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Positive Solutions for Second-Order Singular Difference Equation with Nonlinear Boundary Conditions

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Abstract In this paper, we discuss the existence of positive solutions for the second-order singular difference equation boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = \lambda g(t) f(u), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) = 0, \\ \Delta u(T) + c(u(T+1))u(T+1) = 0, \end{cases}$$

where $\lambda > 0$ is a positive parameter, $f : (0, \infty) \to \mathbb{R}$ is continuous, and is allowed to be singular at 0. The existence of positive solutions is established via introducing a new complete continuous operator.

Keywords difference equation; nonlinear boundary conditions; positive solutions

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1. Introduction

Due to wide applications in engineering, physics and fluid mechanics, boundary value problems (BVPs) of second-order ordinary differential equations have been studied by many authors [1–6]. In 2013, by using fixed-point theorem in cones, Mohamed and Azmi [7] studied singular second order BVP

$$-u'' = \lambda g(t) f(u), \quad t \in (0, 1), \tag{1.1}$$

$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0 \tag{1.2}$$

under the following assumptions:

(A_1) $f : (0, \infty) \to (0, \infty)$ is continuous and $f(u) > 0$ for $u > 0$;

(A_2) $g : [0, 1] \to [0, \infty)$ is continuous and $\int_0^1 G(s, s) g(s) \mathrm{d}s < \infty$;

(A_3) $\alpha, \beta, \gamma, \delta, \lambda > 0$.

They established the following result:

Theorem 1.1 *Let (A_1)–(A_3) hold and $\lim_{u \to 0} f(u) = \infty$. If $f_\infty = \infty$, then for all sufficiently small $\lambda > 0$, (1.1) and (1.2) has two positive solutions, where $f_\infty := \lim_{u \to \infty} \frac{f(u)}{u}$.*

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Note that the existence of positive solutions with linear boundary conditions has been obtained in [7–10], where a unique integral representation of the differential equation with the linear boundary conditions exists. In order to better characterize the safe storage of energetic materials, nuclear waste and even raw garbage, Gordon et al. [11] introduced nonlinear boundary conditions. Subsequently, various interesting results have emerged. For example, Hai and Shivaji [12] discussed the existence of positive solutions for sufficiently small λ to (1.1) with nonlinear boundary conditions

$$u(0) = 0, \quad u'(1) + c(u(1))u(1) = 0, \tag{1.3}$$

where $c : [0, \infty) \to (0, \infty)$ is continuous, $g(t)$ could be singular at $t = 0$, the nonlinearity $f(u)$ is allowed to be the singular at $u = 0$ and extended from nonnegative to $\mathbb{R}$. However, it is difficult to find positive solutions in the singular semi-positive cases due to the lack of maximum principle. We refer the reader to [3–6, 11] and the references therein for literature on singular, nonsingular semi-positive. In the above literature, several different methods have been used, such as, variational methods [4], critical point theory [6], lower and upper solution method [11]. The above results are based on differential equations, and there are few results for the case of difference equations.

Inspired by the above works, in this paper, we aim to consider the existence of positive solutions for the second-order difference equation BVP

$$-\Delta^2 u(t-1) = \lambda g(t) f(u), \quad t \in [1, T]_{\mathbb{Z}}, \tag{1.4}$$

$$u(0) = 0, \quad \Delta u(T) + c(u(T+1))u(T+1) = 0, \tag{1.5}$$

where $T > 1$ is an integer, λ is a positive parameter, $[1, T]_{\mathbb{Z}} = \{1, 2, \ldots, T\}$. Δ is the forward difference operator with $\Delta y(t) = y(t+1) - y(t)$ and $\Delta^2 y(t) = \Delta(\Delta y(t))$. In order to overcome the difficulty of nonlinear boundary conditions, we define a new completely continuous operator T_λ such that $u = T_\lambda v$, where u satisfies the Sturm-Liouville boundary conditions. Through the Krasnoselskii's fixed-point theory, we establish the fixed-point of T_λ, which is the solution of problems (1.4) and (1.5). Furthermore, some results on linear boundary conditions for second-order difference Eq. (1.4) can be found in [13–16].

We make the following hypotheses:

(F1) $g : [1, T]_{\mathbb{Z}} \to (0, \infty)$ with $g > 0$ on $[1, T]_{\mathbb{Z}}$;

(F2) $c : [0, \infty) \to (0, \infty)$ is continuous;

(F3) There exists a constant $0 < \gamma < 1$ such that $\limsup_{u \to 0^+} u^\gamma |f(u)| < \infty$;

(F4) $f : (0, \infty) \to R$ is continuous and $f_\infty = \infty$.

2. Preliminaries

We first recall the following fixed-point theorem of Krasnoselskii's type.

Lemma 2.1 ([17]) *Let E be a Banach space and $T : E \to E$ a completely continuous operator. Suppose there exist $h \in E$, $h \neq 0$ and positive constants r, R with $r \neq R$ such that*

(a) *If $y \in E$ satisfies $y = \theta Ty$, $\theta \in [0,1]$, then $\|y\| \neq r$;*

(b) *If $y \in E$ satisfies $y = Ty + \xi h$, $\xi \geq 0$, then $\|y\| \neq R$.*

Then T has a fixed point $y \in E$ with $\min(r, R) < \|y\| < \max(r, R)$.

In the rest of the article, we let $E = \{u | u : [0, T+1]_{\mathbb{Z}} \to \mathbb{R}\}$ be equipped with the norm

$$\|u\|_E = \max_{t \in [0,T+1]_{\mathbb{Z}}} |u(t)|,$$

then $(E, \|.\|_E)$ is a Banach space. Furthermore, we define $(E_1, \|.\|_1)$ as a Banach space of E with another norm $\|u\|_1 = \sum_{t=1}^{T+1} |u(t)|$.

Lemma 2.2 *Let $k \in E_1$ with $k \geq 0$. Assume that for some constant $\alpha > 0$, u satisfies*

$$\begin{cases} \Delta^2 u(t-1) \leq k(t), & t \in [1, T]_{\mathbb{Z}}, \\ u(0) \geq 0, \ \Delta u(T) + \alpha u(T+1) \geq 0. \end{cases} \tag{2.1}$$

If $\|u\|_E > (T+1)\|k\|_1$ and $\|u\|_E = |u(\tau)|$, then $u(\tau) \geq 0$ and

$$u(t) \geq (\|u\|_E - (T+1)\|k\|_1) q(t), \quad t \in [1, T]_{\mathbb{Z}}, \tag{2.2}$$

where

$$q(t) = \min\{\frac{t}{T+1}, \frac{T+1-t}{T+1}\}.$$

Proof Suppose on the contrary that $u(\tau) < 0, \tau \in [1, T+1]_{\mathbb{Z}}$. Then $\|u\|_E = -u(\tau)$.

Firstly, we consider the case that $\tau \in [1, T]_{\mathbb{Z}}$. Summing both sides of inequality $\Delta^2 u(\gamma - 1) \leq k(\gamma)$ from $\gamma = t$ to $\gamma = s$, we obtain that

$$\Delta u(t-1) \geq \Delta u(s) - \sum_{\gamma=t}^{s} k(\gamma) \geq \Delta u(s) - \|k\|_1. \tag{2.3}$$

Summing (2.3) from τ to T with respect to s, we get

$$\sum_{s=\tau}^{T} \Delta u(t-1) \geq \sum_{s=\tau}^{T} \Delta u(s) - \sum_{s=\tau}^{T} \|k\|_1,$$

which implies that

$$(T - \tau + 1)\Delta u(t-1) \geq u(T+1) - u(\tau) - (T - \tau + 1)\|k\|_1 \geq -(T - \tau + 1)\|k\|_1.$$

Therefore, $\Delta u(t-1) \geq -\|k\|_1$. For $t \in [1, \tau]_{\mathbb{Z}}$, by summing from $t = 1$ to $t = \tau$, we have

$$u(\tau) \geq u(0) - \tau\|k\|_1 \geq -\tau\|k\|_1 \geq -(T+1)\|k\|_1,$$

this leads to a contradiction with $\|u\|_E > (T+1)\|k\|_1$.

Secondly, we consider the case that $\tau = T+1$. From (2.3) with $s = T$, it follows that

$$\Delta u(t-1) \geq \Delta u(T) - \|k\|_1, \quad t \in [1, T+1]_{\mathbb{Z}}.$$

Summing the above inequality from $t = 1$ to $t = T+1$, we have

$$u(T+1) \geq u(0) + (T+1)\Delta u(T) - (T+1)\|k\|_1 \geq u(0) - \alpha(T+1)u(T+1) - (T+1)\|k\|_1.$$

Therefore,

$$u(T+1) \geq \frac{-(T+1)\|k\|_1}{1+\alpha(T+1)} \geq -(T+1)\|k\|_1,$$

a contradiction with $\|u\|_E > (T+1)\|k\|_1$. Thus $u(\tau) \geq 0$.

Suppose $\tau \in [1, T+1]_{\mathbb{Z}}$. For $t \in [1, \tau]_{\mathbb{Z}}$, by summing (2.3) from $s = t-1$ to $s = \tau - 1$, we get

$$(\tau - t + 1)\Delta u(t-1) \geq u(\tau) - u(t-1) - (\tau - t + 1)\|k\|_1,$$

that is

$$\Delta u(t-1) + \frac{u(t-1)}{\tau - t + 1} \geq \frac{\|u\|_E - (\tau - t + 1)\|k\|_1}{\tau - t + 1} \geq \frac{\|u\|_E - (T+1)\|k\|_1}{\tau - t + 1}.$$

Multiplying this inequality by $(\tau - t)^{-1}$, we have

$$\Delta(\frac{u(t-1)}{\tau - t + 1}) \geq \frac{\|u\|_E - (T+1)\|k\|_1}{(\tau - t + 1)(\tau - t)},$$

where $t \neq \tau$. Since $u(0) \geq 0$, we infer that

$$\frac{u(t)}{\tau - t} \geq \frac{u(0)}{\tau} + \sum_{s=1}^{t} \frac{\|u\|_E - (T+1)\|k\|_1}{(\tau - s + 1)(\tau - s)} \geq (\|u\|_E - (T+1)\|k\|_1)\frac{t}{(\tau - t)\tau},$$

i.e.,

$$u(t) \geq (\|u\|_E - (T+1)\|k\|_1)\frac{t}{T+1}, \quad t \in [0, \tau - 1]_{\mathbb{Z}}. \tag{2.4}$$

In particular, for $t = \tau$, it conspicuously satisfyies (2.4). Hence,

$$u(t) \geq (\|u\|_E - (T+1)\|k\|_1)\frac{t}{T+1}, \quad t \in [1, \tau]_{\mathbb{Z}}. \tag{2.5}$$

Next, suppose $t \in [\tau + 1, T+1]_{\mathbb{Z}}$. By (2.3) with $s = T$ and boundary conditions $\Delta u(T) + \alpha u(T+1) \geq 0$, we get

$$\Delta u(t-1) \geq \Delta u(T) - \|k\|_1 \geq -\alpha u(T+1) - \|k\|_1.$$

Summing the above inequality from $t = \tau + 1$ to $t = T+1$, we obtain that

$$\begin{aligned} u(T+1) &\geq \|u\|_E - \alpha(T - \tau + 1)u(T+1) - (T - \tau + 1)\|k\|_1 \\ &\geq \|u\|_E - \alpha(T - \tau + 1)u(T+1) - (T+1)\|k\|_1, \end{aligned}$$

then

$$u(T+1) \geq \frac{\|u\|_E - (T+1)\|k\|_1}{1 + \alpha(T - \tau + 1)}. \tag{2.6}$$

Next, by summing (2.3) from $s = t-1$ to $s = T$, we deduce that

$$(T - t + 2)\Delta u(t-1) \geq u(T+1) - u(t-1) - \sum_{s=t-1}^{T} \sum_{\gamma=t}^{s} k(\gamma),$$

this implies that

$$\Delta u(t-1) + \frac{u(t-1)}{T - t + 2} \geq \frac{u(T+1)}{T - t + 2} - \frac{1}{T - t + 2}\sum_{s=t}^{T}(T + 1 - s)k(s), \quad t \in [\tau + 1, T+1]_{\mathbb{Z}}.$$

Multiplying this inequality by $(T-t+1)^{-1}$, we get

$$\Delta\left(\frac{u(t-1)}{T-t+2}\right) \geq \frac{u(T+1)}{(T-t+2)(T-t+1)} - \frac{1}{(T-t+2)(T-t+1)}\sum_{s=t}^{T}(T+1-s)k(s),$$

where $t \neq T+1$. Therefore,

$$\frac{u(t)}{T-t+1} \geq \frac{u(\tau)}{T-\tau+1} + \sum_{s=\tau+1}^{t}\frac{u(T+1)}{(T-s+2)(T-s+1)} - \sum_{s=\tau+1}^{T}\frac{1}{(T-s+2)(T-s+1)}\sum_{\delta=s}^{T}(T+1-\delta)k(\delta), t\in[\tau+1,T]_{\mathbb{Z}}. \tag{2.7}$$

Since

$$\begin{aligned}
&\sum_{s=\tau+1}^{T}\frac{1}{(T-s+2)(T-s+1)}\sum_{\delta=s}^{T}(T+1-\delta)k(\delta)\\
&= \sum_{z=\tau+1}^{T}\sum_{s=\tau}^{z-1}\frac{1}{(T-s+1)(T-s)}(T+1-z)k(z)\\
&= \sum_{z=\tau+1}^{T}\frac{z-\tau}{T-\tau+1}k(z) \leq \frac{T+1}{T-\tau+1}\|k\|_1,
\end{aligned}$$

it follows from (2.6), (2.7) and the fact that $\|u\|_E \geq (T+1)\|k\|_1$

$$\begin{aligned}
u(t) \geq&(\|u\|_E-(T+1)\|k\|_1)\left(\frac{T-t+1}{T-\tau+1}+\frac{t-\tau}{(T-\tau+1)(1+\alpha(T-\tau+1))}\right)\\
=&(\|u\|_E-(T+1)\|k\|_1)\frac{1+\alpha(T-t+1)}{1+\alpha(T-\tau+1)}.
\end{aligned}$$

Since $t>\tau$ and $\frac{1+\alpha(T-t+1)}{1+\alpha(T-\tau+1)} \geq \frac{T+1-t}{T+1-\tau} \geq \frac{T+1-t}{T+1}$, this implies

$$u(t) \geq (\|u\|_E-(T+1)\|k\|_1)\frac{T+1-t}{T+1}, \quad t\in[\tau+1,T]_{\mathbb{Z}}. \tag{2.8}$$

Combining (2.5) and (2.8), we deduce that

$$u(t) \geq (\|u\|_E-(T+1)\|k\|_1)\min\left\{\frac{t}{T+1},\frac{T+1-t}{T+1}\right\}, \quad t\in[1,T]_{\mathbb{Z}}.$$

If $\tau=0$, then (2.8) holds for all $t\in[1,T]_{\mathbb{Z}}$. This completes the proof of Lemma 2.2. □

3. Main results

In this section, we present and prove our main result.

Theorem 3.1 *Let (F1)–(F4) hold. Then there exists a constant $\lambda_1>0$ such that for $\lambda<\lambda_1$, BVPs (1.4) and (1.5) has a positive solution u with $u\to\infty$ as $\lambda\to 0$ on $[1,T]_{\mathbb{Z}}$.*

Proof Let $\lambda>0$. For $v\in E$, define $T_\lambda v=u$, where u is the solution of the following problem

$$\begin{cases}-\Delta^2u(t-1)=\lambda g(t)f(\tilde{v}), & t\in[1,T]_{\mathbb{Z}},\\ u(0)=0, & \Delta u(T)+\alpha_v u(T+1)=0,\end{cases}$$

where $\tilde{v}(t) = \max(v(t), q(t)), \alpha_v = c(|v(T+1)|)$ and $q(t)$ is defined in Lemma 2.2. Then

$$u(t) = \lambda \sum_{s=1}^{T} G_v(t,s)h(s)f(\tilde{v}), \quad t \in [0, T+1]_{\mathbb{Z}},$$

where the Green's function $G_v(t,s)$ is given by

$$G_v(t,s) = \frac{1}{1+\alpha_v(T+1)} \begin{cases} t(1+\alpha_v(T-s+1)), & s > t-1, \\ s(1+\alpha_v(T-t+1)), & s \le t-1. \end{cases}$$

It is obvious that T_λ is a completely continuous operator.

Let $a > 1$ be such that $f(x) > 0$ for $x \ge a$. By using the fact that $\limsup_{u \to 0^+} u^\gamma |f(u)| < \infty$, there exists a constant $d > 0$ such that $|f(x)| \le \frac{d}{x^\gamma}$, $x \in (0, a)$. Hence,

$$f(x) \ge -\frac{d}{x^\gamma} \tag{3.1}$$

and

$$f(x) \le \frac{d}{x^\gamma} + \hat{f}(\max(x, a)) \tag{3.2}$$

for all $x > 0$, where $\hat{f}(t) = \sup_{a \le x \le t} f(x)$ for $t \ge a$. Note that $\hat{f}$ is nondecreasing.

(a) There exists $r > 0$ such that if $u \in E$ satisfies $u = \theta T_\lambda u$, $\theta \in [0,1]$, then $\|u\|_E \ne r$.

In fact, let $u \in E$ satisfy $u = \theta T_\lambda u$, $\theta \in [0,1]$. Then u satisfies

$$u(t) = \lambda\theta \sum_{s=1}^{T} G_u(t,s)g(s)f(\tilde{u}), \ t \in [0, T+1]_{\mathbb{Z}}.$$

By Lemma 2.2, $q(s) \le 1$, $s \in [1, T]_{\mathbb{Z}}$ and $a > 1$, we have

$$\begin{aligned} |f(\tilde{u}(s))| &\le \frac{d}{\tilde{u}^\gamma(s)} + \hat{f}(\max(\tilde{u}(s), a)) \le \frac{d}{q^\gamma(s)} + \hat{f}(\max(\max(u(s), q(s)), a)) \\ &\le \frac{d}{q^\gamma(s)} + \hat{f}(\max(u(s), a)). \end{aligned}$$

Suppose $\lambda < \frac{a}{2(c_1 + c_2 f(a))}$, where $c_1 = (T+1)d \sum_{s=1}^{T} \frac{g(s)}{q^\gamma(s)}$ and $c_2 = (T+1)\sum_{s=1}^{T} g(s)$. Since $G(t,s) \le T+1$ and $\theta \le 1$, it follows that

$$\begin{aligned} |u(t)| &\le \lambda \Big|\sum_{s=1}^{T} G_u(t,s)g(s)f(\tilde{u})\Big| \\ &\le \lambda\Big(d(T+1)\sum_{s=1}^{T}\frac{g(s)}{q^\gamma(s)} + \hat{f}(\max(\|u\|_E, a))(T+1)\sum_{s=1}^{T} g(s)\Big) \\ &= \lambda(c_1 + c_2\hat{f}(\max(\|u\|_E, a))), \quad t \in [0, T+1]_{\mathbb{Z}}, \end{aligned}$$

which implies

$$\frac{\|u\|_E}{c_1 + c_2\hat{f}(\max(\|u\|_E, a))} \le \lambda. \tag{3.3}$$

Since $\frac{a}{c_1 + c_2\hat{f}(a)} = \frac{a}{c_1 + c_2 f(a)} > 2\lambda$ and $\lim\limits_{x\to\infty} \frac{x}{c_1 + c_2\hat{f}(x)} = 0$ by (F_4), there exists a constant $r > a$ such that

$$\frac{r}{c_1 + c_2\hat{f}(r)} = 2\lambda. \tag{3.4}$$

By (3.3) and (3.4), we deduce that $\|u\|_E \neq r$. Note that $r \to \infty$ as $\lambda \to 0$.

(b) There exists $R > r$ such that if $u = T_\lambda u + \xi$, $\xi \geq 0$, then $\|u\|_E \neq R$.

Let $u \in E$ with $u = T_\lambda u + \xi$ for some $\xi \geq 0$. Then $u - \xi = T_\lambda u$ and so

$$u(t) - \xi = \lambda \sum_{s=1}^{T} G_u(t,s)h(s)f(\tilde{u}).$$

Let $k(t) = \frac{dg(t)}{q^\gamma(t)}, t \in [1,T]_{\mathbb{Z}}$. u satisfies

$$\begin{cases} -\Delta^2 u(t-1) = \lambda g(t) f(\tilde{v}), & t \in [1,T]_{\mathbb{Z}}, \\ u(0) = \xi \geq 0, \\ \Delta u(T) + \alpha_v u(T+1) = \alpha_v \xi \geq 0, \end{cases}$$

and

$$g(t) f(\tilde{v}(t)) \geq -\frac{dg(t)}{\tilde{v}^\gamma(t)} \geq -\frac{dg(t)}{q^\gamma(t)} = -k(t).$$

Suppose $\|u\|_E > \max(3(T+1)\|k\|_1, 6)$, it follows from Lemma 2.2 that

$$u(t) \geq (\|u\|_E - (T+1)\|k\|_1) q(t) \geq \frac{2}{3}\|u\|_E q(t) \geq \frac{\|u\|_E}{6}$$

for $t \in \Gamma = [\frac{T+1}{4}, \frac{3(T+1)}{4}]_{\mathbb{Z}}$. Since

$$G_u(t,s) \geq \frac{(T+1)(4+\alpha_v(T+1))}{16(1+\alpha_v(T+1))} \geq \frac{T+1}{16}, \quad s,t \in \Gamma,$$

we deduce that

$$\begin{aligned} u(t) &\geq \lambda\Big(\sum_{\Gamma} G_u(t,s) g(s) f(\tilde{u}(s)) + \sum_{\Gamma^c} G_u(t,s) g(s) f(\tilde{u}(s))\Big) \\ &\geq \lambda(\frac{T+1}{16} \check{f}(\frac{\|u\|_E}{6}) \sum_{\Gamma} g(s) - (T+1)\|k\|_1), \end{aligned}$$

where $\check{f}(t) = \inf_{z \geq t} f(z)$. Consequently,

$$\frac{\frac{T+1}{16}\check{f}(\frac{\|u\|_E}{6}) \sum_\Gamma g(s) - (T+1)\|k\|_1}{\|u\|_E} \leq \frac{1}{\lambda}.$$

Since $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, it follows that the left side of this inequality goes to ∞ as $\|u\|_E \to \infty$. Then $\|u\|_E < R$ for $R >> 1$.

Hence, by Lemma 2.2, T_λ has a fixed point u with $\|u\|_E > r$. Since (2.2) holds and $r \to \infty$ as $\lambda \to 0$, it implies that u is a positive solution of (1.4) and (1.5) if λ is sufficiently small and $u(t) \to \infty$ as $\lambda \to 0$. □

Example 3.2 Let $g(t) = t^{-\frac{3}{10}}$, $f(u) = 9u^{-\frac{1}{2}} + e^u$, $c(u) = (u^2+3)\ln(2+u)$. It is easy to verify that the hypotheses of Theorem 3.1 hold. Therefore, BVPs (1.4) and (1.5) has a positive solution u with $u \to \infty$ as $\lambda \to 0^+$ on $[1,T]_{\mathbb{Z}}$.

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