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On the Normalized Laplacian Spectra of some Double Join Operations of Graphs

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Abstract This paper is concerned with the normalized Laplacian spectra of four variants of double join operations based on subdivision graph, Q-graph, R-graph and total graph. The results here generalize some well known results about some join operations of graphs.

Keywords normalized Laplacian matrix; normalized Laplacian spectrum; double join operations; regular graph

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1. Introduction

Let $G = G(V,E)$ be a simple undirected graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. The adjacency matrix of G denoted by $A(G) = (a_{ij})_{n\times n}$ is an $n \times n$ matrix defined as $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ elsewhere. Let $D(G) = (d_1, d_2, \ldots, d_n)$ be the diagonal matrix of G, where d_i is the degree of the vertex i. The matrix $D^{-1/2}$ is a diagonal matrix with diagonal entries $\frac{1}{\sqrt{d_i}}$ for $i = 1, 2, \ldots, n$. Denote by M and $Q(G) = D(G) + A(G)$ the vertex-edge incidence matrix and the signless Laplacian matrix of G, respectively. Chung [1] introduced the normalized Laplacian matrix $\mathcal{L}(G)$ of a simple graph G. It is defined to be the matrix $\mathcal{L}(G) = I - D^{-1/2}A(G)D^{-1/2}$, whose $(i,j)^{\text{th}}$-entry is given by

$$\mathcal{L}_{ij} = \begin{cases} 1, & i = j \text{ and } d_i \neq 0, \\ \frac{-1}{\sqrt{d_i d_j}}, & ij \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Since $A(G)$ and $\mathcal{L}(G)$ are all real symmetric matrices, their eigenvalues can be arranged in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, respectively. In [1], Chung proved that all normalized Laplacian eigenvalues of a graph lie in the interval $[0, 2]$, and 0 is always a normalized Laplacian eigenvalue, that is $\sigma_n(G) = 0$. These eigenvalues together with their multiplicities is called normalized Laplacian spectrum or $\mathcal{L}$-spectrum of G.

Determining the spectra of many graph operations is a basic and very meaningful work in spectral graph theory [2,3]. In recent years, there has been tremendous interest in developing nor-

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malized Laplacian spectra of graphs [4–7]. The mathematicians like Chen and Zhang expressed the resistance distance in terms of normalized Laplacian eigenvalues and vectors of the graph G (see [8]). Also they pointed out that degree-Kirchhoff index is closely related to spectrum of the normalized Laplacian. The concept of limit point for the normalized Laplacian eigenvalues are used by Kirkkland in [9]. In [10], Banergee and Jost investigated how the normalized spectrum is affected by some operations like mofit doubling, graph splitting or joining. Varghese and Susha [11] determined the normalized Laplacian spectrum of duplication vertex join of two graphs, duplication graph, splitting graph and double graph of a regular graph. In [12], Das and Panigrahi determined the full normalized Laplacian spectrum of the subdivision-vertex join, subdivision-edge join, R-vertex join, and R-edge join of two regular graphs in terms of the normalized Laplacian eigenvalues of the graphs. Also they described adjacency, Laplacian and normalized Laplacian spectrum of the Q-vertex join and Q-edge join of a connected regular graph with an arbitrary regular graph in terms of their respective eigenvalues in [13]. Tian et al. [14] gave an explicit complete characterization of the Laplacian eigenvalues and the corresponding eigenvectors of four variants of join operations in terms of the Laplacian eigenvalues and the eigenvectors of factor graphs.

Motivated by above results, here we are interested in finding the normalized Laplacian spectra of double join operations of regular graphs, based on subdivision graph, Q-graph, R-graph and total graph, namely, $G^S \vee (G_1^\bullet, G_2^\circ)$, $G^Q \vee (G_1^\bullet, G_2^\circ)$, $G^R \vee (G_1^\bullet, G_2^\circ)$ and $G^T \vee (G_1^\bullet, G_2^\circ)$. The rest of this paper is organized as follows. In Section 2, we determine the normalized Laplacian spectra of the four double join operations of regular graphs. In Section 3, we summarize our work and give some further remarks thereafter.

Definition 1.1 ([15]) *Let G be a connected graph with n vertices and m edges. The related graphs $S(G)$, $Q(G)$, $R(G)$ and $T(G)$ can be defined as follows:*

(a) The subdivision graph $S(G)$ of G is formed by substituting a path of length 2 corresponding to each edge of G.

(b) The Q-graph $Q(G)$ is formed by bringing in a new vertex into each edge of G, then linking the pairs of new vertices through edges on adjacent edges of G.

(c) The R-graph $R(G)$ is constructed by placing a new vertex related to each edge of G, then connecting each new vertex to the end vertices of the corresponding edge.

(d) The total graph $T(G)$ has the edges and vertices of G as its own vertices. Adjacency of $T(G)$ is specified as adjacency or incidence for the corresponding elements of G. This graph is named as the total graph of G.

The four operations, $S(G)$, $Q(G)$, $R(G)$ and $T(G)$ on a graph G are illustrated with sketches in Figure 1.

Definition 1.2 ([14]) *Let G be a connected graph with n vertices and m edges. Also let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. The subdivision double join $G^S \vee (G_1^\bullet, G_2^\circ)$ of G, G_1 and G_2 is the graph obtained from $S(G)$, G_1 and G_2 by joining every vertex of G to every vertex of G_1 and every vertex of $I(G)$ to every vertex of G_2, where $I(G)$ denotes the vertex* *set of the added new vertices in $S(G)$. Replace $S(G)$ by $Q(G)(R(G), T(G))$ in this definition, then the resulting graphs are referred to as Q-graph (R-graph, total, respectively) double join of these graphs. Similarly, we denote them by $G^Q \vee (G_1^\bullet, G_2^\circ)$, $G^R \vee (G_1^\bullet, G_2^\circ)$ and $G^T \vee (G_1^\bullet, G_2^\circ)$, respectively.*

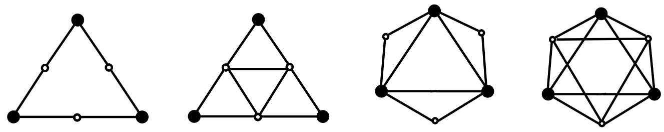

Figure 1 $S(K_3)$,$Q(K_3)$,$R(K_3)$ and $T(K_3)$ for the complete graph K_3

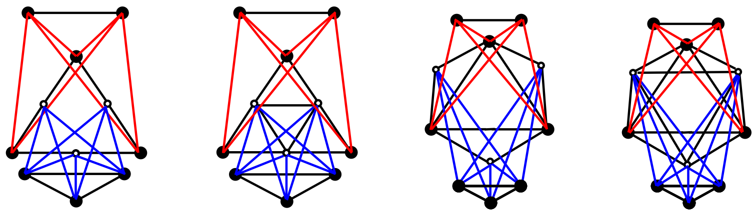

Figure 2 $K_3^S \vee (K_2^\bullet, K_3^\circ)$, $K_3^Q \vee (K_2^\bullet, K_3^\circ)$, $K_3^R \vee (K_2^\bullet, K_3^\circ)$ and $K_3^T \vee (K_2^\bullet, K_3^\circ)$

Example 1.3 Let G, G_1 and G_2 be the complete graph K_3, K_2 and K_3, respectively. Four graphs $K_3^S \vee (K_2^\bullet, K_3^\circ)$, $K_3^Q \vee (K_2^\bullet, K_3^\circ)$, $K_3^R \vee (K_2^\bullet, K_3^\circ)$ and $K_3^T \vee (K_2^\bullet, K_3^\circ)$ are displayed in Figure 2 above.

2. Normalized Laplacian spectra

In this section, we will explore the normalized Laplacian spectrum of subdivision double join, Q-graph double join, R-graph double join and total double join of a regular graph.

Let us first introduce some notations used in the later. Let I_n be the identity matrix of order n, and let $\mathbf{1}_n$ be the column vector with all entries equal to 1. Denote by J and $\mathbf{0}$ the matrix with all entries equal to 1 and 0, respectively. For other notations and terms in this article we refer to [15–17].

Next, we focus on determining the normalized Laplacian spectra of the subdivision double join $G^S \vee (G_1^\bullet, G_2^\circ)$ for the regular graph G and two regular graphs G_1, G_2.

Theorem 2.1 *Let G be a k-regular graph on n vertices and m edges, and let G_i be an r_i-regular graph with n_i vertices, $i = 1, 2$. Then the normalized Laplacian spectrum of $G^S \vee (G_1^\bullet, G_2^\circ)$ consists of:*

(a) $1 - \frac{\lambda_{1i}}{r_1+n}$, *for* $i = 2, 3, \ldots, n_1$;

(b) $1 - \frac{\lambda_{2i}}{r_2+m}$, *for* $i = 2, 3, \ldots, n_2$;

(c) 1, *repeated* $m - n$ *times;*

(d) $1 \pm \sqrt{\frac{b_i^2}{(n_1+k)(n_2+2)}}$, *for* $i = 2, 3, \ldots, n$;

(e) *Four roots of the equation*

$$(r_1+n)\lambda^4 - (2r_1+3n+\frac{m(r_1+n)}{r_2+m})\lambda^3 + (r_1+3n+\frac{3mn+2mr_1}{r_2+m} - \frac{mn_2(r_1+n)}{(n_2+2)(r_2+m)} - \frac{2k(r_1+n)}{(n_1+k)(n_2+2)} - \frac{nn_1}{n_1+k})\lambda^2 + (\frac{mn_2(r_1+n)+mnn_2}{(n_2+2)(r_2+m)} + \frac{2kn}{(n_1+k)(n_2+2)} + \frac{2km(r_1+n)}{(r_2+m)(n_1+k)(n_2+2)} + \frac{mnn_1}{(r_2+m)(n_1+k)} + \frac{nn_1}{n_1+k} - \frac{3mn+mr_1}{r_2+m} - n)\lambda + \frac{mn}{r_2+m} - \frac{mnn_2}{(n_2+2)(r_2+m)} - \frac{mnn_1}{(n_1+k)(r_2+m)} + \frac{mnn_1n_2 - 2kmn}{(n_1+k)(n_2+2)(r_2+m)} = 0,$$

where λ_{1i} $(i = 2, 3, \ldots, n_1)$ *and* λ_{2i} $(i = 2, 3, \ldots, n_2)$ *are the adjacency eigenvalues of* G_1 *and* G_2*, respectively.*

Proof With a suitable labeling of the vertices of $G^S \vee (G_1^\bullet, G_2^\circ)$, the adjacency matrix of it can be written as

$$A\big(G^S \vee (G_1^\bullet, G_2^\circ)\big) = \begin{bmatrix} \mathbf{0}_n & M_{n\times m} & J_{n\times n_1} & \mathbf{0}_{n\times n_2} \\ M_{m\times n}^\top & \mathbf{0}_m & \mathbf{0}_{m\times n_1} & J_{m\times n_2} \\ J_{n_1\times n} & \mathbf{0}_{n_1\times m} & A(G_1) & \mathbf{0}_{n_1\times n_2} \\ \mathbf{0}_{n_2\times n} & J_{n_2\times m} & \mathbf{0}_{n_2\times n_1} & A(G_2) \end{bmatrix}.$$

Similarly, one can get the normalized Laplacian matrix of $G^S \vee (G_1^\bullet, G_2^\circ)$

$$\mathcal{L} = I - D^{-1/2}AD^{-1/2} = \begin{bmatrix} I_n & \frac{-M_{n\times m}}{\sqrt{(n_1+k)(n_2+2)}} & \frac{-J_{n\times n_1}}{\sqrt{(n_1+k)(r_1+n)}} & \mathbf{0}_{n\times n_2} \\ \frac{-M_{m\times n}^\top}{\sqrt{(n_1+k)(n_2+2)}} & I_m & \mathbf{0}_{m\times n_1} & \frac{-J_{m\times n_2}}{\sqrt{(n_2+2)(r_2+m)}} \\ \frac{-J_{n_1\times n}}{\sqrt{(n_1+k)(r_1+n)}} & \mathbf{0}_{n_1\times m} & I_{n_1} - \frac{A(G_1)}{r_1+n} & \mathbf{0}_{n_1\times n_2} \\ \mathbf{0}_{n_2\times n} & \frac{-J_{n_2\times m}}{\sqrt{(n_2+2)(r_2+m)}} & \mathbf{0}_{n_2\times n_1} & I_{n_2} - \frac{A(G_2)}{r_2+m} \end{bmatrix}.$$

Since G_i is r_i-regular, it has an eigenvector $\mathbf{1}_{n_i}$ corresponding to the eigenvalue r_i and other eigenvectors are orthogonal to $\mathbf{1}_{n_i}$. Let λ_{1i} be an eigenvalue of G_1 with eigenvector Z such that $\mathbf{1}_{n_1}^\top Z = 0$. Then $(0, 0, Z, 0)^\top$ is an eigenvector of $\mathcal{L}$ corresponding to the eigenvalue $1 - \frac{\lambda_{1i}}{r_1+n}$.

This is because,

$$\mathcal{L}\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ Z \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ Z - \frac{A(G_1)Z}{r_1+n} \\ \mathbf{0} \end{pmatrix} = \Big(1 - \frac{\lambda_{1i}}{r_1+n}\Big)\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ Z \\ \mathbf{0} \end{pmatrix}.$$

Therefore, $1-\frac{\lambda_{1i}}{r_1+n}$ $(i = 2, 3, \ldots, n_1)$ are eigenvalues corresponding to the eigenvector $(\mathbf{0}, \mathbf{0}, Z, \mathbf{0})^\top$. Similarly, $1 - \frac{\lambda_{2i}}{r_2+m}$ for $i = 2, 3, \ldots, n_2$ are eigenvalues of $\mathcal{L}$ corresponding to the eigenvector $(\mathbf{0}, \mathbf{0}, \mathbf{0}, W)^\top$.

Let X_i and Y_i be the singular vector pairs of M corresponding to the singular values b_i for $i = 1, 2, \ldots, n$, then X_i and Y_i are the orthogonal unit eigenvectors of I_n and I_m. Now consider the following vectors

$$x = \begin{pmatrix} k_1 X_i \\ Y_i \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \text{ for } i = 2, 3, \ldots, n,$$

where k_1 is an unknow constant to be determined. By $\mathcal{L}x = \lambda x$, we obtain

$$\mathcal{L}x = \begin{bmatrix} k_1 X_i - \frac{b_i X_i}{\sqrt{(n_1+k)(n_2+2)}} \\ \frac{-b_i k_1 Y_i}{\sqrt{(n_1+k)(n_2+2)}} + Y_i \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} k_1 X_i \\ Y_i \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

which reduces to the following conditions

$$k_1 - \frac{b_i}{\sqrt{(n_1+k)(n_2+2)}} = \lambda k_1, \quad \frac{-b_i k_1}{\sqrt{(n_1+k)(n_2+2)}} + 1 = \lambda.$$

Eliminating k_1 from above conditions, one obtains quadratic equation

$$\lambda^2 - 2\lambda + 1 - \frac{b_i^2}{(n_1+k)(n_2+2)} = 0$$

with roots $\lambda = 1 \pm \sqrt{\frac{b_i^2}{(n_1+k)(n_2+2)}}$.

Next, we consider the vectors

$$x = \begin{pmatrix} \mathbf{0} \\ Y_j \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \text{ for } j = n+1, n+2, \ldots, m.$$

Notice that $Y_1 = \frac{1}{\sqrt{m}} \mathbf{1}_m, Y_2, \ldots, Y_m$ are orthogonal eigenvectors of the matrix I_m. Then, the equation $\mathcal{L}x = \lambda x$ becomes

$$\mathcal{L}x = \begin{bmatrix} \mathbf{0} \\ c_j Y_i \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{0} \\ Y_j \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Hence, $\lambda = c_j = 1$ $(j = n+1, n+2, \ldots, m)$ are eigenvalues of $\mathcal{L}$. So far we have determinted $n + m + n_1 + n_2 - 4$ eigenvalues of $\mathcal{L}$.

To determine the four remaining eigenvalues and the corresponding eigenvectors, let

$$x = \begin{pmatrix} k_1 \mathbf{1}_n \\ k_2 \mathbf{1}_m \\ k_3 \mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \end{pmatrix},$$

where k_1, k_2, k_3 are three unknown constants to be determined. By $\mathcal{L}x = \lambda x$, we get following conditions.

$$\begin{cases} k_1 - \frac{k_2 b_1 \sqrt{\frac{m}{n}}}{\sqrt{(n_1+k)(n_2+2)}} - \frac{k_3 n_1}{\sqrt{(n_1+k)(r_1+n)}} = \lambda k_1, \\ -\frac{k_1 b_1 \sqrt{\frac{n}{m}}}{\sqrt{(n_1+k)(n_2+2)}} + k_2 - \frac{n_2}{\sqrt{(n_2+2)(r_2+m)}} = \lambda k_2, \\ -\frac{k_1 n}{\sqrt{(n_1+k)(r_1+n)}} + \frac{n k_3}{r_1+n} = \lambda k_3, \\ -\frac{k_2 m}{\sqrt{(n_2+2)(r_2+m)}} + \frac{m}{r_2+m} = \lambda. \end{cases}$$

Eliminating k_1, k_2 and k_3 from above conditions, we have

$$(r_1+n)\lambda^4 - (2r_1 + 3n + \frac{m(r_1+n)}{r_2+m})\lambda^3 + (r_1 + 3n + \frac{3mn + 2mr_1}{r_2+m} - \frac{mn_2(r_1+n)}{(n_2+2)(r_2+m)} - \frac{2k(r_1+n)}{(n_1+k)(n_2+2)} - \frac{nn_1}{n_1+k})\lambda^2 + (\frac{mn_2(r_1+n) + mnn_2}{(n_2+2)(r_2+m)} + \frac{2kn}{(n_1+k)(n_2+2)} + \frac{2km(r_1+n)}{(r_2+m)(n_1+k)(n_2+2)} + \frac{mnn_1}{(r_2+m)(n_1+k)} + \frac{nn_1}{n_1+k} - \frac{3mn + mr_1}{r_2+m} - n)\lambda + \frac{mn}{r_2+m} - \frac{mnn_2}{(n_2+2)(r_2+m)} - \frac{mnn_1}{(n_1+k)(r_2+m)} + \frac{mnn_1n_2 - 2kmn}{(n_1+k)(n_2+2)(r_2+m)} = 0. \quad \square$$

Remark 2.2 The subdivision double join $G^S \vee (G_1^\bullet, G_2^\circ)$ becomes the subdivision-vertex join (resp., subdivision-edge join) defined in [18] whenever G_2 (resp., G_1) is a null graph. In [12], Das and Panigrahi determined the normalized Laplacian spectra of subdivision-vertex join and subdivision-edge join. Clearly, Theorem 2.1 generalizes the results of both Theorems 2.1 and 2.2 in [12].

Next, we give a complete description of the normalized Laplacian spectra of the Q-graph double join $G^Q \vee (G_1^\bullet, G_2^\circ)$ for a regular graph G and two regular graphs G_1, G_2.

Theorem 2.3 *Let G be a k-regular graph on n vertices and m edges, and let G_i be an r_i-regular graph with n_i vertices, $i = 1, 2$. Then the normalized Laplacian spectrum of $G^Q \vee (G_1^\bullet, G_2^\circ)$ consists of:*

(a) $1 - \frac{\lambda_{1i}}{r_1+n}$, *for* $i = 2, 3, \ldots, n_1$;

(b) $1 - \frac{\lambda_{2i}}{r_2+m}$, *for* $i = 2, 3, \ldots, n_2$;

(c) $1 - \frac{\lambda_{lj}}{2k+n_2}$, *for* $j = n+1, n+2, \ldots, m$;

(d) $\frac{2 - \frac{\lambda_{li}}{2k+n_2} \pm \sqrt{(\frac{\lambda_{li}}{2k+n_2})^2 + \frac{4b_i^2}{(n_1+k)(n_2+2k)}}}{2}$, *for* $i = 2, 3, \ldots, n$;

(e) *Four roots of the equation*

$$(r_1+n)\lambda^4 - (r_1 + 2n + \frac{(r_1+n)(n_2+2)}{2k+n_2} + \frac{m(r_1+n)}{r_2+m})\lambda^3 + (\frac{(r_1+n)(n_2+2) + n(n_2+2)}{2k+n_2} + n + \frac{2mn + mr_1}{r_2+m} + \frac{m(r_1+n)(n_2+2) - mn_2(r_1+n)}{(2k+n_2)(r_2+m)} - \frac{2k(r_1+n)}{(2k+n_2)(n_1+k)} - \frac{nn_1}{n_1+k})\lambda^2 +$$

$$(\frac{2kn+nn_1(n_2+2)}{(2k+n_2)(n_1+k)}+\frac{2mnn_2+mn_2r_1-m(r_1+2n)(n_2+2)}{(2k+n_2)(r_2+m)}+ \frac{2km(r_1+n)}{(2k+n_2)(n_1+k)(r_2+m)}+\frac{mnn_1}{(r_2+m)(n_1+k)}- \frac{n(n_2+2)}{2k+n_2}-\frac{mn}{r_2+m})\lambda=0,$$

where λ_{1i} $(i=2,3,\ldots,n_1)$, λ_{2i} $(i=2,3,\ldots,n_2)$ and λ_{lj} $(j=2,3,\ldots,m)$ are the adjacency eigenvalues of G_1, G_2 and the line graph of G, respectively.

Proof The adjacency matrix of $G^Q \vee (G_1^{\bullet}, G_2^{\circ})$ can be expressed as

$$A\big(G^Q \vee (G_1^{\bullet}, G_2^{\circ})\big)=\begin{bmatrix} \mathbf{0}_n & M_{n\times m} & J_{n\times n_1} & \mathbf{0}_{n\times n_2} \\ M_{m\times n}^{\top} & A\big(l(G)\big) & \mathbf{0}_{m\times n_1} & J_{m\times n_2} \\ J_{n_1\times n} & \mathbf{0}_{n_1\times m} & A(G_1) & \mathbf{0}_{n_1\times n_2} \\ \mathbf{0}_{n_2\times n} & J_{n_2\times m} & \mathbf{0}_{n_2\times n_1} & A(G_2) \end{bmatrix},$$

where $l(G)$ denotes the line graph of G. Similarly, the normalized Laplacian matrix of $G^Q \vee (G_1^{\bullet}, G_2^{\circ})$ is

$$\mathcal{L}=I-D^{-1/2}AD^{-1/2}$$
$$=\begin{bmatrix} I_n & \frac{-M_{n\times m}}{\sqrt{(n_1+k)(n_2+2k)}} & \frac{-J_{n\times n_1}}{\sqrt{(n_1+k)(r_1+n)}} & \mathbf{0}_{n\times n_2} \\ \frac{-M_{m\times n}^{\top}}{\sqrt{(n_1+k)(n_2+2k)}} & I_m-\frac{A(l(G))}{n_2+2k} & \mathbf{0}_{m\times n_1} & \frac{-J_{m\times n_2}}{\sqrt{(n_2+2k)(r_2+m)}} \\ \frac{-J_{n_1\times n}}{\sqrt{(n_1+k)(r_1+n)}} & \mathbf{0}_{n_1\times m} & I_{n_1}-\frac{A(G_1)}{r_1+n} & \mathbf{0}_{n_1\times n_2} \\ \mathbf{0}_{n_2\times n} & \frac{-J_{n_2\times m}}{\sqrt{(n_2+2k)(r_2+m)}} & \mathbf{0}_{n_2\times n_1} & I_{n_2}-\frac{A(G_2)}{r_2+m} \end{bmatrix}.$$

Since G_i is r_i-regular, it has an eigenvector $\mathbf{1}_{n_i}$ corresponding to the eigenvalue r_i and other eigenvectors are orthogonal to $\mathbf{1}_{n_i}$. Let λ_{1i} be an eigenvalue of G_1 with eigenvector Z such that $\mathbf{1}_{n_1}^{\top}Z=0$, then it is easy to see $(\mathbf{0},\mathbf{0},Z,\mathbf{0})^{\top}$ is an eigenvector of $\mathcal{L}$ corresponding to the eigenvalue $1-\frac{\lambda_{1i}}{r_1+n}$ $(i=2,3,\ldots,n_1)$. Similarly, $1-\frac{\lambda_{2i}}{r_2+m}$ $(i=2,3,\ldots,n_2)$ are eigenvalues of $\mathcal{L}$ corresponding to the eigenvector $(\mathbf{0},\mathbf{0},\mathbf{0},W)^{\top}$.

Let X_i and Y_i be the singular vector pairs of M corresponding to the singular values b_i for $i=1,2,\ldots,n$, then X_i and Y_i are the orthogonal unit eigenvectors of I_n and $A(l(G))$. Now consider the following vectors

$$x=\begin{pmatrix} k_1X_i \\ Y_i \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad i=2,3,\ldots,n,$$

where k_1 is an unknown constant to be determined. By $\mathcal{L}x = \lambda x$, we obtain

$$\mathcal{L}x = \begin{bmatrix} k_1 X_i - \frac{b_i X_i}{\sqrt{(n_1+k)(n_2+2k)}} \\ \frac{-b_i k_1 Y_i}{\sqrt{(n_1+k)(n_2+2k)}} + (1 - \frac{\lambda_{li}}{n_2+2k})Y_i \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} k_1 X_i \\ Y_i \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

which reduces to the following conditions

$$k_1 - \frac{b_i}{\sqrt{(n_1+k)(n_2+2k)}} = \lambda k_1, \quad \frac{-b_i k_1}{\sqrt{(n_1+k)(n_2+2k)}} + (1 - \frac{\lambda_{li}}{n_2+2k}) = \lambda.$$

Eliminating k_1 from above conditions, one obtains equation

$$\lambda^2 - (2 - \frac{\lambda_{li}}{n_2+2k})\lambda + 1 - \frac{\lambda_{li}}{n_2+2k} - \frac{b_i^2}{(n_1+k)(n_2+2k)} = 0$$

with roots $\lambda = \frac{(2-\frac{\lambda_{li}}{n_2+2k}) \pm \sqrt{(\frac{\lambda_{li}}{n_2+2k})^2 + \frac{4b_i^2}{(n_1+k)(n_2+2)}}}{2}$.

Next, we consider the vectors

$$x = \begin{pmatrix} \mathbf{0} \\ Y_j \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad j = n+1, n+2, \ldots, m.$$

Notice that $Y_1 = \frac{1}{\sqrt{m}} \mathbf{1}_m, Y_2, \ldots, Y_m$ are orthogonal eigenvectors of the matrix $A(l(G))$. Then, the equation $\mathcal{L}x = \lambda x$ becomes

$$\mathcal{L}x = \begin{bmatrix} \mathbf{0} \\ (1 - \frac{\lambda_{lj}}{n_2+2k})Y_j \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{0} \\ Y_j \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Hence, $\lambda = 1 - \frac{\lambda_{lj}}{n_2+2k}$ $(j = n+1, n+2, \ldots, m)$ are eigenvalues of $\mathcal{L}$. So far we have determinted $n + m + n_1 + n_2 - 4$ eigenvalues of $\mathcal{L}$.

To determine the four remaining eigenvalues and the corresponding eigenvectors, let

$$x = \begin{pmatrix} k_1 \mathbf{1}_n \\ k_2 \mathbf{1}_m \\ k_3 \mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \end{pmatrix},$$

where k_1, k_2, k_3 are three unknown constants to be determined. By $\mathcal{L}x = \lambda x$, one gets

$$\begin{cases} k_1 - \frac{k_2 b_1 \sqrt{\frac{m}{n}}}{\sqrt{(n_1+k)(n_2+2k)}} - \frac{k_3 n_1}{\sqrt{(n_1+k)(r_1+n)}} = \lambda k_1, \\ -\frac{k_1 b_1 \sqrt{\frac{n}{m}}}{\sqrt{(n_1+k)(n_2+2k)}} + \frac{k_2(n_2+2)}{n_2+2k} - \frac{n_2}{\sqrt{(n_2+2k)(r_2+m)}} = \lambda k_2, \\ -\frac{k_1 n}{\sqrt{(n_1+k)(r_1+n)}} + \frac{n k_3}{r_1+n} = \lambda k_3, \\ -\frac{k_2 m}{\sqrt{(n_2+2k)(r_2+m)}} + \frac{m}{r_2+m} = \lambda. \end{cases}$$

Eliminating k_1, k_2 and k_3 from above equations, we have

$$\begin{aligned} &(r_1+n)\lambda^4 - (r_1+2n+\frac{(r_1+n)(n_2+2)}{2k+n_2}+\frac{m(r_1+n)}{r_2+m})\lambda^3+ \\ &(\frac{(r_1+n)(n_2+2)+n(n_2+2)}{2k+n_2}+n+\frac{2mn+mr_1}{r_2+m}+\frac{m(r_1+n)(n_2+2)-mn_2(r_1+n)}{(2k+n_2)(r_2+m)}- \\ &\frac{2k(r_1+n)}{(2k+n_2)(n_1+k)}-\frac{nn_1}{n_1+k})\lambda^2+(\frac{2kn+nn_1(n_2+2)}{(2k+n_2)(n_1+k)}+ \\ &\frac{2mnn_2+mn_2r_1-m(r_1+2n)(n_2+2)}{(2k+n_2)(r_2+m)}+\frac{2km(r_1+n)}{(2k+n_2)(n_1+k)(r_2+m)}+ \\ &\frac{mnn_1}{(r_2+m)(n_1+k)}-\frac{n(n_2+2)}{2k+n_2}-\frac{mn}{r_2+m})\lambda = 0. \quad \square \end{aligned}$$

Remark 2.4 If G_2 (resp., G_1) is a null graph, then the Q-graph double join $G^Q \vee (G_1^\bullet, G_2^\circ)$ reduces to Q-graph vertex join (resp., Q-graph edge join) [13]. Naturally, Theorem 2.3 implies the results of both Theorems 2.3 and 2.6 in [13].

The following result describes the normalized Laplacian spectra of the R-graph double join $G^R \vee (G_1^\bullet, G_2^\circ)$ for a regular graph G and two regular graphs G_1, G_2.

Theorem 2.5 *Let G be a k-regular graph on n vertices and m edges, and let G_i be an r_i-regular graph with n_i vertices, $i = 1, 2$. Then the normalized Laplacian spectrum of $G^R \vee (G_1^\bullet, G_2^\circ)$ consists of:*

(a) $1 - \frac{\lambda_{1i}}{r_1+n}$, *for* $i = 2, 3, \ldots, n_1$;
(b) $1 - \frac{\lambda_{2i}}{r_2+m}$, *for* $i = 2, 3, \ldots, n_2$;
(c) 1, *repeated* $m - n$ *times;*
(d) $\frac{(2-\frac{\lambda_i}{2k+n_1})\pm\sqrt{(\frac{\lambda_i}{2k+n_1})^2+\frac{4b_i^2}{(2k+n_1)(n_2+2)}}}{2}$, *for* $i = 2, 3, \ldots, n$;
(e) Four roots of the equation

$$\begin{aligned} &(r_1+n)\lambda^4 - (r_1+2n+\frac{(k+n_1)(r_1+n)}{2k+n_1}+\frac{m(r_1+n)}{r_2+m})\lambda^3+ \\ &(n+\frac{(k+n_1)(r_1+n)-nn_1}{2k+n_1}+\frac{m(r_1+2n)}{r_2+m}+\frac{m(k+n_1)(r_1+n)}{(2k+n_1)(r_2+m)}- \\ &\frac{mn_2(r_1+n)}{(2+n_2)(r_2+m)}-\frac{2k(r_1+n)}{(2k+n_1)(2+n_2)})\lambda^2+ \\ &(\frac{n_1n_2-n(k+n_1)}{2k+n_1}+\frac{2kn}{(2k+n_1)(2+n_2)}+\frac{2km(r_1+n)+mn_2(k+n_1)(r_1+n)}{(2k+n_1)(2+n_2)(r_2+m)}- \end{aligned}$$

$$\frac{kmn+m(k+n_1)(r_1+n)}{(2k+n_1)(r_2+m)}+\frac{mnn_2}{(2+n_2)(r_2+m)}-\frac{mn}{r_2+m})\lambda+$$
$$\frac{kmn}{(2k+n_1)(r_2+m)}-\frac{2kmn+kmnn_2}{(2k+n_1)(r_2+m)(n_2+2)}=0,$$

where λ_{1i} $(i=2,3,\ldots,n_1)$ *and* λ_{2i} $(i=2,3,\ldots,n_2)$ *are the adjacency eigenvalues of* G_1 *and* G_2*, respectively.*

Proof With a proper labeling of vertices, the adjacency matrix of $G^R\vee(G_1^\bullet,G_2^\circ)$ can be written as

$$A\big(G^R\vee(G_1^\bullet,G_2^\circ)\big)=\begin{bmatrix} A(G) & M_{n\times m} & J_{n\times n_1} & \mathbf{0}_{n\times n_2}\\ M_{m\times n}^\top & \mathbf{0}_m & \mathbf{0}_{m\times n_1} & J_{m\times n_2}\\ J_{n_1\times n} & \mathbf{0}_{n_1\times m} & A(G_1) & \mathbf{0}_{n_1\times n_2}\\ \mathbf{0}_{n_2\times n} & J_{n_2\times m} & \mathbf{0}_{n_2\times n_1} & A(G_2)\end{bmatrix}.$$

Similarly, the normalized Laplacian matrix of $G^R\vee(G_1^\bullet,G_2^\circ)$

$$\mathcal{L}=I-D^{-1/2}AD^{-1/2}$$
$$=\begin{bmatrix} I_n-\frac{A(G)}{n_1+2k} & \frac{-M_{n\times m}}{\sqrt{(n_1+2k)(n_2+2)}} & \frac{-J_{n\times n_1}}{\sqrt{(n_1+2k)(r_1+n)}} & \mathbf{0}_{n\times n_2}\\ \frac{-M_{m\times n}^\top}{\sqrt{(n_1+2k)(n_2+2)}} & I_m & \mathbf{0}_{m\times n_1} & \frac{-J_{m\times n_2}}{\sqrt{(n_2+2)(r_2+m)}}\\ \frac{-J_{n_1\times n}}{\sqrt{(n_1+2k)(r_1+n)}} & \mathbf{0}_{n_1\times m} & I_{n_1}-\frac{A(G_1)}{r_1+n} & \mathbf{0}_{n_1\times n_2}\\ \mathbf{0}_{n_2\times n} & \frac{-J_{n_2\times m}}{\sqrt{(n_2+2)(r_2+m)}} & \mathbf{0}_{n_2\times n_1} & I_{n_2}-\frac{A(G_2)}{r_2+m}\end{bmatrix}.$$

Using the same technique as the proof of Theorem 2.1, we can obtain the desired result. □

Remark 2.6 Similarly, if G_2 (resp., G_1) is a null graph, then our R-graph double join $G^R\vee(G_1^\bullet,G_2^\circ)$ reduces to R-graph vertex join (R-graph edge join) [19]. Naturaly, Theorem 2.5 implies the results of both Theorems 2.3 and 2.4 in [12].

For the total double join $G^T\vee(G_1^\bullet,G_2^\circ)$, we describe the normalized Laplacian spectra in the following results.

Theorem 2.7 *Let* G *be a* k*-regular graph on* n *vertices and* m *edges, and let* G_i *be an* r_i*-regular graph with* n_i *vertices,* $i=1,2$*. Then the normalized Laplacian spectrum of* $G^T\vee(G_1^\bullet,G_2^\circ)$ *consists of:*

(a) $1-\frac{\lambda_{1i}}{r_1+n}$, *for* $i=2,3,\ldots,n_1$;

(b) $1-\frac{\lambda_{2i}}{r_2+m}$, *for* $i=2,3,\ldots,n_2$;

(c) $1-\frac{\lambda_{lj}}{2k+n_2}$, *for* $j=n+1,n+2,\ldots,m$;

(d) $\frac{(2-\frac{\lambda_i}{2k+n_1}-\frac{\lambda_{li}}{2k+n_2})\pm\sqrt{(\frac{\lambda_i}{2k+n_1}+\frac{\lambda_{li}}{2k+n_2})^2-\frac{4(\lambda_{li}\lambda_i-b_i^2)}{(n_1+2k)(n_2+2k)}}}{2}$, *for* $i=2,3,\ldots,n$;

(e) *Four roots of the equation*

$$(r_1+n)\lambda^4-(n+\frac{(k+n_1)(r_1+n)}{2k+n_1}+\frac{m(r_1+n)}{r_2+m}+\frac{(n_2+2)(r_1+n)}{2k+n_2})\lambda^3+$$
$$(\frac{k}{2k+n_1}+\frac{mn}{r_2+m}+\frac{n(n_2+2)}{2k+n_2}+\frac{m(k+n_1)(r_1+n)}{(r_2+m)(2k+n_1)}+$$

$$\frac{(kn_2+n_1n_2+2n_1)(r_1+n)}{(2k+n_1)(2k+n_2)}+\frac{2m(r_1+n)}{(r_2+m)(2k+n_2)})\lambda^2-(\frac{2mn}{(2k+n_2)(r_2+m)}+$$
$$\frac{2mn_1(r_1+n)}{(2k+n_1)(2k+n_2)(r_2+m)}+\frac{knn_2}{(2k+n_1)(2k+n_2)}+\frac{kmn}{(r_2+m)(2k+n_1)})\lambda=0,$$

where λ_{1i} $(i=2,3,\ldots,n_1)$, λ_{2i} $(i=2,3,\ldots,n_2)$ *and* λ_{lj} $(j=2,3,\ldots,m)$ *are the adjacency eigenvalues of* G_1, G_2 *and the line graph of* G, *respectively.*

Proof With a suitable labeling of the vertices of $G^T\vee(G_1^\bullet,G_2^\circ)$, we can write the adjacency matrix of $G^T\vee(G_1^\bullet,G_2^\circ)$ as

$$A\big(G^T\vee(G_1^\bullet,G_2^\circ)\big)=\begin{bmatrix} A(G) & M_{n\times m} & J_{n\times n_1} & \mathbf{0}_{n\times n_2}\\ M_{m\times n}^\top & A\big(l(G)\big) & \mathbf{0}_{m\times n_1} & J_{m\times n_2}\\ J_{n_1\times n} & \mathbf{0}_{n_1\times m} & A(G_1) & \mathbf{0}_{n_1\times n_2}\\ \mathbf{0}_{n_2\times n} & J_{n_2\times m} & \mathbf{0}_{n_2\times n_1} & A(G_2)\end{bmatrix}.$$

Then the normalized Laplacian matrix of $G^T\vee(G_1^\bullet,G_2^\circ)$

$$\mathcal{L}=I-D^{-1/2}AD^{-1/2}$$
$$=\begin{bmatrix} I_n-\frac{A(G)}{n_1+2k} & \frac{-M_{n\times m}}{\sqrt{(n_1+2k)(n_2+2k)}} & \frac{-J_{n\times n_1}}{\sqrt{(n_1+2k)(r_1+n)}} & \mathbf{0}_{n\times n_2}\\ \frac{-M_{m\times n}^\top}{\sqrt{(n_1+2k)(n_2+2k)}} & I_m-\frac{A(l(G))}{n_2+2k} & \mathbf{0}_{m\times n_1} & \frac{-J_{m\times n_2}}{\sqrt{(n_2+2k)(r_2+m)}}\\ \frac{-J_{n_1\times n}}{\sqrt{(n_1+2k)(r_1+n)}} & \mathbf{0}_{n_1\times m} & I_{n_1}-\frac{A(G_1)}{r_1+n} & \mathbf{0}_{n_1\times n_2}\\ \mathbf{0}_{n_2\times n} & \frac{-J_{n_2\times m}}{\sqrt{(n_2+2k)(r_2+m)}} & \mathbf{0}_{n_2\times n_1} & I_{n_2}-\frac{A(G_2)}{r_2+m}\end{bmatrix}.$$

Using the similar technique to the proof of Theorem 2.2, we get the expected result. □

3. Conclusion

Here, we give an explicit complete characterization of the normalized Laplacian spectra of four variants of double join operations of graphs in terms of the normalized Laplacian spectra of the factor graphs. In addition, these results describe completely the eigenvectors corresponding to all the normalized Laplacian eigenvalues of these graphs.

Before the end of this paper, it should be pointed out that the normalized Laplacian matrix of usual join graph of regular graph can be obtained by choosing $M=\mathbf{0}_{n\times m}, n_2=0$ and the block which I_m lied in equals $\mathbf{0}_{m\times m}$ in $\mathcal{L}\big(G^F\vee(G_1^\bullet,G_2^\circ)\big)$, $F\in\{S,Q,R,T\}$. Thus the nonzero normalized Laplacian eigenvalues of the join graph of regular graphs can be obtained from the corresponding eigenvalues of $\mathcal{L}\big(G^F\vee(G_1^\bullet,G_2^\circ)\big)$. Hence, the Corollary 3.3 in [20] can also be obtained from the results here.

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