Journal of Mathematical Research with Applications
Mar., 2023, Vol. 43, No. 2, pp. 150–160
DOI:10.3770/j.issn:2095-2651.2023.02.003
Http://jmre.dlut.edu.cn

Optimal $L(2,1,1)$-Labelings of Caterpillars

Xiaoling ZHANG
Teachers College, Jimei University, Fujian 361021*, P. R. China*

Abstract An $L(2,1,1)$-labeling of a graph G is an assignment of non-negative integers (labels) to the vertices of G such that adjacent vertices receive labels with difference at least 2, and vertices at distance 2 or 3 receive distinct labels. The span of such a labeling is the difference between the maximum and minimum labels used, and the minimum span over all $L(2,1,1)$-labelings of G is called the $L(2,1,1)$-labeling number of G, denoted by $\lambda_{2,1,1}(G)$. In this paper, we investigate the $L(2,1,1)$-labelings of caterpillars. Some useful sufficient conditions for $\lambda_{2,1,1}(T)=\Delta_2(T)=\max_{uv\in E(T)}(d(u)+d(v)))$ are given. Furthermore, we show that the sufficient conditions we provide are also necessary for caterpillars with $\Delta_2(T)=6$.

Keywords channel assignment; $L(2,1,1)$-labeling; span; caterpillar

MR(2020) Subject Classification 05C78; 05C15

1. Introduction

Multilevel distance labeling can be regarded as an extension of distance two labeling, and both of them are motivated by the channel assignment problem introduced by Hale [1]. The channel assignment problem addresses the assignment of a channel, known as a frequency, to each transmitter in a network. The channels assigned to transmitters must satisfy certain distance restrictions to avoid interference between nearby transmitters. If there is high usage of wireless communication networks, we have to find an appropriate channel assignment solution, so that the range of channels used is minimized.

Griggs and Yeh [2] firstly proposed the notation of distance two labeling of a graph, and they generalized it to p-levels of interference, specifically for given positive integers $k_1,k_2,\ldots,k_p$, an $L(k_1,k_2,\ldots,k_p)$-labeling of a graph G is a function f from the vertices of G to non-negative integers (labels), such that for each pair of distinct vertices u, v of G, $|f(u)-f(v)|\geq k_t$ if $\text{dist}(u,v)=t$, where $\text{dist}(u,v)$ is the distance between u and v. The span of f is the maximum difference $f(u)-f(v)$ of any pair of vertices u, v of G. Without loss of generality, we will always assume $\min_{v\in V(G)} f(v)=0$. So the span of f is defined as $\max_{v\in V(G)} f(v)$. The $L(k_1,k_2,\ldots,k_p)$-labeling number, denoted by $\lambda_{k_1,k_2,\ldots,k_p}(G)$, is the minimum span of all $L(k_1,k_2,\ldots,k_p)$-labelings of G. If an $L(k_1,k_2,\ldots,k_p)$-labeling uses labels in the set $\{0,1,\ldots,k\}$, it will be called a k-$L(k_1,k_2,\ldots,k_p)$-labeling.

Received February 24, 2022; Accepted May 21, 2022
Supported by the National Natural Science Foundation of China (Grant No. 11601265) and the Scientific Research Foundation of Jimei University (Grant No. Q202201).
E-mail address: xml000999@163.com

The $L(k_1, k_2, \ldots, k_p)$-labeling problem above is interesting in both theory and practical applications. For instance, when $p = 1$, $k_1 = 1$, it becomes the ordinary vertex-coloring problem. When $p = 2$, many interesting results [2–5] have been obtained for various families of finite graphs, especially for the case $(k_1, k_2) = (2, 1)$. For more details, one may refer to the surveys [6, 7].

More recently, researchers began to investigate the $L(k_1, k_2, k_3)$-labeling problem [8–13]. For example, Zhou studied the problem for hypercubes Q_n in [10]. The $L(h, 1, 1)$-labeling problem for outer-planar graphs was investigated in [11]. Shao and Vesel [12] determined the $L(3, 2, 1)$-labeling numbers for toroidal grids and triangular grids. In [13], King et al. studied the $L(h, 1, 1)$-labeling problem for trees. They proved that $\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T)$ and proposed the following questions: To characterize finite trees T with diameter at least 3 such that $\lambda_{2,1,1}(T) = \Delta_2(T)$ (see [13, Question 10]). In addition, they conjectured that almost all trees have the $L(2, 1, 1)$-labeling number attaining the lower bound. Recently, the result in [14, 15] asserted that deciding whether a given tree has the $L(2, 1, 1)$-labeling number attaining the lower bound is *NP*-complete. Therefore, providing some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2(T)$ or giving a characterization result for the subclass of trees becomes a meaningful topic.

Based on the above topics, some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2(T)$ were provided in [16]. Moreover, the sufficient conditions are also necessary for trees with diameter at most 6. And in [17], the authors determined the $L(2, 1, 1)$-labeling numbers of caterpillars (as a subclass of trees) with $\Delta_2(T) \leq 5$. But we found the case for $\Delta_2(T) \geq 6$ is more difficult than the case for $\Delta_2(T) \leq 5$.

In this paper, we continue to study the $L(2, 1, 1)$-labelings of caterpillars. We provide some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2(T)$ in Section 2, which gives a partial answer in [13, Question 10]. Furthermore, in Section 3, we show that the sufficient conditions we provide are also necessary for caterpillars with $\Delta_2(T) = 6$. This means that the problem of deciding whether the $L(2, 1, 1)$-labeling number of a caterpillar T is $\Delta_2(T) - 1$ is polynomial when $\Delta_2(T) = 6$.

2. Some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2(T)$

In this paper, we always suppose that T is a finite tree with diameter at least 3. Define $\Delta_2(T) := \max_{uv \in E(T)}(d(u)+d(v))$, where $d(u)$ is the degree of u. In the following, we abbreviate $\Delta_2(T)$ to Δ_2. An edge $e = uv$ is said to be heavy if $d(u) + d(v) = \Delta_2$, light if $d(u) + d(v) < \Delta_2$. A vertex v is said to be bad if $d(v) = \Delta_2 - 2$.

King et al. [13] studied the $L(2, 1, 1)$-labelings of trees and gave the following result.

Lemma 2.1 ([13]) *Let T be a finite tree with diameter at least 3. Then $\Delta_2 - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2$.*

For a vertex u in T, let $N_0(u) = \{w|uw \text{ is light}\}$, $N_1(u) = \{w|uw \text{ is heavy}\}$ and $d_0(u) = |N_0(u)|$, $d_1(u) = |N_1(u)|$. Then $N(u) = N_0(u) \cup N_1(u)$ and $d(u) = d_0(u) + d_1(u)$. Let $N[u] = N(u) \cup \{u\}$. For integers i and j with $i \leq j$, we denote $[i, j]$ as the set $\{i, i+1, \ldots, j-1, j\}$. Let $F = [0, \Delta_2 - 1]$.

Before providing some sufficient conditions for $\lambda_{2,1,1}(T) = \Delta_2$, we give some useful lemmas as follows.

Lemma 2.2 ([16]) *Let f be an $L(2,1,1)$-labeling of T with span Δ_2-1. Let uv be heavy. Then $f(N(u))\cup f(N(v))=F$ and $|f(u)-f(v)|>2$.*

Lemma 2.3 ([17]) *Let f be an $L(2,1,1)$-labeling of T with span Δ_2-1. If there exists a path vuw in T such that $d(u)=2$, uv is heavy and uw is light, then either $f(v)=0$, $f(w)=1$ and $f(N(v))=[2,\Delta_2-1]$, or $f(v)=\Delta_2-1$, $f(w)=\Delta_2-2$ and $f(N(v))=[0,\Delta_2-3]$. What is more, if $d(w)=d(v)-1$, then $f(N(w))=[3,\Delta_2-1]$ or $[0,\Delta_2-4]$.*

A tree is called a caterpillar if the removal of all vertices of degree 1 results in a path, called the spline. In view of the above results, we now give some sufficient conditions for caterpillars with $\Delta_2=6$.

Theorem 2.4 ([17]) *Let T be a caterpillar with $\Delta_2=6$. If T contains one of the following configurations, then $\lambda_{2,1,1}(T)=6$.*

(C1) There exist two bad vertices u and v such that $\mathrm{dist}(u,v)=2$ or 6;

(C2) There exist three bad vertices u, v and w such that $\mathrm{dist}(u,v)=\mathrm{dist}(v,w)=3$.

Theorem 2.5 *Let T be a caterpillar with $\Delta_2=6$. Let u and v be two consecutive bad vertices with $\mathrm{dist}(u,v)=10$ and $uu_1u_2\ldots u_9v$ induce a path between u and v. If T contains one of the following configurations, then $\lambda_{2,1,1}(T)=6$.*

(C1) $d(u_i)=3$ for each $i\in\{4,5,6\}$;

(C2) $d(u_i)=3$ for each $i\in\{3,4,6,7\}$;

(C3) $d(u_i)=3$ for each $i\in\{2,3,4,6\}$ or $\{4,6,7,8\}$.

Proof Suppose T contains one of the configurations (C1)–(C3). Let f be a 5-$L(2,1,1)$-labeling of T. Then $f(u)=0$ or 5 in view of Lemma 2.3. Without loss of generality, we assume that $f(u)=0$. This implies that $f(u_2)=1$. Therefore, $f(u_4)\notin\{1,3,4\}$ since $d(u_4)=3$. By symmetry, $f(u_6)\notin\{1,4\}$.

(C1) By Lemma 2.3, $|f(u_4)-f(u_5)|>2$ and $|f(u_5)-f(u_6)|>2$ since u_4u_5 and u_5u_6 are heavy. This means $f(u_5)\notin\{2,3\}$. So $f(u_5)\in\{0,4,5\}$. If $f(u_5)=0$, then $f(u_4)=5$, $f(u_6)=4$. But it is impossible since $f(u_6)\notin\{1,4\}$.

(C2) Firstly, we have $f(u_3)\in\{3,4,5\}$. Next, we treat the following three cases to prove.

Case 1. If $f(u_3)=3$, then $f(u_4)=0$. Thus $f(u_5)\in\{2,4\}$. If $f(u_5)=2$, then $f(u_6)=5, f(u_7)=1$ and u_6's pendant neighbor must be labeled by 3. So $f(u_8)=4$. But now there is no proper label for u_7's pendant neighbor. If $f(u_5)=4$, then $f(u_6)=1$, a contradiction.

Case 2. If $f(u_3)=4$, then $f(u_4)=0$. So $f(u_5)\in\{3,5\}$ and $f(u_6)=1$, a contradiction.

Case 3. If $f(u_3)=5$, then $f(u_4)\in\{0,2\}$. In the case, if $(f(u_4),f(u_5))=(0,2)$, then u_3's pendant neighbor and u_4's pendant neighbor must be labeled by 3 and 4, respectively. Now there is no proper label for u_6. If $(f(u_4),f(u_5))=(0,3)$ or $(0,4)$, then $f(u_6)=1$, a contradiction. If $(f(u_4),f(u_5))=(2,0)$, then $f(u_6)=3$. But there is no proper label for u_7. If $(f(u_4),f(u_5))=(2,4)$, then $f(u_6)=1$, a contradiction.

(C3) Let $d(u_i)=3$ for each $i\in\{2,3,4,6\}$. In the case, $f(u_4)\in\{0,2,5\}$. If $f(u_4)=5$, then there is no proper label for u_3. If $f(u_4)=2$, then u_3's pendant neighbor must be labeled by 0 and $f(N(u_4)) = \{0,4,5\}$. It is a contradiction since any vertex in $N(u_4)$ is distance at most 3 with u_3's pendant neighbor. Thus $f(u_4) = 0$ and u_3's pendant neighbor must be labeled by 2. So $f(N[u_4]) = \{0,3,4,5\}$. Therefore, $f(u_6) \in \{1,2\}$. If $f(u_6) = 2$, then $f(N(u_6)) = \{0,4,5\}$, again a contradiction to $f(u_4) = 0$. Thus $f(u_6) = 1$. But it is impossible. A similar argument can be made for $d(u_i) = 3$ for each $i \in \{4,6,7,8\}$. □

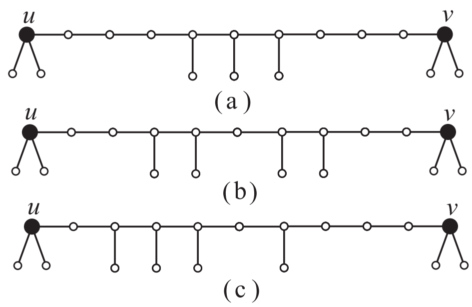

Figure 1 (a) for (C1), (b) for (C2), (c) for (C3)

Theorem 2.6 *Let T be a caterpillar with $\Delta_2 = 6$. Let u and v be two consecutive bad vertices with $\mathrm{dist}(u,v) = 4k+2$ ($k \geq 3$) and $uu_1u_2\cdots u_{4k+1}v$ is the path between u and v. If T contains all the following configurations, then $\lambda_{2,1,1}(T) = 6$.*

(I) $d(u_i) = 3$ for each $i \in \{4,6,8,\ldots,4k-2\}$;

(II) $d(u_2) = d(u_3) = 3$, or $d(u_5) = 3$, or $d(u_7) = 3$;

(III) $d(u_{4k-5}) = 3$, or $d(u_{4k-3}) = 3$, or $d(u_{4k-1}) = d(u_{4k}) = 3$;

(IV) $d(u_i) = 3$, or $d(u_{i+2}) = 3$, or $d(u_{i+4}) = 3$ for all $i \in \{7,11,\ldots,4k-9\}$.

Proof Assume the following conditions hold.

(I) $d(u_i) = 3$ for each $i \in \{4,6,8,\ldots,4k-2\}$;

(II′) Exactly one of $d(u_2) = d(u_3) = 3$, $d(u_5) = 3$ and $d(u_7) = 3$ holds;

(III′) Exactly one of $d(u_{4k-5}) = 3$, $d(u_{4k-3}) = 3$ and $d(u_{4k-1}) = d(u_{4k}) = 3$ holds;

(IV′) Exactly one of $d(u_i) = 3$, $d(u_{i+2}) = 3$ and $d(u_{i+4}) = 3$ holds, for each $i \in \{7,11,\ldots,4k-9\}$.

It is enough to show the assumption derives $\lambda_{2,1,1}(T) = 6$, since any subgraph of T has the $L(2,1,1)$-labeling number smaller than T.

Suppose f is a 5-$L(2,1,1)$-labeling of T. Without loss of generality, suppose $f(u) = 0$. Then $f(u_2) = 1$ by Lemma 2.3. Similarly, $f(u_{4k}) \in \{1,4\}$. Next, we have the following claims.

Claim 1. If $d(u_2) = d(u_3) = 3$ and $d(u_5) = d(u_7) = 2$, then $f(u_6) = 1$ and $f(u_8) \in \{0,2\}$.

According to the proof of (C3) in Theorem 2.5, we have $f(u_6) = 1$. So $f(u_8) \in \{0,2\}$.

Claim 2. If $d(u_5) = 3$ and $d(u_2) = d(u_3) = d(u_7) = 2$, then (a) $f(u_6) = 1$ and $f(u_8) \in \{0,2\}$; or (b) $f(u_6) = 4$ and $f(u_8) \in \{3,5\}$.

By Lemma 2.3, $|f(u_4) - f(u_5)| > 2$ and $|f(u_5) - f(u_6)| > 2$ since u_4u_5 and u_5u_6 are heavy. Thus $f(u_5) \notin \{2,3\}$. This means $f(u_5) \in \{0,4,5\}$. If $f(u_5) = 0$, then $f(u_4) = 5$ in view of $f(u_2) = 1$. So $f(u_6) = 4$ and $f(u_8) \in \{3,5\}$. If $f(u_5) = 4$ or 5, then we have $f(u_6) = 1$ and $f(u_8) \in \{0,2\}$.

Claim 3. If $d(u_7) = 3$ and $d(u_2) = d(u_3) = d(u_5) = 2$, then (a) $f(u_6) = 1$ and $f(u_8) = 0$; or (b) $f(u_6) = 4$ and $f(u_8) = 5$.

By Lemma 2.3, $|f(u_6) - f(u_7)| > 2$ and $|f(u_7) - f(u_8)| > 2$ since u_6u_7 and u_7u_8 are heavy. This means $f(u_7) \in \{0, 1, 4, 5\}$. If $f(u_7) = 0$, then $\{f(u_6), f(u_8)\} \in \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$. Firstly, it is not difficult to see that $\{f(u_6), f(u_8)\} \neq \{3, 5\}$. Secondly, $\{f(u_6), f(u_8)\} \neq \{3, 4\}$. Otherwise, u_7's pendant neighbor has no proper label. Thus $\{f(u_6), f(u_8)\} = \{4, 5\}$. If $f(u_6) = 5$, $f(u_8) = 4$, then u_6's and u_7's pendant neighbor must be labeled by 1 and 3, respectively. Thus $f(u_5) = 2, f(u_4) = 4$, a contradiction. So $f(u_6) = 4$, $f(u_8) = 5$. By symmetry, $f(u_6) = 1$, $f(u_8) = 0$ when $f(u_7) = 5$. If $f(u_7) = 1$, then $\{f(u_6), f(u_8)\} = \{4, 5\}$. If $f(u_6) = 5$, $f(u_8) = 4$, then $f(u_4) = 3$ and $f(N(u_4)) = \{0, 1, 5\}$. It is a contradiction since any vertex in $N(u_4)$ is distance at most 3 with u_6. Thus $f(u_6) = 4$, $f(u_8) = 5$. By symmetry, we have $f(u_6) = 1$, $f(u_8) = 0$ when $f(u_7) = 4$.

Let $i \in \{7, 11, \ldots, 4k - 9\}$. Then we have the following claims.

Claim 4. Suppose $d(u_i) = 2$, $f(u_{i-1}) = 1$ and $f(u_{i+1}) \in \{0, 2\}$. If $d(u_{i+2}) = 3$, $d(u_{i+4}) = 2$, then $f(u_{i+3}) = 1$ and $f(u_{i+5}) \in \{0, 2\}$. If $d(u_{i+2}) = 2$, $d(u_{i+4}) = 3$, then (a) $f(u_{i+3}) = 1$ and $f(u_{i+5}) = 0$; or (b) $f(u_{i+3}) = 4$ and $f(u_{i+5}) = 5$.

If $d(u_{i+2}) = 3$, then by Lemma 2.3, $|f(u_{i+1}) - f(u_{i+2})| > 2$ and $|f(u_{i+2}) - f(u_{i+3})| > 2$ since $u_{i+1}u_{i+2}$ and $u_{i+2}u_{i+3}$ are heavy. Thus $f(u_{i+3}) = 1$ and $f(u_{i+5}) \in \{0, 2\}$. Similarly, if $d(u_{i+2}) = 2$, $d(u_{i+4}) = 3$, then (a) $f(u_{i+3}) = 1$ and $f(u_{i+5}) = 0$; or (b) $f(u_{i+3}) = 4$ and $f(u_{i+5}) = 5$.

By symmetry, it is easy to obtain the following claim.

Claim 5. Suppose $d(u_i) = 2$, $f(u_{i-1}) = 4$ and $f(u_{i+1}) \in \{3, 5\}$. If $d(u_{i+2}) = 3$, $d(u_{i+4}) = 2$, then $f(u_{i+3}) = 4$ and $f(u_{i+5}) \in \{3, 5\}$. If $d(u_{i+2}) = 2$, $d(u_{i+4}) = 3$, then (a) $f(u_{i+3}) = 1$ and $f(u_{i+5}) = 0$; or (b) $f(u_{i+3}) = 4$ and $f(u_{i+5}) = 5$.

Claim 6. Suppose $d(u_i) = 3$, $d(u_{i+3}) = d(u_{i+5}) = 2$. If $f(u_{i-1}) = 1$ and $f(u_{i+1}) = 0$, then $f(u_{i+3}) = 1$ and $f(u_{i+5}) \in \{0, 2\}$. If $f(u_{i-1}) = 4$ and $f(u_{i+1}) = 5$, then $f(u_{i+3}) = 4$ and $f(u_{i+5}) \in \{3, 5\}$.

If $f(u_{i-1}) = 1$ and $f(u_{i+1}) = 0$, then u_i's pendant neighbor must be labeled by 2. Thus $f(N(u_{i+1})) = \{3, 4, 5\}$. So $f(u_{i+3}) = 1$ and $f(u_{i+5}) \in \{0, 2\}$. Similarly, if $f(u_{i-1}) = 4$ and $f(u_{i+1}) = 5$, then $f(u_{i+3}) = 4$ and $f(u_{i+5}) \in \{3, 5\}$.

We have another three claims.

Claim 7. Suppose $d(u_{4k-5}) = 2$, $f(u_{4k-6}) = 1$ and $f(u_{4k-4}) \in \{0, 2\}$. If $d(u_{4k-3}) = 3$, $d(u_{4k-1}) = d(u_{4k}) = 2$, then $f(u_{4k-2}) = 1$ and $f(u_{4k}) \in \{0, 2\}$. If $d(u_{4k-3}) = 2$, $d(u_{4k-1}) = d(u_{4k}) = 3$, then $f(u_{4k-2}) = 1$ and $f(u_{4k}) = 0$.

According to the proof of Claim 4, we have $f(u_{4k-2}) = 1$ and $f(u_{4k}) \in \{0, 2\}$. If $d(u_{4k-3}) = 2$, $d(u_{4k-1}) = d(u_{4k}) = 3$, then $f(u_{4k-2}) = 1$ and $f(u_{4k}) = 0$ by the proof of Claim 5.

By symmetry, it is easy to obtain the following claim.

Claim 8. Suppose $d(u_{4k-5}) = 2$, $f(u_{4k-6}) = 4$ and $f(u_{4k-4}) \in \{3, 5\}$. If $d(u_{4k-3}) = 3$, $d(u_{4k-1}) = d(u_{4k}) = 2$, then $f(u_{4k-2}) = 4$ and $f(u_{4k}) \in \{3, 5\}$. If $d(u_{4k-3}) = 2$, $d(u_{4k-1}) = d(u_{4k}) = 3$, then $f(u_{4k-2}) = 4$ and $f(u_{4k}) = 5$.

Claim 9. Suppose $d(u_{4k-5}) = 3$, $d(u_{4k-3}) = d(u_{4k-1}) = d(u_{4k}) = 2$. If $f(u_{4k-6}) = 1$ and $f(u_{4k-4}) = 0$, then $f(u_{4k-2}) = 1$ and $f(u_{4k}) \in \{0, 2\}$. If $f(u_{4k-6}) = 4$ and $f(u_{4k-4}) = 5$, then $f(u_{4k-2}) = 4$ and $f(u_{4k}) \in \{3, 5\}$.

Using a similar argument to the proof of Claim 4, we have the results hold.

By Claims 1–9, we conclude that $f(u_{4k-2}) \in \{1, 4\}$, which is a contradiction. Thus

$$\lambda_{2,1,1}(T) = 6. \quad \square$$

3. A characterization result for caterpillars with $\Delta_2 = 6$

Let T be a tree with diameter at least 3. Then by the definition of Δ_2, we have $\Delta_2 \geq 4$. For $\Delta_2 = 4$ and $\Delta_2 = 5$, we have given a complete characterization in [17]. In this section, we always suppose T is a caterpillar with $\Delta_2 = 6$.

Theorem 3.1 ([17]) *Let T be a caterpillar without bad vertex or with a unique bad vertex. Then $\lambda_{2,1,1}(T) = \Delta_2 - 1$.*

Now we consider that T is a caterpillar with at least two bad vertices.

Theorem 3.2 ([17]) *Let T be a caterpillar with no bad vertices of distance 3 or $4k+2$ for some integer $k \geq 0$. Then $\lambda_{2,1,1}(T) = \Delta_2 - 1$.*

In the following, we will give a complete characterization of caterpillars with $\Delta_2 = 6$.

Theorem 3.3 *Let T be a caterpillar with $\Delta_2 = 6$. Then $\lambda_{2,1,1}(T) = 6$ if and only if one of the followings holds.*

(1) T contains one of the configurations (C1)–(C2) in Theorem 2.4;

(2) T contains one of the configurations (C1)–(C3) in Theorem 2.5;

(3) T contains all the configurations in Theorem 2.6.

Proof Sufficiency. Obviously, if one of (1)–(3) holds, then $\lambda_{2,1,1}(T) = 6$ by Theorems 2.4–2.6.

Necessity. Suppose that T has no configurations of Theorems 2.4 and 2.5. And suppose for any two consecutive bad vertices u, v with $\text{dist}(u, v) = 4k + 2$ $(k \geq 3)$, we have one of the followings holds:

(I) $d(u_i) = 2$ for some $i \in \{4, 6, 8, \ldots, 4k-2\}$;

(II) $d(u_2) = d(u_5) = d(u_7) = 2$ or $d(u_3) = d(u_5) = d(u_7) = 2$;

(III) $d(u_{4k-5}) = d(u_{4k-3}) = d(u_{4k-1}) = 2$ or $d(u_{4k-5}) = d(u_{4k-3}) = d(u_{4k}) = 2$;

(IV) $d(u_i) = d(u_{i+2}) = d(u_{i+4}) = 2$ for some $i \in \{7, 11, \ldots, 4k-9\}$,

where $uu_1u_2 \cdots u_{4k+1}v$ is the path between u and v.

Let $v_1, v_2, \ldots, v_b$ be all bad vertices of T. For any bad vertex v_j, let V_j^p be the set of vertices on the $v_j - v_{j+1}$ path. Let $V_j = V_j^p \cup N(V_j^p)$. For a 5-$L(2,1,1)$-labeling f on T, if $f(v_j) = 0$, $f(v_j^r) = 3$ (or 4), then we call v_j is of A-style, where v_j^r is the right-hand side neighbor of v_j; If $f(v_j) = 5$, $f(v_j^r) = 2$ (or 3), then we call v_j is of B-style. If there exists a 5-$L(2,1,1)$-labeling f on $G(V_j)$, such that v_j is X-style under f and v_{j+1} is Y-style under f, then we call $G(V_j)$ is of

XY-style, for $X, Y\in \{A, B\}$. By symmetry of i and $5-i$, $G(V_j)$ is of XY-style, if and only if $G(V_j)$ is of YX-style.

Before giving a 5-$L(2,1,1)$-labeling of T, we first show that $G(V_j)$ is of certain style, where $\mathrm{dist}(v_j, v_{j+1}) = 4k+2$ for some $k \ge 3$.

Case 1. If (I) holds, that is, $d(u_i) = 2$ for some $i \in \{4, 6, 8, \ldots, 4k-2\}$, we give a 5-$L(2,1,1)$-labeling on $G(V_j)$ as follows, which implies $G(V_j)$ is of AA-style (also BB-style by symmetry), see Figure 2.

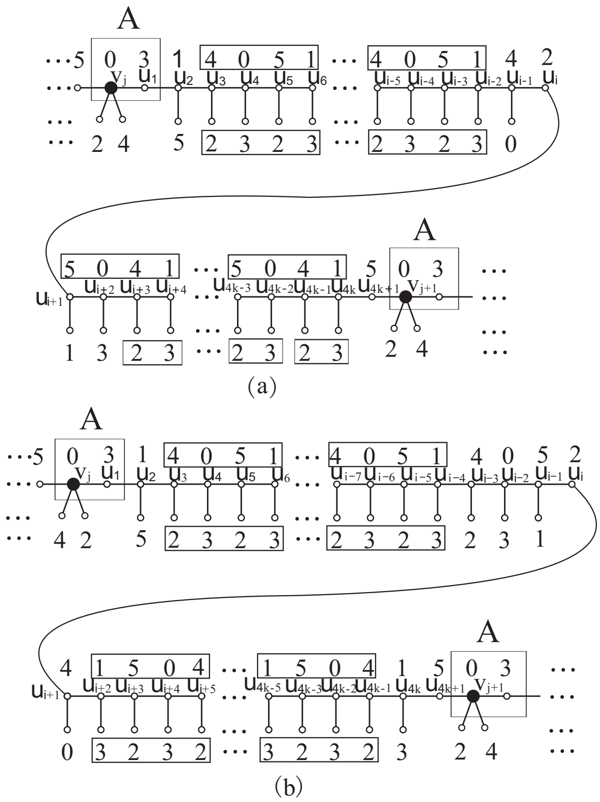

Figure 2 'AA' labeling style of the $4k+2$ segment in Case 1

(1) If $i \mid 4$, let $f(v_j)f(u_1)\cdots f(u_{i-1}) = 031\underline{4051}\cdots\underline{4051}4$, $f(u_i) = 2$, and $f(u_{i+1})f(u_{i+2})\cdots f(u_{4k})f(u_{4k+1})f(v_{j+1}) = \underline{5041}\cdots\underline{5041}50$;

If $i \nmid 4$, let $f(v_j)f(u_1)\cdots f(u_{i-1}) = 031\underline{4051}\cdots\underline{4051}405$, $f(u_i) = 2$, and $f(u_{i+1})f(u_{i+2})\cdots f(u_{4k})f(u_{4k+1})f(v_{j+1}) = 4\underline{1504}\cdots\underline{1504}150$.

(2) For $j \notin \{2, i-1, i+1\}$, if $f(u_j) = 0$ or 1, then label u_j's pendant neighbor by 3; If $f(u_j) = 4$ or 5, then label u_j's pendant neighbor by 2.

(3) For $j \in \{i-1, i+1\}$, if $f(u_j) = 4$, then label u_j's pendant neighbor by 0; If $f(u_j) = 5$, then label u_j's pendant neighbor by 1; Label u_2's pendant neighbor by 5.

One can verify that it is a 5-$L(2,1,1)$-labeling on the segment between v_j and v_{j+1}.

Case 2. If (II) holds, that is, $d(u_2) = d(u_5) = d(u_7) = 2$ or $d(u_3) = d(u_5) = d(u_7) = 2$, then we give a 5-$L(2,1,1)$-labeling on $G(V_j)$ as follows, which implies $G(V_j)$ is of AB-style (also

BA-style by symmetry), see Figure 3.

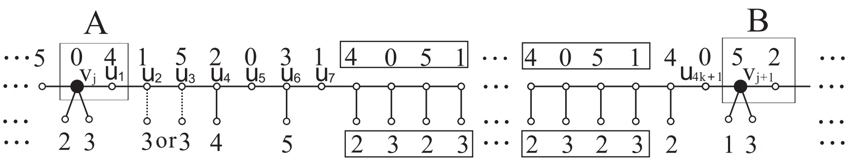

Figure 3 'AB' labeling style of the $4k+2$ segment in Case 2

Case 3. If (III) holds, that is, $d(u_{4k-5}) = d(u_{4k-3}) = d(u_{4k-1}) = 2$ or $d(u_{4k-5}) = d(u_{4k-3}) = d(u_{4k}) = 2$, we give a 5-$L(2,1,1)$-labeling on $G(V_j)$ as follows, which implies $G(V_j)$ is of AB-style (also BA-style by symmetry), see Figure 4.

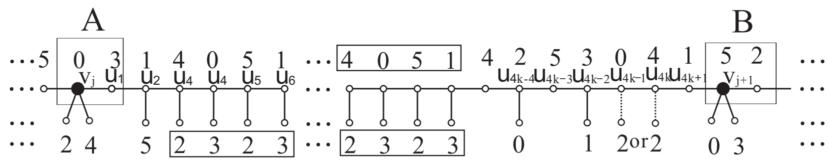

Figure 4 'AB' labeling style of the $4k+2$ segment in Case 3

Case 4. If (IV) holds, that is, $d(u_i) = d(u_{i+2}) = d(u_{i+4}) = 2$ for some $i \in \{7, 11, \ldots, 4k-9\}$, we give a 5-$L(2,1,1)$-labeling on $G(V_j)$ as follows, which implies $G(V_j)$ is of AB-style (also BA-style by symmetry), see Figure 5.

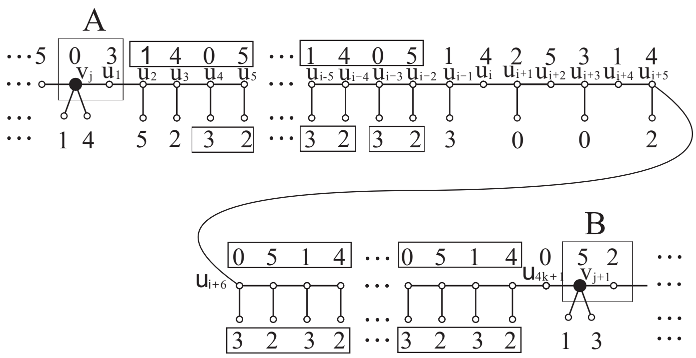

Figure 5 'AB' labeling style of the $4k+2$ segment in Case 4

Secondly, we show that $G(V_j)$ is of certain style, where $\text{dist}(v_j, v_{j+1}) = 4k, 4k+1, 4k+3$ for some $k \geq 1$.

We can label $G(V_j)$ as Figure 6, when $\text{dist}(v_j, v_{j+1}) = 4k$. This implies $G(V_j)$ is of AA-style (also BB-style by symmetry).

We can label $G(V_j)$ as Figure 7, when $\text{dist}(v_j, v_{j+1}) = 4k+1$. This implies $G(V_j)$ is of AB-style (also BA-style by symmetry).

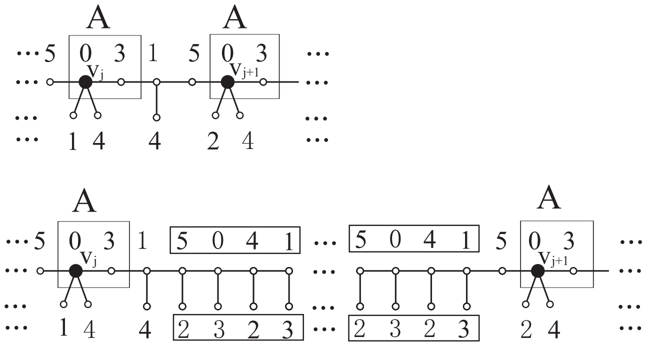

Figure 6 'AA' labeling style of the $4k$ segment

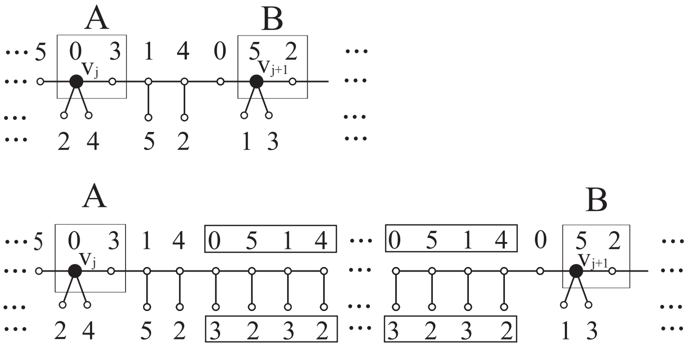

Figure 7 'AB' labeling style of the $4k+1$ segment

We can label $G(V_j)$ as Figure 8, when $\mathrm{dist}(v_j, v_{j+1}) = 4k+3$. This implies $G(V_j)$ is of AB-style (also BA-style by symmetry).

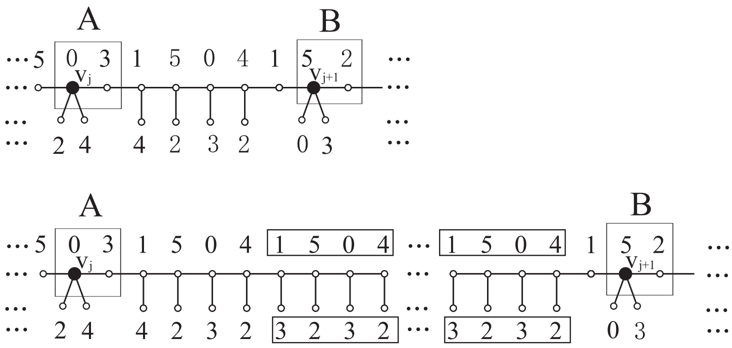

Figure 8 'AB' labeling style of the $4k+3$ segment

Now we give a 5-$L(2,1,1)$-labeling of T by the following three steps.

Step 1. Label the vertices in the left-hand side of v_1 as follows, such that v_1 is of A-style, see Figure 9.

Step 2. Suppose that $G(V_0 \cup V_1 \cup \cdots \cup V_j)$ has an $L(2,1,1)$-labeling with span 5 such that v_j is of A or B-style, where V_0 is the set of vertices on the left hand side of v_1^r (include v_1^r). Then by the discussion above we can extend the f to $G(V_0 \cup V_1 \cup \cdots \cup V_j \cup V_{j+1})$ such that v_{j+1} is of A or B-style. Going on with the above process, we can extend f to $G(V_0 \cup V_1 \cup \cdots \cup V_b)$.

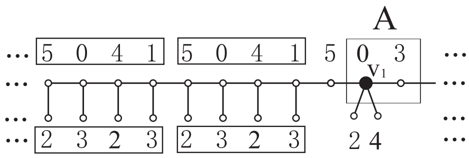

Figure 9 Label the vertices in the left-hand side of v_1 such that v_1 is of A-style

Step 3. Label the vertices on the right hand side of v_b as Figure 10, when v_b is of A or B-style.

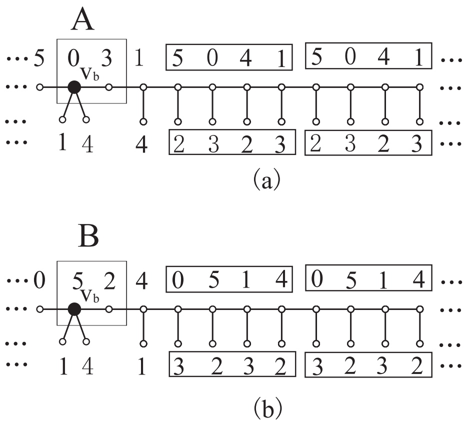

Figure 10 Label the vertices on the right hand side of v_b^r

Thus, f is a 5-$L(2,1,1)$-labeling of T. This completes the proof of Theorem 3.3. □

4. Concluding remarks

Golovach et al. [14] asserted that deciding whether a given tree has the $L(2,1,1)$-labeling number attaining the lower bound is NP-complete. Therefore, giving a characterization result for the subclass of trees is a meaningful topic. In this paper, we completely characterize the $L(2,1,1)$-labelings of caterpillars (as a subclass of trees) with $\Delta_2(T) = 6$. We also try to characterize the $L(2,1,1)$-labelings of caterpillars with $\Delta_2(T) \geq 7$. But we found it very difficult. This leads us to the following question: what is the computational complexity of $L(2,1,1)$-labeling for caterpillars?

Acknowledgements The authors would like to thank the anonymous referees for careful reading and comments.

References

[1] W. K. HALE. *Frequency assignment: Theory and applications*. Proc. IEEE, 1980, **68**(12): 1497–1514.

[2] J. R. GRIGGS, R. K. YEH. *Labeling graph with a condition at distance two*. SIAM J. Discrete Math., 1992, **5**(4): 586–595.

[3] G. J. CHANG, D. KUO. *The $L(2,1)$-labeling problem on graphs*. SIAM J. Discrete Math., 1996, **9**(2): 309–316.

[4] J. P. GEORGES, D. M. MAURO. *Labeling trees with a condition at distance two*. Discrete Math., 2003, **269**(1-3): 127–148.

[5] Weifan WANG. *The $L(2,1)$-labelling of trees*. Discrete Appl. Math., 2007, **154**(3): 598–603.

[6] R. K. YEH. *A survey on labeling graphs with a condition at distance two*. Discrete Math., 2006, **306**(12): 1217–1231.

[7] T. CALAMONERI. *The $L(h,k)$-labelling problem: an updated survey and annotated bibliography*. Computer J., 2011, **54**(8): 1344–1371.

[8] M. L. CHIA, D. KUO, Hongya LIAO, et al. *$L(3,2,1)$-labeling of graphs*. Taiwanese J. Math., 2011, **15**(6): 2439–2457.

[9] B. M. KIM, B. C. SONG, W. HWANG. *Distance three labelings for direct products of three complete graphs*. Taiwanese J. Math., 2013, **17**(1): 207–219.

[10] Sanming ZHOU. *A distance labeling problem for hypercubes*. Discrete Appl. Math., 2008, **156**(15): 2846–2854.

[11] T. CALAMONERI, E. G. FUSCO, R. B. TAN, et al. *$L(h,1,1)$-labeling of outerplanar graphs*. Math. Methods Oper. Res., 2009, **69**(2): 307–321.

[12] Zehui SHAO, A. VESEL. *$L(3,2,1)$-labeling of triangular and toroidal grids*. CEJOR Cent. Eur. J. Oper. Res., 2015, **23**(3): 659–673.

[13] D. KING, C. J. RAS, Sanming ZHOU. *The $L(h,1,1)$-labelling problem for trees*. Europ. J. Combin., 2010, **31**(5): 1295–1306.

[14] P. A. GOLOVACH, B. LIDICKÝ, D. PAULUSMA. *$L(2,1,1)$-Labeling is NP-Complete for Trees*. Springer, Berlin, 2010.

[15] J. FIALA, P. A. GOLOVACH, J. KRATOCHVÍl, et al. *Distance three Labelings of trees*. Discrete Appl. Math., 2012, **160**(6): 764–779.

[16] Xiaoling ZHANG, Kecai DENG. *Characterization results for the $L(2,1,1)$-labeling problem on trees*. Discuss. Math. Graph Theory, 2017, **37**(3): 611–622.

[17] Xiaoling ZHANG. *The $L(2,1,1)$-Labelings of caterpillars*. Computer Mathematics: Computer Systems Theory, 2016, **1**(3-4): 85–97.