* $\Phi(A)_2 \times \Phi(B)_2 \le Z(A) \times Z(\Phi(B)_2)$;

*(3)* $\Phi(A)_2 \times \Phi(B)_2 \le Z(\Phi(A)_2) \times Z(B)$.

**Proof** By Lemma 2.1 and Theorem 3.1, $A \times B$ is solvable. Furthermore, $A, B$ are $CS_2$-groups by Lemma 2.1. By Theorem 3.4, $\Phi(A)_2 \trianglelefteq A$. So $\Phi(A)_2 A_{2'}$ is 2-nilpotent by Theorem 3.2 since $\Phi(A)_2 A_{2'}$ is a $CS_2$-group. Therefore, $A_{2'}$ centralizes $\Phi(A)_2$. Similarly, $B_{2'}$ centralizes $\Phi(B)_2$. On the other hand, since $A \times B$ is a $CS_2$-group, then $A_2, B_2$ are $c$-supplemented groups. By Lemma 2.5, we have $|A_2 : C_{A_2}(\Phi(A_2))| \le 2$ and $|B_2 : C_{B_2}(\Phi(B_2))| \le 2$. Now Theorem 3.2 implies that $\Phi(A)_2 = \Phi(A_2)$ and $\Phi(B)_2 = \Phi(B_2)$, therefore, $|A_2 : C_{A_2}(\Phi(A)_2)| \le 2$ and $|B_2 : C_{B_2}(\Phi(B)_2)| \le 2$. If $|A_2 : C_{A_2}(\Phi(A)_2)| = |B_2 : C_{B_2}(\Phi(B)_2)| = 1$, then $\Phi(A)_2 \le Z(A_2)$ and $\Phi(B)_2 \le Z(B_2)$. Hence, (1) holds. If $|A_2 : C_{A_2}(\Phi(A)_2)| = 1$ and $|B_2 : C_{B_2}(\Phi(B)_2)| = 2$, then $\Phi(A)_2 \le Z(A_2)$ and $\Phi(B)_2 \le Z(\Phi(B_2))$. Hence $\Phi(B)_2 = Z(\Phi(B)_2)$. So (2) holds. Similarly, we have (3) if $|A_2 : C_{A_2}(\Phi(A)_2)| = 2$ and $|B_2 : C_{B_2}(\Phi(B)_2)| = 1$. □

**Theorem 3.10** *Let* $A, B$ *be two* $p$*-solvable* $CS_p$*-groups,* $p \in \pi(\Phi(A)) \cap \pi(\Phi(B))$. *Denote* $p^e = \mathrm{Exp}(\Phi(A)_p), p^f = \mathrm{Exp}(\Phi(B)_p)$. *Then* $A \times B$ *is a* $CS_p$*-group if one of the following holds:*

*(1)* $\Phi(A)_p \times \Phi(B)_p \le Z(A \times B)$;

*(2) If* $p > 2$, *either* $\Phi(A)_p \le Z(A)$ *and* $\Phi(B)_p$ *centralizes* $B_{p'}$ *and* $e < f$ *or* $\Phi(B)_p \le Z(B)$ *and* $\Phi(A)_p$ *centralizes* $A_{p'}$ *and* $f < e$;

*(3) If* $p = \min(\pi(A) \cup \pi(B))$, $\min\{e, f\} \le 1$.

**Proof** Obviously, $A \times B$ is a $CS_p$-group if (1) holds by Theorem 3.7. In case (2), its proof is similar to [8, Theorem 4.6].

(3) Obviously, $p \in \pi(A) \cap \pi(B)$. So $p = \min(\pi(A))$ and $p = \min(\pi(B))$. Without loss of generality, we may assume $e \le 1$. Then $\Phi(A)_p \le Z(A_p)$ since $A$ is a $CS_p$-group. Note that $B$ is a $CS_p$-group, then $\Phi(B)_p \trianglelefteq B$ by Theorem 3.4 and $B_{p'}$ centralizes $\Phi(B)_p$ by Theorem 3.2, respectively. Similarly, $A_{p'}$ centralizes $\Phi(A)_p$. So $\Phi(A)_p \le Z(A)$. If $p$ is odd and $f \le 1$, then $\Phi(B)_p \le Z(B_p)$, this is case (1). If $p$ is odd and $f \ge 2$, this is case (2). So we may assume that $p = 2$. By Theorem 3.7, we only need to prove that $\langle ab \rangle \trianglelefteq A \times B$ for any $a \in \Phi(A)_2$ and $b \in \Phi(B)_2$. Note that $\langle a \rangle \trianglelefteq A$ and $\langle b \rangle \trianglelefteq B$ by Theorem 3.4. Assume $o(b) = 2$. It is easy to see that $b \in Z(B)$ and $\langle ab \rangle \trianglelefteq A \times B$. So we may assume that $b \in \Phi(B)_2 \backslash Z(B_2)$ and $o(b) \ge 4$. By Theorem 3.2, $b \in \Phi(B_2)$. In order to prove $\langle ab \rangle \trianglelefteq A \times B$, we can pick $x \in B_2$ and claim that $(ab)^x \in \langle ab \rangle$ since $B_{2'}$ centralizes $\Phi(B)_2 = \Phi(B_2)$. By Lemma 2.6, if $b^x = b$, obviously, $(ab)^x = a^x b^x = ab \in \langle ab \rangle$. If $b^x = b^{-1}$, then $(ab)^x = a^x b^x = ab^{-1} = (ab)^{-1}$ since $e \le 1$. So in any case, we have $\langle ab \rangle \trianglelefteq A \times B$ for any $ab \in \Phi(A)_2 \times \Phi(B)_2$. □

**Acknowledgements** We are very grateful to the referees for their time and valuable comments.

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