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A Condition of Boundary Layer Separation for 2-D Incompressible Magnetohydrodynamic Equations

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Abstract In this paper, we consider the solutions of 2-D incompressible magnetohydrodynamic (MHD) equations with homogenous Dirichlet boundary condition for velocity and with nonhomogenous Dirichlet boundary condition for magnetic field. We obtain a condition of boundary layer separation by Taylor expansion of functions in the MHD equations and by structural bifurcation theory for divergence free flows with Dirichlet boundary conditions. Furthermore, the condition, determined by external forces, initial values and the value of magnetic field on the boundary, can predict when and where boundary layer separation for the magnetic fluid will occur.

Keywords Boundary layer separation; Magnetohydrodynamic equations; Incompressible flow; Structural bifurcation

MR(2020) Subject Classification 35B32; 76D10

1. Introduction

Magnetohydrodynamic(MHD) equations govern the motion of electrically conducting fluids in the presence of a magnetic field such as plasmas, liquid metals and electrolytes, which are formed by coupling the hydrodynamics equations and the magnetic field equations. There are many researches in MHD equations over past three decades [1–7]. He and Xin [1] studied partial regularity for MHD equations. Xiao, Xin and Wu [2] investigated zero viscosity and diffusion vanishing limit for MHD equations. Jiang, Ju and Li [3] concerned the incompressible limit of the compressible MHD equations with vanishing viscosity coefficients. Ju, Li and Li [4] obtained asymptotic limits for the full compressible MHD equations. Ai, Tan and Zhou [5] got global well-posedness for MHD equations. Lin, Ji, Wu and Boardman [6,7] investigated stabilization of a background magnetic field on a 2D MHD flow.

MHD equations are the special kind of hydrodynamic equations. In hydrodynamics, boundary layer is an important research topic. There are many results about boundary layer theory of hydrodynamic equations [8–11] and on boundary layer of magnetic fluid [12]. Wang and Xin [12]

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considered the zero magnetic diffusion limit problem of Dirichlet initial boundary value problem of viscous diffusion MHD system with general initial value in viscous solids.

An important bifurcation phenomenon in hydrodynamics is boundary layer separation, which is a very common phenomenon in geophysical dynamics. There are many researches about boundary layer separation [13–19]. Chorin and Marsden [13] proposed the key question of how the separation or boundary layer separation takes place. Ghil, Ma and Wang [14, 15] provided a first rigorous account on boundary layer separation. Ghil and Liu [16] studied the bifurcation process in the flow's topological structure for a two-dimensional incompressible flow subject to the Dirichlet boundary conditions and its connection with boundary layer separation. Gargano and Sammartino [17] analyzed boundary layer separation with 2-D incompressible fluid by a rectilinear vortex. Luo, Wang and Ma [18, 19] obtained the separation location and time of solutions of the Navier-Stokes for straight boundary and curved boundary, respectively.

Enlightened by the researches of boundary layer for the MHD equation and the study of boundary layer separation for the Navier-Stokes, we investigated the boundary layer separation of MHD equations in this paper. So, we study the following MHD equations:

$$u_t + (u \cdot \nabla)u - R_e^{-1}\Delta u - S(\nabla \times b) \times b + \nabla p = g(x), \quad \text{in } \Omega_T, \tag{1.1}$$

$$b_t - \nabla \times (u \times b) + R_m^{-1}\nabla \times (\nabla \times b) = 0, \quad \text{in } \Omega_T, \tag{1.2}$$

$$\operatorname{div} u = 0,\ \operatorname{div} b = 0, \quad \text{in } \Omega_T, \tag{1.3}$$

$$u|_{\partial\Omega} = 0,\ b|_{\partial\Omega} = h(x), \quad \text{in } [0, T], \tag{1.4}$$

$$u(x, 0) = \alpha(x),\ b(x, 0) = \beta(x), \quad \text{in } \Omega, \tag{1.5}$$

where Ω is bounded and open domain of R^2 with boundary $\partial\Omega$, $0 < T < \infty$, $\Omega_T := \Omega \times (0, T]$. u, b and p are the velocity of the fluid, the magnetic field and the pressure, respectively. And $\alpha(x)$ and $\beta(x)$ are initial values of the velocity and the magnetic field, respectively. $g(x)$ is external force and $h(x)$ is the value of the magnetic field on the boundary. Here $R_e > 0$ and $R_m > 0$ are the Reynolds number and the magnetic Reynolds number, and $S = M^2/(R_eR_m)$ with M being the Hartman number.

We can get a condition of boundary layer separation for MHD equations by analyzing the solutions of (1.1)–(1.5) and by using the lemma of boundary layer separation [14, 15, 20], and the condition is determined by initial values, external forces and the value of the magnetic field on the boundary. Then we can predict when and where boundary layer separation will be found for MHD equations.

The paper is organized as follows. In Section 2, we introduce preliminaries containing the concept of boundary layer separation, boundary singularity and lemma of boundary layer separation. In Section 3, we draw a main conclusion and present some remarks.

2. Preliminaries

Let Ω be a bounded and open domain of R^2 with boundary $\partial\Omega$, which is C^{r+1}. $C^r(\Omega)$ is the space of all C^r fields on Ω and $B_0^r(\Omega) = \{u \in C^r(\Omega) | \operatorname{div} u = 0, u|_{\partial\Omega} = 0\}$. We use n and τ to denote the unit normal and tangent vector of $\partial\Omega$, respectively, and $u_\tau = u \cdot \tau$.

Before stating our main results, we recall some basic concepts.

Definition 2.1 ([14,15,20]) *Suppose $u \in B_0^r(\Omega)$ $(r \geq 2)$. $\bar{x} \in \partial\Omega$ is called a boundary singularity of u, if $\frac{\partial u_\tau}{\partial n}(\bar{x}) = 0$.*

Definition 2.2 ([14,15,20]) *We call that the boundary layer separation governed by a 2-D vector field $u \in C^1([0,T]; B_0^2(\Omega))$ occurs at t_0, if $u(x,t)$ is topologically equivalent to the structure of Figure 1 (a) for any $t < t_0$, but $u(x,t)$ is topologically equivalent to the structure of Figure 1 (c) for $t > t_0$. That is, if $t < t_0$, $u(x,t)$ is topologically equivalent to a parallel flow, and if $t > t_0$, $u(x,t)$ separates a vortex. Furthermore, we call that boundary layer separation occurs at $\bar{x} \in \partial\Omega$, if $\bar{x}$ is an isolated boundary singularity at time $t = t_0$.*

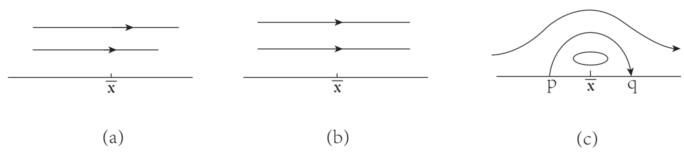

Figure 1 Definition of boundary layer separation

Lemma 2.3 ([14,15,20]) *Let $u \in C^1([0,T]; B_0^2(\Omega))$ be 2-D vector field and $\bar{x} \in \partial\Omega$. Boundary layer separation represented by u occurs at $(\bar{x}, t_0)$, if there exists $0 < t_0 < T$ such that*

$$\frac{\partial u_\tau}{\partial n}(x,t) \neq 0, \quad t < t_0, \ x \in \partial\Omega, \tag{2.1}$$

$$\frac{\partial u_\tau}{\partial n}(\bar{x}, t_0) = 0, \tag{2.2}$$

$$\frac{\partial u_\tau}{\partial t \partial n}(\bar{x}, t) \neq 0, \tag{2.3}$$

where $\bar{x}$ is an isolated boundary singularity of $u(\cdot, t_0)$ on $\partial\Omega$.

3. Main result

Choose one party $\Gamma \subset \partial\Omega$. We take a coordinate system (x_1, x_2) with $\bar{x}$ at the origin and Γ given by

$$\Gamma = \{(x_1, 0) \mid 0 < |x_1| < \delta\}$$

for some $\delta > 0$. Obviously, the tangent and normal vectors on Γ are the unit vectors in the x_1- and x_2-directions, respectively.

If $\alpha(x) \in C^3(\overline{\Omega}; R^2)$ satisfying $\alpha(x)|_{\partial\Omega} = 0$, $\operatorname{div} \alpha(x) = 0$, then we get

$$\alpha_1(x) = \alpha_{11}(x_1)x_2 + \alpha_{12}(x_1)x_2^2 + \alpha_{13}(x_1)x_2^3 + o(x_2^3), \tag{3.1}$$

$$\alpha_2(x) = \alpha_{21}(x_1)x_2^2 + \alpha_{22}(x_1)x_2^3 + o(x_2^3). \tag{3.2}$$

Let $\beta(x) \in C^2(\overline{\Omega}; R^2)$. Because $\beta(x)|_{\partial\Omega} = h(x)$, $\operatorname{div} \beta(x) = 0$, we obtain

$$\beta_1(x) = h_1(x_1) + \beta_{11}(x_1)x_2 + \beta_{12}(x_1)x_2^2 + o(x_2^2), \tag{3.3}$$

$$\beta_2(x) = h_2(x_1) + \beta_{21}(x_1)x_2 + \beta_{22}(x_1)x_2^2 + o(x_2^2). \tag{3.4}$$

Let $g(x) \in C^1(\overline{\Omega}; R^2)$. We present the Taylor expansion of $g(x)$ at $x_2 = 0$,

$$g_1(x) = g_{10}(x_1) + g_{11}(x_1)x_2 + o(x_2), \tag{3.5}$$

$$g_2(x) = g_{20}(x_1) + g_{21}(x_1)x_2 + o(x_2). \tag{3.6}$$

After these preparations, we present the main result.

Theorem 3.1 *Let $\alpha(x) \in C^3(\overline{\Omega}, R^2)$, $\beta(x) \in C^2(\overline{\Omega}, R^2)$, and $g(x) \in C^1(\overline{\Omega}, R^2)$. If*

$$0 < \min_{\Gamma} \frac{-\alpha_{11}}{2R_e^{-1}\alpha_{11}'' + 6R_e^{-1}\alpha_{13} - SR_m h_2^2\alpha_{11} + g_{11} - g_{20}'} \ll 1, \tag{3.7}$$

Then there exist $t_0 > 0$ and $\bar{x} \in \Gamma$ such that boundary layer separation of the solution to (1.1)–(1.5) occurs at $(t_0, \bar{x})$, where $\alpha_{11}, \alpha_{13}, g_{11}, g_{20}$ and h_2 are as (3.1)–(3.6).

Proof From the formula of cross product, we know

$$(\nabla \times b) \times b = b \cdot \nabla b - \frac{1}{2}\nabla(|b|^2), \tag{3.8}$$

$$\nabla \times (\nabla \times b) = \nabla \operatorname{div} b - \Delta b, \tag{3.9}$$

$$\nabla \times (u \times b) = b \cdot \nabla u - u \cdot \nabla b + u \operatorname{div} b - b \operatorname{div} u. \tag{3.10}$$

Combining (3.8)–(3.10), we can rewrite (1.1) and (1.2) as follows.

$$u_t - R_e^{-1}\Delta u + u \cdot \nabla u - Sb \cdot \nabla b + \frac{S}{2}\nabla(|b|^2) + \nabla p = g(x), \tag{3.11}$$

$$b_t - R_m^{-1}\Delta b + u \cdot \nabla b - b \cdot \nabla u = 0. \tag{3.12}$$

Let u and b have the Taylor expansion at $t = 0$,

$$u = \alpha + t\psi + o(t), \quad b = \beta + t\zeta + o(t), \tag{3.13}$$

where $\psi = (\psi_1, \psi_2)$ and $\zeta = (\zeta_1, \zeta_2)$ satisfy, respectively,

$$\psi|_\Gamma = 0, \ \operatorname{div} \psi = 0, \ \zeta|_\Gamma = 0, \ \operatorname{div} \zeta = 0.$$

Let

$$\psi_1 = \psi_{11}x_2 + o(x_2), \quad \psi_2 = \psi_{21}x_2^2 + o(x_2^2),$$

$$\zeta_1 = \zeta_{11}x_2 + o(x_2), \quad \zeta_2 = \zeta_{21}x_2^2 + o(x_2^2).$$

Thus,

$$\psi = (\psi_{11}x_2 + o(x_2), \quad \psi_{21}x_2^2 + o(x_2^2)), \tag{3.14}$$

$$\zeta = (\zeta_{11}x_2 + o(x_2), \quad \zeta_{21}x_2^2 + o(x_2^2)). \tag{3.15}$$

Let

$$p = p_0 + tp_1 + o(t), \quad p_0 = p_{01} + p_{02}x_2 + o(x_2). \tag{3.16}$$

Substituting (3.1)–(3.6), (3.13), (3.14) and (3.16) in (3.11), we can get

$$\psi_{11}x_2 - R_e^{-1}\alpha_{11}''x_2 - R_e^{-1}(2\alpha_{12} + 6\alpha_{13}x_2) - Sh_2\beta_{11} - 2Sh_2\beta_{12}x_2 -$$

$$\begin{aligned} & S\beta_{11}\beta_{21}x_2 + Sh_2h_2' + Sh_2'\beta_{21}x_2 + Sh_2\beta_{21}'x_2 + p_{01}' + p_{02}'x_2 \\ & = g_{10} + g_{11}x_2 + o(x_2), \end{aligned} \tag{3.17}$$

$$-2R_e^{-1}\alpha_{21} - Sh_1h_2' + S\beta_{11}h_1 + p_{02} = g_{20} + o(1). \tag{3.18}$$

Substituting (3.1)–(3.4), (3.13) and (3.15) in (3.12), we obtain

$$-R_m^{-1}h_1'' - 2R_m^{-1}\beta_{12} - h_2\alpha_{11} + o(1) = 0, \tag{3.19}$$

$$-R_m^{-1}h_2'' - 2R_m^{-1}\beta_{22} + o(1) = 0. \tag{3.20}$$

From (3.17) and (3.18), we get

$$\begin{aligned} \psi_{11} = & g_{11} + R_e^{-1}\alpha_{11}'' + 6R_e^{-1}\alpha_{13} + 2S\beta_{12}h_2 + S\beta_{11}\beta_{21} \\ & - Sh_2'\beta_{21} - Sh_2\beta_{21}' - p_{02}', \end{aligned} \tag{3.21}$$

$$p_{02} = g_{20} + 2R_e^{-1}\alpha_{21} + Sh_1h_2' - S\beta_{11}h_1. \tag{3.22}$$

By (3.19) and (3.20), we have

$$h_1'' = -2\beta_{12} - R_mh_2\alpha_{11}, \quad h_2'' = -2\beta_{22} = \beta_{11}'. \tag{3.23}$$

Putting (3.22) into (3.21), we get

$$\psi_{11} = g_{11} + 2R_e^{-1}\alpha_{11}'' + 6R_e^{-1}\alpha_{13} + 2S\beta_{12}h_2 + Sh_2h_1'' - g_{20}' - Sh_1h_2'' + S\beta_{11}'h_1. \tag{3.24}$$

Combining (3.23) and (3.24), we obtain

$$\psi_{11} = 2R_e^{-1}\alpha_{11}'' + 6R_e^{-1}\alpha_{13} - SR_mh_2^2\alpha_{11} + g_{11} - g_{20}'. \tag{3.25}$$

From (3.1), (3.13), (3.14) and (3.25), we have

$$\begin{aligned} \frac{\partial u_\tau}{\partial n}|_\Gamma &= \frac{\partial u_1}{\partial x_2}|_{x_2=0} = \frac{\partial(\alpha_1 + t\psi_1 + o(t))}{\partial x_2}|_{x_2=0} = (\frac{\partial \alpha_1}{\partial x_2} + t\frac{\partial \psi_1}{\partial x_2} + o(t))|_{x_2=0} \\ &= \alpha_{11} + t(2R_e^{-1}\alpha_{11}'' + 6R_e^{-1}\alpha_{13} - SR_mh_2^2\alpha_{11} + g_{11} - g_{20}') + o(t). \end{aligned}$$

If

$$0 < \min_\Gamma \frac{-\alpha_{11}}{2R_e^{-1}\alpha_{11}'' + 6R_e^{-1}\alpha_{13} - SR_mh_2^2\alpha_{11} + g_{11} - g_{20}'} \ll 1,$$

then we can get there exists t_0 such that

$$\frac{\partial u_\tau}{\partial n}|_{\Gamma, t=t_0} = 0.$$

In other words, if there is a point $\bar{x} \in \Gamma$ such that

$$\frac{-\alpha_{11}}{2R_e^{-1}\alpha_{11}'' + 6R_e^{-1}\alpha_{13} - SR_mh_2^2\alpha_{11} + g_{11} - g_{20}'}$$

can take the minimum value, then

$$\frac{\partial u_\tau(\bar{x}, t_0)}{\partial n} = 0,$$

which implies (2.2).

Combining (3.13) and (3.14), we get

$$\frac{\partial^2 u_\tau}{\partial t \partial n} = \frac{\partial^2 u_1}{\partial t \partial x_2} = \psi_{11}.$$

Obviously,

$$\frac{\partial^2 u_\tau}{\partial t \partial n}(\bar{x}, t_0) \neq 0,$$

which satisfies (2.3).

From (3.1), (3.13), (3.14) and (3.25), we have

$$\begin{aligned}\frac{\partial u_\tau}{\partial n}|_\Gamma &= \frac{\partial u_1}{\partial x_2}|_{x_2=0} = \frac{\partial(\alpha_1 + t\psi_1 + o(t))}{\partial x_2}|_{x_2=0} \\ &= (\frac{\partial \alpha_1}{\partial x_2} + t\frac{\partial \psi_1}{\partial x_2} + o(t))|_{x_2=0} \\ &= \alpha_{11} + t(2R_e^{-1}\alpha''_{11} + 6R_e^{-1}\alpha_{13} - SR_m h_2^2 \alpha_{11} + g_{11} - g'_{20}) + o(t) \neq 0\end{aligned}$$

at $t < t_0$, which satisfies (2.1). □

Remark 3.2 The condition (3.7) for boundary layer separation can be observed by g_{11}, α_{11}, α''_{11}, α_{13}, h_2 and g'_{20}. And the condition is related to the value of magnetic field on the boundary, the initial value of the velocity and external force.

Remark 3.3 When the magnetic field is zero, the condition of boundary layer separation for MHD equations will become

$$0 < \min_\Gamma \frac{-\alpha_{11}}{g_{11} + 2R_e^{-1}\alpha''_{11} + 6R_e^{-1}\alpha_{13} - g'_{20}} \ll 1,$$

which is the condition of boundary layer separation of 2-D incompressible fluid flows in [18].

Remark 3.4 From the proof of Theorem 3.1, it can be seen that the separation time is approximately obtained, but the result is of great significance in practical application. To be more specific, we can get the separation time if the value of magnetic field on the boundary, the initial value of the velocity and external force are known.

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