Journal of Mathematical Research with Applications
Jan., 2023, Vol. 43, No. 1, pp. 91–100
DOI:10.3770/j.issn:2095-2651.2023.01.010
Http://jmre.dlut.edu.cn

Positive Solutions of Fourth-Order Equations under Nonlocal Boundary Value Conditions of Sturm-Liouville Type

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Abstract In this paper, we study the fourth-order problem with the first and second derivatives in nonlinearity under nonlocal boundary value conditions of Sturm-Liouville type involving Stieltjes integrals. Some inequality conditions on nonlinearity are presented that guarantee the existence of positive solutions to the problem by the theory of fixed point index on a special cone. Some examples are provided to support the main results under mixed boundary conditions containing multi-point with sign-changing coefficients and integral with sign-changing kernel.

Keywords positive solution; fixed point index; cone

MR(2020) Subject Classification 34B18; 34B10; 34B15

1. Introduction

In this paper, we investigate the existence of positive solutions for fourth-order boundary value problem (BVP) with dependence on the first and second derivatives in nonlinearity subject to boundary conditions of Stieltjes integral type

$$\begin{cases} u^{(4)}(t) = h(t)f(t, u(t), u'(t), u''(t)), \quad t \in (0, 1), \\ u(0) = u(1) = \beta_1[u],\ au''(0) - bu'''(0) + \beta_2[u] = 0,\ cu''(1) + du'''(1) + \beta_3[u] = 0, \end{cases} \tag{1.1}$$

where a, b, c, d are nonnegative constants with $\delta = ad + bc + ac \neq 0$, $\beta_i[u] = \int_0^1 u(t)\mathrm{d}\mathcal{B}_i(t)$ is Stieltjes integral with $\mathcal{B}_i$ of bounded variation $(i = 1, 2, 3)$.

For the case where $\mathcal{B}_i = 0$ $(i = 1, 2, 3)$, BVP (1.1) is investigated respectively by [1] with $h = 1$ and $f(t, u)$ which relies on a nonlinear alternative of Leray-Schauder type, and by [2] with h sign-changing and $f(u, u'')$ which applies the Avery-Peterson fixed point theorem in a cone. In fact, in [1,2] they consider the more general conditions $au''(\xi_1) - bu'''(\xi_1) = 0$, $cu''(\xi_2) + du'''(\xi_2) = 0$, $0 \le \xi_1 < \xi_2 \le 1$.

For the case where $a = c = 1$, $b = d = 0$, the existence of positive solutions to BVP (1.1) is also studied by [3] with $h = 1$, $\mathcal{B}_2 = \mathcal{B}_3$ and $f(t, u, u'')$ in which the method of fixed point index

Received March 12, 2022; Accepted May 22, 2022
Supported by the National Training Program of Innovation and Entrepreneurship for Undergraduates (Grant No. S202210145149).
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is used, by [4] with $h=1$ and $f(t,u,u'')$, and by [5] in which the computations of fixed point index in [6] are applied.

Let E be a real Banach space with the zero element denoted by θ. A nonempty closed convex set $P\subset E$ is called a cone if it satisfies the following two conditions: (i) $\lambda x\in P$ for $x\in P$ and $\lambda\geq 0$; (ii) $\pm x\in P$ implies $x=\theta$. For the properties of cones and fixed point index we refer to [7–10]. Denote $\mathbb{R}_+=[0,\infty)$ and $\mathbb{R}_-=(-\infty,0]$. A functional $\alpha:P\to\mathbb{R}_+$ is called to be sublinear if $\alpha(tx)\leq t\alpha(x)$ for all $x\in P,\ t\in[0,1]$.

Lemma 1.1 ([6]) *Let P be a cone in E and Ω be a bounded open subset relative to P with $\theta\in\Omega$, $S:\overline{\Omega}\to P$ be a completely continuous operator. Suppose that $\alpha:P\to\mathbb{R}_+$ is a continuous and sublinear functional with $\alpha(\theta)=0$, $\alpha(x)\neq 0$ for $x\neq\theta$. If $Sx\neq x$ and $\alpha(Sx)\leq\alpha(x)$ for all $x\in\partial\Omega$, then the fixed point index $i(S,\Omega,P)=1$.*

Lemma 1.2 ([6]) *Let P be a cone in E and Ω be a bounded open subset relative to P with $\theta\in\Omega$, $S:\overline{\Omega}\to P$ be a completely continuous operator. Suppose that $\alpha:P\to\mathbb{R}_+$ is a continuous and sublinear functional with $\alpha(\theta)=0$, $\alpha(x)\neq 0$ for $x\neq\theta$, and $\inf_{x\in\partial\Omega}\alpha(x)>0$. If $Sx\neq x$, $\alpha(Sx)\geq\alpha(x)$ for all $x\in\partial\Omega$, then the fixed point index $i(S,\Omega,P)=0$.*

2. Preliminaries

Take $\gamma_1(t)=1$, $\gamma_2(t)=\frac{1}{6\delta}t(1-t)(2c+3d-ct)$ and $\gamma_3(t)=\frac{1}{6\delta}t(1-t)(a+3b+at)$, they are the solutions to $u^{(4)}(t)=0$, respectively, subject to following boundary conditions:

$$u(0)=u(1)=1,\ au''(0)-bu'''(0)=0,\ cu''(1)+du'''(1)=0;$$

$$u(0)=u(1)=0,\ au''(0)-bu'''(0)+1=0,\ cu''(1)+du'''(1)=0;$$

$$u(0)=u(1)=0,\ au''(0)-bu'''(0)=0,\ cu''(1)+du'''(1)+1=0.$$

Let

$$G_0(t,s)=\int_0^1 G_1(t,\xi)G_2(\xi,s)\mathrm{d}\xi, \tag{2.1}$$

where

$$G_1(t,\xi)=\begin{cases}\xi(1-t), & 0\leq\xi\leq t\leq 1,\\ t(1-\xi), & 0\leq t<\xi\leq 1,\end{cases} \tag{2.2}$$

$$G_2(\xi,s)=\frac{1}{\delta}\begin{cases}(as+b)(c(1-\xi)+d), & 0\leq s\leq\xi\leq 1,\\ (a\xi+b)(c(1-s)+d), & 0\leq\xi<s\leq 1.\end{cases} \tag{2.3}$$

$G_0(t,s)$ is the Green's function associated with

$$\begin{cases}u^{(4)}(t)=0, \quad t\in[0,1],\\ u(0)=u(1)=0,\ au''(0)-bu'''(0)=0,\ cu''(1)+du'''(1)=0.\end{cases}$$

We assume that

(C_1) $f:[0,1]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_-\to\mathbb{R}_+$ is continuous and $h\in L^1(0,1)$ with $h(t)\geq 0$ and $\int_0^1 h(t)\mathrm{d}t>0$.

(C_2) For each $i \in \{1,2,3\}$, $\mathcal{B}_i$ is of bounded variation and

$$\mathcal{K}_i(s) := \int_0^1 G_0(t,s)\mathrm{d}\mathcal{B}_i(t) \geq 0, \quad \forall s \in [0,1].$$

(C_3) $\beta_i[\gamma_j] \geq 0$ $(i,j = 1,2,3)$ and for the 3×3 matrix

$$[B] = \begin{pmatrix} \beta_1[\gamma_1] & \beta_1[\gamma_2] & \beta_1[\gamma_3] \\ \beta_2[\gamma_1] & \beta_2[\gamma_2] & \beta_2[\gamma_3] \\ \beta_3[\gamma_1] & \beta_3[\gamma_2] & \beta_3[\gamma_3] \end{pmatrix},$$

its spectral radius $r([B]) < 1$.

Let $E = C^2[0,1]$ be the Banach space consisting of all twice continuously differentiable functions on $[0,1]$ with the norm

$$\|u\|_{C^2} = \max\{\|u\|_C, \|u'\|_C, \|u''\|_C\},$$

where $\|u\|_C = \max\{|u(t)| : t \in [0,1]\}$ for $u \in C[0,1]$. Define an operator in $C^2[0,1]$ as

$$(Tu)(t) = \sum_{i=1}^{3} \beta_i[u]\gamma_i(t) + \int_0^1 G_0(t,s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s,$$

where $\beta_i[u] = \int_0^1 u(t)\mathrm{d}\mathcal{B}_i(t)$ $(i = 1,2,3)$. We set

$$(Bu)(t) =: \sum_{i=1}^{3} \beta_i[u]\gamma_i(t), \ (Fu)(t) =: \int_0^1 G_0(t,s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s,$$

so $(Tu)(t) = (Bu)(t) + (Fu)(t)$. Writing $\langle \beta, \gamma \rangle = \sum_{i=1}^{3} \beta_i\gamma_i$ for the inner product in $\mathbb{R}^3$, we define the operator S in $C^2[0,1]$ as

$$(Su)(t) = \langle (I - [B])^{-1}\beta[Fu], \gamma(t)\rangle + (Fu)(t),$$

where $\beta[Fu] = (\beta_1[Fu], \beta_2[Fu], \beta_3[Fu])^T$ is the transposed vector. Similar to [11] we have the following lemmas.

Lemma 2.1 *Suppose that (C_1) holds. Then BVP (1.1) has a solution if and only if there exists a fixed point of T in $C^2[0,1]$.*

Lemma 2.2 *Suppose that (C_1)–(C_3) hold. Then S can be written as*

$$\begin{aligned}(Su)(t) &= ((I - B)^{-1}Fu)(t) \\ &= \int_0^1 (\langle (I - [B])^{-1}\mathcal{K}(s), \gamma(t)\rangle + G_0(t,s))h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s \\ &=: \int_0^1 G_S(t,s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s, \end{aligned} \tag{2.4}$$

where $\mathcal{K}(s) = (\mathcal{K}_1(s), \mathcal{K}_2(s), \mathcal{K}_3(s))^T$, i.e.,

$$G_S(t,s) = \langle (I - [B])^{-1}\mathcal{K}(s), \gamma(t)\rangle + G_0(t,s) = \sum_{i=1}^{3} \kappa_i(s)\gamma_i(t) + G_0(t,s) \tag{2.5}$$

and $\kappa_i(s)$ is the ith component of $(I - [B])^{-1}\mathcal{K}(s)$.

Lemma 2.3 *Let* $\Gamma = \max\{\max_{t\in[0,1]}\gamma_1(t), \max_{t\in[0,1]}\gamma_2(t), \max_{t\in[0,1]}\gamma_3(t)\}$. *If* (C_2) *and* (C_3) *hold, then* $\kappa_i(s) \geq 0$ $(i = 1, 2, 3)$,

$$G_S(0,s) = G_S(1,s) = \kappa_1(s), \tag{2.6}$$

and for $t, s \in [0,1]$,

$$c_0(t)\Phi_0(s) \leq G_S(t,s) \leq \Phi_0(s), \tag{2.7}$$

where

$$\Phi_0(s) = \Gamma\sum_{i=1}^{3}\kappa_i(s) + \int_0^1 G_1(\xi,\xi)G_2(\xi,s)\mathrm{d}\xi, \tag{2.8}$$

$$c_0(t) = \min\{\frac{1}{\Gamma}\gamma_1(t), \frac{1}{\Gamma}\gamma_2(t), \frac{1}{\Gamma}\gamma_3(t), t, 1-t\}, \tag{2.9}$$

$$c_1(t)\Phi_1(s) \leq -\frac{\partial^2 G_S(t,s)}{\partial t^2} \leq \Phi_1(s), \tag{2.10}$$

where

$$\Phi_1(s) = \frac{1}{\delta}\max\{a+b, c+d\}(\kappa_2(s)+\kappa_3(s)) + \frac{1}{\delta}(as+b)(c(1-s)+d), \tag{2.11}$$

$$c_1(t) = \frac{\min\{c(1-t)+d, at+b\}}{\max\{a+b, c+d\}}. \tag{2.12}$$

Proof By [11], we have $\kappa_i(s) \geq 0$ $(i = 1, 2, 3)$, and (2.6) holds from (2.5). It follows from (2.2) that $G_1(t,\xi) \leq G_1(\xi,\xi)$ for $t, \xi \in [0,1]$, then from (2.1) we have $G_S(t,s) \leq \Phi_0(s)$. Since $G_1(t,\xi) \geq \min\{t, 1-t\}G_1(\xi,\xi)$, by (2.1) we have $G_0(t,s) \geq \min\{t, 1-t\}\int_0^1 G_1(\xi,\xi)G_2(\xi,s)\mathrm{d}\xi$, and thus

$$G_S(t,s) = \Gamma\sum_{i=1}^{3}\kappa_i(s)(\frac{1}{\Gamma}\gamma_i(t)) + G_0(t,s) \geq c_0(t)\Phi_0(s).$$

Moreover,

$$\begin{aligned}-\frac{\partial^2 G_S(t,s)}{\partial t^2} &= -\sum_{i=2}^{3}\kappa_i(s)\gamma_i''(t) - \frac{\partial^2 G_0(t,s)}{\partial t^2}\\ &= \frac{1}{\delta}((c(1-t)+d)\kappa_2(s) + (at+b)\kappa_3(s)) + G_2(t,s) \leq \Phi_1(s).\end{aligned}$$

As for $-\frac{\partial^2 G_S(t,s)}{\partial t^2} \geq c_1(t)\Phi_1(s)$, it can be checked easily. □

Define a cone P in E as follows:

$$\begin{aligned}P =\Big\{&u \in E: u(0) = u(1),\quad u(t) \geq c_0(t)\|u\|_C,\\ &-u''(t) \geq c_1(t)\|u''\|_C,\ \forall\, t \in [0,1];\ \beta_i[u] \geq 0\ (i = 1,2,3)\Big\}.\end{aligned} \tag{2.13}$$

By the method due to Webb and Infante [11] we have the following lemma.

Lemma 2.4 *Suppose that* (C_1)–(C_3) *hold. Then* $S: P \to P$ *is a completely continuous operator,* S *and* T *have the same fixed points in* P. *As a result, BVP (1.1) has a positive solution if and only if* S *has a fixed point in* P.

3. Main results

Take $\tau \in (0, 1/2)$ such that $\int_\tau^{1-\tau} h(t)\mathrm{d}t > 0$ and denote

$$h_0 = \max\Big\{\int_0^1 \Phi_0(t)h(t)\mathrm{d}t,\ \int_0^1 \Phi_1(t)h(t)\mathrm{d}t\Big\},$$

$$h_\tau = \min\Big\{\int_\tau^{1-\tau} \Phi_0(t)h(t)\mathrm{d}t,\ \int_\tau^{1-\tau} \Phi_1(t)h(t)\mathrm{d}t\Big\}.$$

Lemma 3.1 *If (C_2) and (C_3) hold, define a functional $\alpha : P \to \mathbb{R}_+$ as*

$$\alpha(u) = \max\{\max_{\tau\le t\le 1-\tau} |u(t)|,\ \max_{\tau\le t\le 1-\tau} |u''(t)|\},$$

then α is a continuous and sublinear functional with $\alpha(\theta) = 0$, $\alpha(u) \neq 0$ for $u \neq \theta$.

Denote several constants by

$$\overline{c}_0 = \max_{\tau\le t\le 1-\tau} c_0(t),\ \overline{c}_1 = \max_{\tau\le t\le 1-\tau} c_1(t),\ \underline{c}_0 = \min_{\tau\le t\le 1-\tau} c_0(t),\ \underline{c}_1 = \min_{\tau\le t\le 1-\tau} c_1(t)$$

and $\overline{c} = \min\{\overline{c}_0,\ \overline{c}_1\}$.

Theorem 3.2 *Suppose that (C_1)–(C_3) are satisfied. If there exist constants a_1 and b_1 with $0 < b_1 < a_1$ satisfying $b_1 < a_1 \min\{\overline{c}_0\underline{c}_0,\ \overline{c}_1\underline{c}_1\}$, such that*

$$f(t, x_1, x_2, x_3) \le \frac{b_1}{h_0} \tag{3.1}$$

for $(t, x_1, x_2, x_3) \in D_1 = [0, 1] \times [0, b_1/\overline{c}_0] \times [-b_1/\overline{c}_1, b_1/\overline{c}_1] \times [-b_1/\overline{c}_1, 0]$, and

$$f(t, x_1, x_2, x_3) \ge \frac{a_1}{\overline{c}h_\tau} \tag{3.2}$$

for $(t, x_1, x_2, x_3) \in D_2 \cup D_3$, where

$$D_2 = [0, 1] \times [\underline{c}_0 a_1, a_1] \times [-a_1/\overline{c}_1, a_1/\overline{c}_1] \times [-a_1, 0],$$

$$D_3 = [0, 1] \times [0, a_1] \times [-a_1/\overline{c}_1, a_1/\overline{c}_1] \times [-a_1, -\underline{c}_1 a_1],$$

then BVP (1.1) has at least one positive solution.

Proof Obviously, $D_1 \cap (D_2 \cup D_3) = \emptyset$ since $b_1 < a_1 \min\{\overline{c}_0\underline{c}_0,\ \overline{c}_1\underline{c}_1\}$. Let

$$\Omega_1 = \{u \in P :\ \alpha(u) < b_1\},\quad \Omega_2 = \{u \in P :\ \alpha(u) < a_1\}.$$

It is clear that $\overline{\Omega}_1 \subset \Omega_2$, both Ω_1 and Ω_2 are open sets in P with $\theta \in \Omega_1$.

If $u \in \Omega_2$, by Lemma 2.3, we have

$$a_1 > \max_{\tau\le t\le 1-\tau} |u(t)| \ge (\max_{\tau\le t\le 1-\tau} c_0(t))\|u\|_C = \overline{c}_0\|u\|_C,$$

$$a_1 \ge \max_{\tau\le t\le 1-\tau} |u''(t)| \ge (\max_{\tau\le t\le 1-\tau} c_1(t))\|u''\|_C = \overline{c}_1\|u''\|_C.$$

Since $u(0) = u(1)$, there exists $\eta \in (0, 1)$ such that $u'(\eta) = 0$ and thus

$$\|u'\|_C = \max_{0\le t\le 1} |u'(t)| \le \max_{0\le t\le 1} \Big|\int_\eta^t |u''(s)|\mathrm{d}s\Big| \le \|u''\|_C \le \frac{a_1}{\overline{c}_1}.$$

Therefore, Ω_2 is bounded. Similarly, $\|u\|_C \le b_1/\overline{c}_0$, $\|u'\|_C \le b_1/\overline{c}_1$, $\|u''\|_C \le b_1/\overline{c}_1$ for $u \in \Omega_1$.

If $u \in \partial\Omega_1$, then $\alpha(u) = b_1$. From Lemma 2.3 and (3.1) it follows that

$$\max_{\tau\le t\le 1-\tau} |(Su)(t)| \le \frac{b_1}{h_0}\int_0^1 \Phi_0(s)h(s)\mathrm{d}s \le b_1,$$

$$\max_{\tau\le t\le 1-\tau} |(Su)''(t)| \le \frac{b_1}{h_0}\int_0^1 \Phi_1(s)h(s)\mathrm{d}s \le b_1,$$

and hence $\alpha(Su) \le \alpha(u)$. So by Lemma 1.1 the fixed point index

$$i(S, \Omega_1, P) = 1 \tag{3.3}$$

if $Su \ne u$ for $u \in \partial\Omega_1$.

If $u \in \partial\Omega_2$, then $\alpha(u) = a_1$ and by Lemma 2.3 for $t \in [\tau, 1-\tau]$,

$$a_1 \ge u(t) \ge c_0(t)\|u\|_C \ge (\min_{\tau\le t\le 1-\tau} c_0(t))\|u\|_C \ge \underline{c}_0 \max_{\tau\le t\le 1-\tau} |u(t)|, \tag{3.4}$$

$$a_1 \ge -u''(t) \ge c_1(t)\|u''\|_C \ge (\min_{\tau\le t\le 1-\tau} c_1(t))\|u''\|_C \ge \underline{c}_1 \max_{\tau\le t\le 1-\tau} |u''(t)|. \tag{3.5}$$

When $\alpha(u) = a_1 = \max_{\tau\le t\le 1-\tau} |u(t)|$, it follows from Lemma 2.3, together with (3.2) and (3.4), that

$$\begin{aligned}
\max_{\tau\le t\le 1-\tau} |(Su)(t)| &= \max_{\tau\le t\le 1-\tau} \Big|\int_0^1 G_S(t,s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s\Big| \\
&\ge (\max_{\tau\le t\le 1-\tau} c_0(t)) \int_\tau^{1-\tau} \Phi_0(s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s \\
&\ge \overline{c}_0 \times \frac{a_1}{\overline{c}h_\tau}\int_\tau^{1-\tau} \Phi_0(s)h(s)\mathrm{d}s \ge \overline{c} \times \frac{a_1}{\overline{c}h_\tau}\int_\tau^{1-\tau} \Phi_0(s)h(s)\mathrm{d}s \ge a_1, \\
\max_{\tau\le t\le 1-\tau} |(Su)''(t)| &= \max_{\tau\le t\le 1-\tau} \Big|\int_0^1 \frac{\partial^2 G_S(t,s)}{\partial t^2}h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s\Big| \\
&\ge (\max_{\tau\le t\le 1-\tau} c_1(t)) \int_\tau^{1-\tau} \Phi_1(s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s \\
&\ge \overline{c}_1 \times \frac{a_1}{\overline{c}h_\tau}\int_\tau^{1-\tau} \Phi_1(s)h(s)\mathrm{d}s \ge \overline{c} \times \frac{a_1}{\overline{c}h_\tau}\int_\tau^{1-\tau} \Phi_1(s)h(s)\mathrm{d}s \ge a_1,
\end{aligned}$$

and hence $\alpha(Su) \ge \alpha(u)$; when $\alpha(u) = a_1 = \max_{\tau\le t\le 1-\tau} |u''(t)|$, it similarly follows from Lemma 2.3, together with (3.2) and (3.5), that $\alpha(Su) \ge \alpha(u)$. So by Lemma 1.2 and since $\inf_{x\in\partial\Omega_2} \alpha(u) = a_1 > 0$, the fixed point index

$$i(S, \Omega_2, P) = 0 \tag{3.6}$$

if $Su \ne u$ for $u \in \partial\Omega_2$.

From (3.3) and (3.6) it follows that S has at least one fixed point, and hence BVP (1.1) has at least one positive solution by Lemma 2.4. □

Theorem 3.3 *Suppose that* (C_1)–(C_3) *are satisfied. If there exist constants* a_2 *and* b_2 *with* $0 < b_2 < a_2$ *satisfying* $b_2 < \overline{c}h_\tau h_0^{-1} a_2$, *such that*

$$f(t, x_1, x_2, x_3) \ge \frac{b_2}{\overline{c}h_\tau} \tag{3.7}$$

for $(t, x_1, x_2, x_3) \in D_4 \cup D_5$, *where*

$$D_4 = [0,1] \times [\underline{c}_0 b_2, b_2] \times [-b_2/\overline{c}_1, b_2/\overline{c}_1] \times [-b_2, 0],$$

$$D_5 = [0,1] \times [0, b_2] \times [-b_2/\overline{c}_1, b_2/\overline{c}_1] \times [-b_2, -\underline{c}_1 b_2],$$

$$f(t, x_1, x_2, x_3) \leq \frac{a_2}{h_0} \tag{3.8}$$

for $(t, x_1, x_2, x_3) \in D_6 = [0,1] \times [0, a_2/\overline{c}_0] \times [-a_2/\overline{c}_1, a_2/\overline{c}_1] \times [-a_2/\overline{c}_1, 0]$, *then BVP* (1.1) *has at least one positive solution.*

Proof Obviously, $D_4 \cup D_5 \subset D_6$ due to $\overline{c}_0 \leq 1$ and $\overline{c}_1 \leq 1$; however (3.7) and (3.8) are well-posed since $b_2 < \overline{c} h_\tau h_0^{-1} a_2$. Let $\Omega_1 = \{u \in P : \alpha(u) < b_2\}$, $\Omega_2 = \{u \in K : \alpha(u) < a_2\}$, we know from the proof of Theorem 3.2 that Ω_1 and Ω_2 are bounded open sets in P with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Moreover, $\|u\|_C \leq b_2/\overline{c}_0, \|u'\|_C \leq b_2/\overline{c}_1, \|u''\|_C \leq b_2/\overline{c}_1$ for $u \in \Omega_1$; $\|u\|_C \leq a_2/\overline{c}_0, \|u'\|_C \leq a_2/\overline{c}_1, \|u''\|_C \leq a_2/\overline{c}_1$ for $u \in \Omega_2$.

If $u \in \partial\Omega_1$, then $\alpha(u) = b_2$ and by Lemma 2.3 for $t \in [\tau, 1-\tau]$,

$$b_2 \geq u(t) \geq c_0(t)\|u\|_C \geq (\min_{\tau \leq t \leq 1-\tau} c_0(t))\|u\|_C \geq \underline{c}_0\|u\|_C \geq \underline{c}_0 \max_{\tau \leq t \leq 1-\tau} |u(t)|, \tag{3.9}$$

$$b_2 \geq -u''(t) \geq c_1(t)\|u''\|_C \geq (\min_{\tau \leq t \leq 1-\tau} c_1(t))\|u\|_C \geq \underline{c}_1 \max_{\tau \leq t \leq 1-\tau} |u''(t)|. \tag{3.10}$$

When $\alpha(u) = b_2 = \max_{\tau \leq t \leq 1-\tau} |u(t)|$, it follows from Lemma 2.3, as well as (3.7) and (3.9), that

$$\begin{aligned}
\max_{\tau \leq t \leq 1-\tau} |(Su)(t)| &= \max_{\tau \leq t \leq 1-\tau} \Big| \int_0^1 G_S(t,s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s \Big| \\
&\geq (\max_{\tau \leq t \leq 1-\tau} c_0(t)) \int_\tau^{1-\tau} \Phi_0(s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s \\
&\geq \overline{c}_0 \times \frac{b_2}{\overline{c}h_\tau} \int_\tau^{1-\tau} \Phi_0(s)h(s)\mathrm{d}s \geq \overline{c} \times \frac{b_2}{\overline{c}h_\tau} \int_\tau^{1-\tau} \Phi_0(s)h(s)\mathrm{d}s \geq b_2, \\
\max_{\tau \leq t \leq 1-\tau} |(Su)''(t)| &= \max_{\tau \leq t \leq 1-\tau} \Big| \int_0^1 \frac{\partial^2 G_S(t,s)}{\partial t^2} h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s \Big| \\
&\geq (\max_{\tau \leq t \leq 1-\tau} c_1(t)) \int_\tau^{1-\tau} \Phi_1(s)h(s)f(s,u(s),u'(s),u''(s))\mathrm{d}s \\
&\geq \overline{c}_1 \times \frac{b_2}{\overline{c}h_\tau} \int_\tau^{1-\tau} \Phi_1(s)h(s)\mathrm{d}s \geq \overline{c} \times \frac{b_2}{\overline{c}h_\tau} \int_\tau^{1-\tau} \Phi_1(s)h(s)\mathrm{d}s \geq b_2,
\end{aligned}$$

and hence $\alpha(Su) \geq \alpha(u)$; when $\alpha(u) = b_2 = \max_{\tau \leq t \leq 1-\tau} |u''(t)|$, it similarly follows from Lemma 2.3, together with (3.7) and (3.10), that $\alpha(Su) \geq \alpha(u)$. So by Lemma 1.2 and since $\inf_{x \in \partial\Omega_1} \alpha(u) = b_2 > 0$, the fixed point index

$$i(S, \Omega_1, P) = 0 \tag{3.11}$$

if $Su \neq u$ for $u \in \partial\Omega_1$.

If $u \in \partial\Omega_2$, then $\alpha(u) = a_2$ and from Lemma 2.3 and (3.8) it follows that

$$\max_{\tau \leq t \leq 1-\tau} |(Su)(t)| \leq \frac{a_2}{h_0} \int_0^1 \Phi_0(s)h(s)\mathrm{d}s \leq a_2,$$

$$\max_{\tau\le t\le 1-\tau}|(Su)''(t)|\le \frac{a_2}{h_0}\int_0^1\Phi_1(s)h(s)\mathrm{d}s\le a_2,$$

and hence $\alpha(Su)\le\alpha(u)$. So by Lemma 1.1 the fixed point index

$$i(S,\Omega_2,P)=1 \tag{3.12}$$

if $Su\ne u$ for $u\in\partial\Omega_2$.

From (3.11) and (3.12) it follows that S has at least one fixed point, and hence BVP (1.1) has at least one positive solution by Lemma 2.4. □

Remark 3.4 If $a=c=1,\ b=d=0$, BVP (1.1) is

$$\begin{cases} u^{(4)}(t)=h(t)f(t,u(t),u'(t),u''(t)), & t\in(0,1),\\ u(0)=u(1)=\beta_1[u],\ u''(0)+\beta_2[u]=0,\ u''(1)+\beta_3[u]=0, \end{cases} \tag{3.13}$$

then $\delta=1$, $\gamma_1(t)=1,\ \gamma_2(t)=\frac16t(1-t)(2-t),\ \gamma_3(t)=\frac16t(1-t)(1+t)$ and $\Gamma=1$,

$$c_0(t)=\min\{\frac16t(1-t)(2-t),\frac16t(1-t)(1+t)\},\ c_1(t)=\min\{t,1-t\}.$$

Then $\overline{c}_0=\frac1{16},\ \overline{c}_1=\frac12,\ \underline{c}_0=\frac16\tau(1-\tau)(1+\tau),\ \underline{c}_1=\tau,\ \overline{c}=\frac1{16}$ for $\tau\in(0,\frac12)$. For this case we can see [5] and [4, Remark 3.2]. □

Now as the examples we consider fourth-order problems under mixed boundary conditions involving multi-point with sign-changing coefficients and integral with sign-changing kernel

$$\begin{cases} u^{(4)}(t)=\frac{1}{\sqrt{t(1-t)}}f(t,u(t),u'(t),u''(t)), & t\in(0,1),\\ u(0)=u(1)=\frac14u(\frac14)-\frac1{12}u(\frac34),\\ u''(0)-u'''(0)-\int_0^1u(t)\cos(2\pi t)\mathrm{d}t=0,\ u''(1)+u'''(1)+\frac12u(\frac12)-\frac14u(\frac34)=0, \end{cases} \tag{3.14}$$

that is, $\beta_1[u]=\frac14u(\frac14)-\frac1{12}u(\frac34),\ \beta_2[u]=-\int_0^1u(t)\cos(2\pi t)\mathrm{d}t,\ \beta_3[u]=\frac12u(\frac12)-\frac14u(\frac34)$, and $a=b=c=d=1$, $\delta=3$, $\gamma_1(t)=1,\ \gamma_2(t)=\frac1{18}t(1-t)(5-t),\ \gamma_3(t)=\frac1{18}t(1-t)(4+t)$, $\Gamma=1$,

$$G_0(t,s)=\frac1{18}\begin{cases}(1-t)(t(5-t)(1+s)-3s^3), & 0\le s\le t\le1,\\ t(5-9s^2+3s^3-6t-2t^2+s(5+3t+t^2)), & 0\le t<s\le1.\end{cases}$$

Hence for $s\in[0,1]$,

$$\begin{aligned}0\le\mathcal{K}_1(s)&=\frac14G_0(\frac14,s)-\frac1{12}G_0(\frac34,s)\\&=\begin{cases}-\frac1{36}s^3+\frac5{576}s+\frac5{576}, & 0\le s\le\frac14,\\ \frac1{72}s^3-\frac1{32}s^2+\frac{19}{1152}s+\frac{37}{4608}, & \frac14<s\le\frac34,\\ -\frac1{144}s+\frac1{72}, & \frac34<s\le1,\end{cases}\end{aligned}$$

$$\mathcal{K}_2(s)=-\int_0^1G_0(t,s)\cos(2\pi t)\mathrm{d}t=\frac{1+\pi^2(2+2s-2s^2)-\cos(2\pi s)}{16\pi^4}\ge0,$$

$$0 \le \mathcal{K}_3(s) = \frac{1}{2}G_0(\frac{1}{2}, s) - \frac{1}{4}G_0(\frac{3}{4}, s)$$
$$= \begin{cases} -\frac{1}{32}s^3 + \frac{31}{1536}s + \frac{31}{1536}, & 0 \le s \le \frac{1}{2}, \\ \frac{5}{96}s^3 - \frac{1}{8}s^2 + \frac{127}{1536}s + \frac{5}{512}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{1}{96}s^3 - \frac{1}{32}s^2 + \frac{19}{1536}s + \frac{7}{256}, & \frac{3}{4} < s \le 1 \end{cases}$$

and the 3×3 matrix

$$[B] = \begin{pmatrix} \beta_1[\gamma_1] & \beta_1[\gamma_2] & \beta_1[\gamma_3] \\ \beta_2[\gamma_1] & \beta_2[\gamma_2] & \beta_2[\gamma_3] \\ \beta_3[\gamma_1] & \beta_3[\gamma_2] & \beta_3[\gamma_3] \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{5}{576} & \frac{1}{144} \\ 0 & \frac{1}{8\pi^2} & \frac{1}{8\pi^2} \\ \frac{1}{4} & \frac{31}{1536} & \frac{29}{1536} \end{pmatrix}.$$

Its spectral radius is $r([B]) \approx 0.1787 < 1$. This means that (C_2) and (C_3) are satisfied. Moreover,

$$\kappa_1(s) \approx 1.2026\mathcal{K}_1(s) + 0.0107\mathcal{K}_2(s) + 0.0087\mathcal{K}_3(s),$$
$$\kappa_2(s) \approx 0.0039\mathcal{K}_1(s) + 1.0131\mathcal{K}_2(s) + 0.0131\mathcal{K}_3(s),$$
$$\kappa_3(s) \approx 0.3065\mathcal{K}_1(s) + 0.0236\mathcal{K}_2(s) + 1.0217\mathcal{K}_3(s),$$

$$\Phi_0(s) = \sum_{i=1}^{3} \kappa_i(s) + \frac{1}{12}(1 + s - 2s^3 + s^4), \ \Phi_1(s) = \frac{2}{3}\sum_{i=2}^{3} \kappa_i(s) + \frac{1}{3}(1+s)(2-s),$$

$$c_0(t) = \min\{\frac{1}{18}t(1-t)(5-t), \frac{1}{18}t(1-t)(4+t)\}, \ c_1(t) = \frac{1}{2}\min\{2-t, 1+t\}.$$

Take $\tau = 1/4$ and then $\overline{c}_0 = \frac{1}{16}$, $\overline{c}_1 = \frac{3}{4}$, $\underline{c}_0 = \frac{17}{384}$, $\underline{c}_1 = \frac{5}{8}$, $\overline{c} = \frac{1}{16}$,

$$h_0 = \max\left\{\int_0^1 \Phi_0(t)h(t)\mathrm{d}t, \ \int_0^1 \Phi_1(t)h(t)\mathrm{d}t\right\} \approx \max\{0.4658, 2.3121\} = 2.3121,$$

$$h_\tau = \min\left\{\int_{1/4}^{3/4} \Phi_0(t)h(t)\mathrm{d}t, \ \int_{1/4}^{3/4} \Phi_1(t)h(t)\mathrm{d}t\right\} \approx \min\{0.1737, 0.8105\} = 0.1737.$$

Example 3.5 If $f(t, x_1, x_2, x_3) = 123x_1^2 + tx_2^2 + x_3^2$, then BVP (3.14) has a positive solution.

Proof For $a_1 = 384$, $b_1 = 3 \times 10^{-6}$, it is clear that $b_1 < a_1 \min\{\overline{c}_0\underline{c}_0, \ \overline{c}_1\underline{c}_1\} = \frac{17}{16} = 1.0625$. Moreover,

$$f(t, x_1, x_2, x_3) \le 123 \times (48 \times 10^{-6})^2 + 2 \times (4 \times 10^{-6})^2 \approx 0.2834 \times 10^{-6} < \frac{b_1}{h_0} \approx 1.2975 \times 10^{-6}$$

for $(t, x_1, x_2, x_3) \in D_1 = [0, 1] \times [0, 48 \times 10^{-6}] \times [-4 \times 10^{-6}, 4 \times 10^{-6}] \times [-4 \times 10^{-6}, 0]$;

$$f(t, x_1, x_2, x_3) \ge 123 \times 17^2 = 35547 > \frac{a_1}{\overline{c}h_\tau} \approx 35371.3$$

for $(t, x_1, x_2, x_3) \in D_2 = [0, 1] \times [17, 384] \times [-512, 512] \times [-384, 0]$;

$$f(t, x_1, x_2, x_3) \ge (-240)^2 = 57600 > \frac{a_1}{\overline{c}h_\tau} \approx 35371.3$$

for $(t,x_1,x_2,x_3)\in D_3=[0,1]\times[0,384]\times[-512,512]\times[-384,-240]$. Then BVP (3.14) has a positive solution by Theorem 3.2. $\square$

Example 3.6 If $f(t,x_1,x_2,x_3)=123000(1-\frac{1}{1+x_1^2+tx_2^2+x_3^2})$, then BVP (3.14) has a positive solution.

Proof For $a_2=284400,\ b_2=0.384$, it is clear that $b_2<\overline{c}h_\tau h_0^{-1}a_2\approx 1335.37$. Moreover,

$$f(t,x_1,x_2,x_3)\geq 123000(1-\frac{1}{1+0.017^2})\approx 35.5367>\frac{b_2}{\overline{c}h_\tau}\approx 35.3713$$

for $(t,x_1,x_2,x_3)\in D_4=[0,1]\times[0.017,0.384]\times[-0.512,0.512]\times[-0.384,0]$;

$$f(t,x_1,x_2,x_3)\geq 123000(1-\frac{1}{1+0.24^2})\approx 6698.94>\frac{b_2}{\overline{c}h_\tau}\approx 35.3713$$

for $(t,x_1,x_2,x_3)\in D_5=[0,1]\times[0,0.384]\times[-0.512,0.512]\times[-0.384,-0.24]$;

$$f(t,x_1,x_2,x_3)\leq 123000<\frac{a_2}{h_0}\approx 123005$$

for $(t,x_1,x_2,x_3)\in D_6=[0,1]\times[0,4550400]\times[-379200,379200]\times[-379200,0]$. Then BVP (3.14) has a positive solution by Theorem 3.3. $\square$

Acknowledgements We thank the referees for their time and comments.

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