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On Properties of Meromorphic Solutions for Certain q-Difference Equation

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Abstract Let q be a finite nonzero complex number, let the q-difference equation

$$f(qz)f(\frac{z}{q}) = R(z,f(z)) = \frac{P(z,f(z))}{Q(z,f(z))} = \frac{\sum_{j=0}^{\tilde{p}} a_j(z)f^j(z)}{\sum_{k=0}^{\tilde{q}} b_k(z)f^k(z)} \qquad (\dagger)$$

admit a nonconstant meromorphic solution f, where $\tilde{p}$ and $\tilde{q}$ are nonnegative integers, a_j with $0 \le j \le \tilde{p}$ and b_k with $0 \le k \le \tilde{q}$ are polynomials in z with $a_{\tilde{p}} \not\equiv 0$ and $b_{\tilde{q}} \not\equiv 0$ such that $P(z,f(z))$ and $Q(z,f(z))$ are relatively prime polynomials in $f(z)$ and let $m = \tilde{p} - \tilde{q} \ge 3$. Then, ($\dagger$) has no transcendental meromorphic solution when $|q| = 1$, and the lower bound of the lower order of f is obtained when $m \ge 3$ and $|q| \ne 1$.

Keywords complex q-difference equation; transcendental meromorphic function; order of growth; existence

MR(2020) Subject Classification 30D35; 39A45

1. Introduction

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [1, 2] for the more details. Let f be meromorphic function and let $\rho(f)$ and $\mu(f)$ denote the order and the lower order of meromorphic function f, respectively.

Recently, many researchers have considered the properties of solutions of certain complex difference equations, including the existence and the growth, and a lot of interesting results have been obtained, such as [3–7] and so on. Simultaneously, the existence and the growth of meromorphic solutions of q-difference equations have been considered by different researchers, such as [8–14] and so on.

In [5], Peng and Chen studied the growth of transcendental meromorphic solutions of difference Painlevé IV equation, and the lower bound of maximum module of solutions of the equation was obtained as follows.

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Theorem 1.1 ([5, Theorem 1.1]) *Suppose that f is a transcendental meromorphic solution of the equation*

$$(f(z+1)+f(z))(f(z)+f(z-1)) = R(z,f(z)) = \frac{P(z,f(z))}{Q(z,f(z))} = \frac{\sum_{j=0}^{\tilde{p}} a_j(z)f^j(z)}{\sum_{k=0}^{\tilde{q}} b_k(z)f^k(z)}, \tag{1.1}$$

where $\tilde{p}$ and $\tilde{q}$ are nonnegative integers, and a_j with $0 \le j \le \tilde{p}$ and b_k with $0 \le k \le \tilde{q}$ are polynomials in z with $a_{\tilde{p}} \not\equiv 0$ and $b_{\tilde{q}} \not\equiv 0$ such that $P(z,f)$ and $Q(z,f)$ are relatively prime polynomials in f. Let $m = \tilde{p} - \tilde{q} \ge 3$, then one of the following cases can occur:

(1) If f is an entire function or a meromorphic function with finitely many poles in the complex plane, then there exist some two positive constants K and r_0 such that

$$\log M(r,f) \ge K(\frac{m}{2})^r \quad \text{for} \quad r \ge r_0.$$

(2) If f has infinitely many poles in the complex plane, then there exist some two positive constants $K > 0$ and $r_0 > 0$ such that

$$n(r,f) \ge K(m-1)^r \quad \text{for} \quad r \ge r_0.$$

In [14], Zhang and Korhonen considered the existence of transcendental meromorphic solutions of the q-difference equation and obtained the following theorem.

Theorem 1.2 ([14, Theorem 3.1]) *Let $q_1, q_2, \ldots, q_n$ be n distinct finite nonzero complex numbers. If the q-difference equation*

$$\sum_{j=1}^{n} f(q_j z) = R(z,f(z)) = \frac{P(z,f(z))}{Q(z,f(z))} = \frac{\sum_{j=0}^{\tilde{p}} a_j(z)f^j(z)}{\sum_{k=0}^{\tilde{q}} b_k(z)f^k(z)} \tag{1.2}$$

admits a transcendental meromorphic solution f of zero order, where n is a positive integer, $\tilde{p}$ and $\tilde{q}$ are nonnegative integers, a_j with $0 \le j \le \tilde{p}$ and b_k with $0 \le k \le \tilde{q}$ are rational functions in z with $a_{\tilde{p}} \not\equiv 0$ and $b_{\tilde{q}} \not\equiv 0$ such that $P(z,f)$ and $Q(z,f)$ are relatively, prime polynomials in f, then $\max\{\tilde{p}, \tilde{q}\} \le n$.

In [13], the existence and growth of transcendental meromorphic solutions of q-difference Painlevé IV equation were investigated by Peng-Huang, who proved the following result.

Theorem 1.3 ([13, Theorem 1.1]) *Let q be a finite nonzero complex numbers, let the q-difference equation*

$$(f(qz)+f(z))(f(\frac{z}{q})+f(z)) = R(z,f(z)) = \frac{P(z,f(z))}{Q(z,f(z))} = \frac{\sum_{j=0}^{\tilde{p}} a_j(z)f^j(z)}{\sum_{k=0}^{\tilde{q}} b_k(z)f^k(z)} \tag{1.3}$$

admit a nonconstant meromorphic solution f, where $\tilde{p}$ and $\tilde{q}$ are nonnegative integers, and a_j with $0 \le j \le \tilde{p}$ and b_k with $0 \le k \le \tilde{q}$ are polynomials in z with $a_{\tilde{p}} \not\equiv 0$ and $b_{\tilde{q}} \not\equiv 0$ such that $P(z,f)$ and $Q(z,f)$ are relatively prime polynomials in f, and let $m = \tilde{p} - \tilde{q} \ge 3$. Then, one of the following cases can occur:

(I) If $|q| = 1$, then Eq. (1.3) has no transcendental meromorphic solution.

(II) If $|q| \ne 1$ and f is a transcendental meromorphic solution of (1.3), then one of the following cases can occur:

(i) If f is an entire function or a meromorphic function with finitely many poles in the complex plane, then there exist some two positive constants K and r_0 such that

$$\log M(r, f) \geq K(\frac{m}{2})^{\frac{\log r}{|\log |q||}} \quad \text{for} \quad r \geq r_0.$$

Thus, the lower order $\mu(f)$ of f satisfies $\mu(f) \geq \frac{\log \frac{m}{2}}{|\log |q||}$.

(ii) If f has infinitely many poles in the complex plane, then there exist some two positive constants $K > 0$ and $r_0 > 0$ such that

$$n(r, f) \geq K(m-1)^{\frac{\log r}{|\log |q||}} \quad \text{for} \quad r \geq r_0.$$

Thus, the lower order $\mu(f)$ of f satisfies $\mu(f) \geq \frac{\log(m-1)}{|\log |q||}$.

(iii) Thus, the lower order $\mu(f)$ of f satisfies $\mu(f) \geq \frac{\log(m-1)}{|\log |q||}$ when $|q| \neq 1$.

Regarding Theorem 1.3, one may ask, what can be said about the conclusion of Theorem 1.3, if we replace the q-difference Eq. (1.3) with the q-difference Painlevé equation

$$f(qz)f(\frac{z}{q}) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))} = \frac{\sum_{j=0}^{\tilde{p}} a_j(z) f^j(z)}{\sum_{k=0}^{\tilde{q}} b_k(z) f^k(z)},$$

which is one of the q-difference Painlevé III, where q is a finite nonzero complex number, $\tilde{p}$ and $\tilde{q}$ are nonnegative integers, and a_j with $0 \leq j \leq \tilde{p}$ and b_k with $0 \leq k \leq \tilde{q}$ are polynomials in z with $a_{\tilde{p}} \not\equiv 0$ and $b_{\tilde{q}} \not\equiv 0$ such that $P(z, f)$ and $Q(z, f)$ are relatively, prime polynomials in f? In this direction, we will prove the following result:

Theorem 1.4 *Let q be a finite nonzero complex number, let the q-difference equation*

$$f(qz)f(\frac{z}{q}) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))} = \frac{\sum_{j=0}^{\tilde{p}} a_j(z) f^j(z)}{\sum_{k=0}^{\tilde{q}} b_k(z) f^k(z)} \tag{1.4}$$

admit a nonconstant meromorphic solution f, where $\tilde{p}$ and $\tilde{q}$ are nonnegative integers, a_j with $0 \leq j \leq \tilde{p}$ and b_k with $0 \leq k \leq \tilde{q}$ are polynomials in z with $a_{\tilde{p}} \not\equiv 0$ and $b_{\tilde{q}} \not\equiv 0$ such that $P(z, f)$ and $Q(z, f)$ are relatively, prime polynomials in f, and let $m = \tilde{p} - \tilde{q} \geq 3$. Then, one of the following subcases can occur:

(I) If $|q| = 1$, then Eq. (1.4) has no any transcendental meromorphic solution.

(II) If $|q| \neq 1$ and f is a transcendental meromorphic solution of (1.4), then one of the following cases can occur:

(i) If f is an entire function or a meromorphic function with finitely many poles in the complex plane, then there exist some two positive constants K and r_0 such that

$$\log M(r, f) \geq K(\frac{m}{2})^{\frac{\log r}{|\log |q||}} \quad \text{for} \quad r \geq r_0.$$

Thus, the lower order $\mu(f)$ of f satisfies $\mu(f) \geq \frac{\log \frac{m}{2}}{|\log |q||}$.

(ii) If f has infinitely many poles in the complex plane such that for any pole $z_0 \in \mathbb{C}$ of f, neither $\frac{z_0}{q}$ for $|q| > 1$ nor qz_0 for $0 < |q| < 1$ is a pole of f, then there exist some two positive constants K and r_0 such that

$$n(r, f) \geq K(m-1)^{\frac{\log r}{|\log |q||}} \quad \text{for} \quad r \geq r_0.$$

Thus, the lower order $\mu(f)$ of f satisfies $\mu(f) \geq \frac{\log(m-1)}{|\log|q||}$.

2. Proof of Theorem 1.4

In this section, we start the proof from the following result, which can be found in [8].

Lemma 2.1 ([8]) *Let f be a meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$M(r, f(qz)) = M(|q|r, f(z)), \quad N(r, f(qz)) = N(|q|r, f(z)) + O(1)$$

and

$$T(r, f(qz)) = T(|q|r, f(z)) + O(1)$$

hold.

Proof (I) Suppose that Eq. (1.4) has a transcendental meromorphic solution f when $|q| = 1$, and then show a contradiction. We divide into the following three cases.

Case I (1). Suppose that f, a solution of (1.4), is a transcendental entire function. Set $t_j = \deg a_j, l_k = \deg b_k$, $v = 1 + \max\{l_0, l_1, \ldots, l_{\tilde{q}}\}$, with $0 \leq j \leq \tilde{p}$ and $0 \leq k \leq \tilde{q}$. Then we get

$$M(r, \frac{P(z, f(z))}{Q(z, f(z))}) = M(r, f(qz)f(\frac{z}{q})) \leq M(|q|r, f^2(z)) = M^2(|q|r, f(z)) \tag{2.1}$$

holds for the large positive number $|z| = r$ and $|q| \geq 1$. For the large positive number $|z| = r$, we get

$$\begin{aligned}\Big|\sum_{j=0}^{\tilde{p}} a_j(z)f^j(z)\Big| &\geq |a_{\tilde{p}}(z)f^{\tilde{p}}(z)| - (|a_{\tilde{p}-1}(z)f^{\tilde{p}-1}(z)| + \cdots + |a_0(z)|)\\ &\geq \frac{1}{2}|a_{\tilde{p}}(z)f^{\tilde{p}}(z)| = \frac{1}{2}r^{t_{\tilde{p}}}|f(z)|^{\tilde{p}}(1 + o(1)),\end{aligned}$$

$$\Big|\sum_{k=0}^{\tilde{q}} b_k(z)f^k(z)\Big| \leq \sum_{k=0}^{\tilde{q}} |b_k(z)f^k(z)| \leq \sum_{k=0}^{\tilde{q}} r^v|f(z)|^{\tilde{q}}.$$

It follows from above inequalities and (1.4) that

$$\begin{aligned}|\frac{P(z, f(z))}{Q(z, f(z))}| &= |\frac{\sum_{j=0}^{\tilde{p}} a_j(z)f^j(z)}{\sum_{k=0}^{\tilde{q}} b_k(z)f^k(z)}|\\ &\geq \frac{|a_{\tilde{p}}(z)f^{\tilde{p}}(z)| - (|a_{\tilde{p}-1}(z)f^{\tilde{p}-1}(z)| + \cdots + |a_0(z)|)}{|b_{\tilde{q}}(z)f^{\tilde{q}}(z)| + \cdots + |b_1(z)f(z)| + |b_0(z)|}\\ &\geq \frac{1}{2(\tilde{q}+1)}r^{(t_{\tilde{p}}-v)}|f(z)|^{(\tilde{p}-\tilde{q})}(1 + o(1))\end{aligned}$$

and

$$M(r, \frac{P(z, f(z))}{Q(z, f(z))}) \geq \frac{r^{(t_{\tilde{p}}-v)}M(r, f)^m}{2(\tilde{q}+1)} \tag{2.2}$$

for the large positive number r. By (2.1) and (2.2) we have for the large positive number r that

$$2\log M(|q|r, f(z)) \geq m\log M(r, f) + g(r), \tag{2.3}$$

where $g(r) = \log \frac{r^{(t_{\tilde{p}}-v)}}{2(\tilde{q}+1)}$, and $|g(r)| < K \log r$ for some $K > 0$.

By (2.3) and $|q| = 1$, we get

$$2 \log M(r, f) = 2 \log M(|q|r, f) \geq m \log M(r, f) + g(r),$$

which contradicts the fact $m = (\tilde{p} - \tilde{q}) \geq 3$.

Case I (2). Suppose that f, a solution of (1.4), is transcendental meromorphic with finitely many poles. Then there exists a polynomial $P(z)$ such that $F(z) = P(z)f(z)$ is transcendental entire. Substituting $f(z) = \frac{F(z)}{P(z)}$ into (1.4) and multiplying the denominators, we will have an equation similar to (1.4). Applying the same reasons as above Case I (1) to F, for enough large r, we have

$$2 \log M(r, f) = 2 \log M(r, F) + O(1) \geq m \log M(r, F) + g(r),$$

which contradicts the fact $m = (\tilde{p} - \tilde{q}) \geq 3$.

Case I (3). Suppose that f, a solution of (1.4), is a meromorphic with infinitely many poles. Since $a_j(z)$ $(j = 0, 1, \ldots, \tilde{p})$ and $b_k(z)$ $(k = 0, 1, \ldots, \tilde{q})$ are polynomials, there is a constant $R > 0$ such that all zeros of $a_j(z)$ $(j = 0, 1, \ldots, \tilde{p})$ and $b_k(z)$ $(k = 0, 1, \ldots, \tilde{q})$ are not in $D = \{z : |z| > R\}$. Since f has infinitely many poles, there exists a pole $z_0 \in D$ of f which has changed into having multiplicity $k_0 \geq 1$. Then the right-hand side of (1.4) has a pole of multiplicity mk_0 at z_0. Thus there exists at least one index $l \in \{q, \frac{1}{q}\}$ such that lz_0 is a pole of f of multiplicity $k_1 \geq \frac{\tilde{p}k_0}{2}$. Without loss of generality, suppose that $l = q$. Since $|q| = |\frac{1}{q}| = 1$, qz_0 is a pole of f of multiplicity k_1 and $qz_0 \in D$. Substituting qz_0 for z in (1.4) gives

$$f(q^2 z_0) f(\frac{qz_0}{q}) = \frac{a_0(qz_0) + a_1(qz_0)f(qz_0) + \cdots + a_{\tilde{p}}(qz_0)f^{\tilde{p}}(qz_0)}{b_0(qz_0) + b_1(qz_0)f(qz_0) + \cdots + b_{\tilde{q}}(qz_0)f^{\tilde{q}}(qz_0)},$$

that is

$$f(q^2 z_0) f(z_0) = \frac{a_0(qz_0) + a_1(qz_0)f(qz_0) + \cdots + a_{\tilde{p}}(qz_0)f^{\tilde{p}}(qz_0)}{b_0(qz_0) + b_1(qz_0)f(qz_0) + \cdots + b_{\tilde{q}}(qz_0)f^{\tilde{q}}(qz_0)}. \tag{2.4}$$

By (2.4) and $m = (\tilde{p} - \tilde{q}) \geq 3$, we get that $q^2 z_0$ is a pole of f of multiplicity $k_2 > (m-1)k_1$. Since f has infinitely many poles, obviously $q^2 z_0 \in D$. Substituting $q^2 z_0$ for z in (1.4), we obtain

$$f(q^3 z_0) f(qz_0) = \frac{a_0(q^2 z_0) + a_1(q^2 z_0)f(q^2 z_0) + \cdots + a_{\tilde{p}}(q^2 z_0)f^{\tilde{p}}(q^2 z_0)}{b_0(q^2 z_0) + b_1(q^2 z_0)f(q^2 z_0) + \cdots + b_{\tilde{q}}(q^2 z_0)f^{\tilde{q}}(q^2 z_0)}. \tag{2.5}$$

By (2.5) and $m = (\tilde{p} - \tilde{q}) \geq 3$, we get that $q^3 z_0$ is a pole of f of multiplicity $k_3 > (m-1)k_2 = (m-1)^2 k_1$. Since f has infinitely many poles, obviously $q^3 z_0 \in D$.

Applying the same analysis above, $q^d z_0 \in D$ is a pole of f of multiplicity $k_d > (m-1)k_{d-1} > \cdots > (m-1)^{d-1}k_1 \to \infty$ as $d \to \infty$, and $|q^d z_0| = |z_0|$, which contradicts the fact that f does not have essential singularities in the finite complex plane. In conclusion, part (I) is proved.

We prove the part (II) next.

(II) (i) Suppose that f, a solution of (1.4), is transcendental entire. Due to condition $|q| \neq 1$, we respectively discuss the results with $|q| > 1$ and $|q| < 1$.

Case i (1). $|q| > 1$. Set $t_j = \deg a_j(z), l_k = \deg b_k(z)$, $v = 1 + \max\{l_0, l_1, \ldots, l_{\tilde{q}}\}$ with

$0 \le j \le \tilde{p}$ and $0 \le k \le \tilde{q}$. Then for large enough r and $|q| > 1$, we get

$$M(r, \frac{P(z, f(z))}{Q(z, f(z))}) = M(r, f(qz)f(\frac{z}{q})) \le M(|q|r, f^2(z)) = M^2(|q|r, f(z)). \tag{2.6}$$

By using the similar reasons as proving the Case I (1) in part (I), we have (2.3). Iterating (2.3), by direct calculation, we have that

$$\log M(|q|^p r, f(z)) \ge (\frac{m}{2})^p \log M(r, f) + E_p(r), \tag{2.7}$$

where

$$\begin{aligned}|E_p(r)| &= \frac{1}{2}|(\frac{m}{2})^{p-1} g(r) + (\frac{m}{2})^{p-1} g(|q|r) + \cdots + g(|q|^{p-1} r)| \\ &\le \frac{K}{2}(\frac{m}{2})^{p-1} \sum_{k=0}^{p-1} \frac{\log(|q|^k r)}{(\frac{h}{2})^k} \le \frac{K}{2}(\frac{m}{2})^{p-1} \sum_{k=0}^{\infty} \frac{\log(|q|^k r)}{(\frac{h}{2})^k},\end{aligned}$$

and $g(r) = \log \frac{r^{(t_{\tilde{p}} - v)}}{2(\tilde{q}+1)}$. Similarly, we also get the form of $g(|q|^k r)$ for $k = 0, 1, \ldots, (p-1)$.

Since $\log(|q|^k r) = \log |q|^k + \log r \le (\log r)(\log |q|^k)$ for sufficiently large r and k, we have

$$\sum_{k=0}^{\infty} \frac{\log(|q|^k r)}{(\frac{m}{2})^k} \le \sum_{k=0}^{\infty} \frac{(\log r)(\log |q|^k)}{(\frac{m}{2})^k} = \log r \log |q| \sum_{k=0}^{\infty} \frac{k}{(\frac{m}{2})^k}.$$

Note

$$I = \sum_{k=0}^{\infty} a_k = \log |q| \sum_{k=0}^{\infty} \frac{k}{(\frac{m}{2})^k}.$$

By the ratio test, the series $I = \log |q| \sum_{k=0}^{\infty} \frac{k}{(\frac{m}{2})^k}$ is convergent. Hence

$$|E_p(r)| \le K'(\frac{m}{2})^p \log r, \tag{2.8}$$

where K' is some constant.

Since f is transcendental entire, we get the inequality $\log M(r, f(z)) \ge 2K' \log r$ holds for sufficiently large r. By (2.7) and (2.8), there exists $r_0 \ge e$ such that for $r > r_0$,

$$\begin{aligned}\log M(|q|^p r, f(z)) &\ge (\frac{m}{2})^p \log M(r, f(z)) + E_p(r) \\ &\ge (\frac{m}{2})^p 2K' \log r - K'(\frac{m}{2})^p \log r = K'(\frac{m}{2})^p \log r.\end{aligned} \tag{2.9}$$

Thus, for each sufficiently large s, there exists a $p \in \mathbb{N}$ such that $s \in [|q|^p r_0, |q|^{p+1} r_0)$, i.e., $p > \frac{\log s - \log(|q| r_0)}{\log |q|}$. It follows from (2.9) that

$$\log M(s, f(z)) \ge \log M(|q|^p r_0, f(z)) \ge K'(\frac{m}{2})^p \log r_0 \ge K''(\frac{m}{2})^{\frac{\log s}{\log |q|}},$$

where $K'' = K' \log r_0 (\frac{m}{2})^{-\frac{\log(|q| r_0)}{\log |q|}}$. Therefore, the assertion holds for the case that f is entire function.

Suppose now that f, a solution of (1.4), is meromorphic with finitely many poles. Then there exists a polynomial $P(z)$ such that $F(z) = P(z)f(z)$ is entire. Substituting $f(z) = \frac{F(z)}{P(z)}$ into (1.4) and multiplying the denominators, we get the following equation on F

$$F(qz)F(\frac{z}{q}) = \frac{A_0(z) + A_1(z)F(z) + \cdots + A_{\tilde{p}}(z)F^{\tilde{p}}(z)}{B_0(z) + B_1(z)F(z) + \cdots + B_{\tilde{q}}(z)F^{\tilde{q}}(z)},$$

where $A_j(z) = P(qz)P(\frac{z}{q})a_j(z)P^{\tilde{p}-j}(z)$, $j = 0, 1, \ldots, \tilde{p}$ and $B_k(z) = b_k(z) \cdot P(z)^{\tilde{p}-k}$, $j = 0, 1, \ldots, \tilde{q}$. Applying the same reasons as above Case I (2) to F, we obtain that for enough large r,

$$\log M(r, f) = \log M(r, F) + O(1) \geq (K'' - \varepsilon)(\frac{m}{2})^{\frac{\log r}{\log |q|}} = K'''(\frac{m}{2})^{\frac{\log r}{\log |q|}},$$

where $\varepsilon > 0$ and $K''' > 0$ is some constant. The assertion holds for the case that f is meromorphic with finitely many poles.

Case i (2). $|q| < 1$. Set $q_1 = \frac{1}{q}$. Then $|q_1| > 1$. (1.4) yields

$$f(\frac{z}{q_1})f(q_1 z) = \frac{P(z, f(z))}{Q(z, f(z))}.$$

Applying the same reasoning as Case i (1), we have

$$\log M(r, f(z)) \geq K(\frac{m}{2})^{\frac{\log r}{\log |q_1|}} = K(\frac{m}{2})^{\frac{\log r}{|\log |q||}}.$$

From Cases i (1) and i (2), we obtain

$$\log M(r, f(z)) \geq K(\frac{m}{2})^{\frac{\log r}{|\log |q||}}.$$

Finally, since

$$K(\frac{m}{2})^{\frac{\log r}{|\log |q||}} \leq \log M(r, f(z)) \leq 3T(2r, f)$$

holds for all $r \geq r_0$, we obtain $\mu(f) \geq \frac{\log(\frac{m}{2})}{|\log |q||}$. Thus part (i) is proved.

(II) (ii) Suppose that f, the solution of (1.4), is meromorphic with infinitely many poles. Since $a_j(z)$ $(j = 0, 1, \ldots, \tilde{p})$ and $b_k(z)$ $(k = 0, 1, \ldots, \tilde{q})$ in (1.4) are polynomials, there are two constants $R > 0$ and $M > 0$ such that all nonzero zeros of $a_j(z)$ $(j = 0, 1, \ldots, \tilde{p})$ and $b_k(z)$ $(k = 0, 1, \ldots, \tilde{q})$ are in $D_1 = \{z : M \leq |z| \leq R\}$. Set $D = \{z : |z| > R\}$.

Since f has infinitely many poles, there exists a pole $z_0 \in D$ of f having multiplicity $k_0 \geq 1$. Then the right-hand side of (1.4) has a pole of multiplicity mk_0 at z_0. Thus there exists at least one index $l \in \{q, \frac{1}{q}\}$ such that lz_0 is a pole of f of multiplicity $k_1 \geq \frac{mk_0}{2}$. By the hypothesis, without loss of generality, suppose that $l = q$, $|q| > 1$. Then qz_0 is a pole of f of multiplicity k_1 and $qz_0 \in D$. Substituting qz_0 for z in (1.4) gives (2.4). By (2.4) and $m = (\tilde{p} - \tilde{q}) \geq 3$, we get that $q^2 z_0$ is a pole of f of multiplicity $k_2 > (m-1)k_1$. Since f has infinitely many poles, obviously $q^2 z_0 \in D$. By repeating the process, we conclude that $q^p z_0 \in D$ is a pole of f of multiplicity $k_p > (m-1)k_{p-1} > \cdots > (m-1)^{p-1}k_1$, thus, there is a sequence $\{q^p z_0 \in D$, $p = 1, 2, \ldots\}$ which are the poles of f. Since $k_p > (m-1)^{p-1}k_1 \to \infty$ as $p \to \infty$ and f does not have essential singularities in the finite complex plane, we must have $|q^p z_0| \to \infty$ as $p \to \infty$. It is clear that, for sufficiently enough p, we have

$$\begin{aligned}(m-1)^{p-1}k_1 &\leq k_1[1 + (m-1) + \cdots + (m-1)^{p-1}] \\ &\leq n(|q^p z_0|, f(z)) = n(|q|^p|z_0|, f(z)).\end{aligned} \tag{2.10}$$

Thus for each sufficiently enough r, there exists a $p \in \mathbb{N}$ such that $r \in [|q|^p r_0, |q|^{p+1} r_0)$. It follows from (2.10) that

$$n(r, f(z)) \geq (m-1)^{p-1}k_1 \geq k_1(m-1)^{-1+\frac{\log r - \log |qz_0|}{\log |q|}} = K(m-1)^{\frac{\log r}{\log |q|}},$$

where $K = k_1(m-1)^{-1-\frac{\log|qz_0|}{\log|q|}}$.

Finally, for all $r \geq r_0$,

$$K(m-1)^{\frac{\log r}{|\log|q||}} \leq n(r, f(z)) \leq \frac{1}{\log 2}N(2r, f) \leq \frac{1}{\log 2}T(2r, f),$$

which implies $\mu(f) \geq \frac{\log(m-1)}{|\log|q||}$. Thus part (ii) is proved.

In conclusion, Theorem 1.4 is proved. $\square$

We have the following question from Theorem 1.4.

Question Whether there is the same result as (II) (ii) of Theorem 1.4 if f has infinitely many poles when $|q| \neq 1$?

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