Journal of Mathematical Research with Applications
Mar., 2023, Vol. 43, No. 2, pp. 161–165
DOI:10.3770/j.issn:2095-2651.2023.02.004
Http://jmre.dlut.edu.cn

Bounds of the Signed Edge Domination Number of Complete Multipartite Graphs

Yancai ZHAO[1,2]
1. *Wuxi City College of Vocational Technology, Jiangsu* 214153*, P. R. China;*
2. *Wuxi Environmental Science and Engineering Research Center, Jiangsu* 214153*, P. R. China*

Abstract A function $f: E(G) \to \{-1,1\}$ is called a signed edge dominating function (SEDF for short) of G if $f[e]=f(N[e])=\sum_{e'\in N[e]} f(e') \geq 1$, for every edge $e \in E(G)$. $w(f)=\sum_{e\in E} f(e)$ is called the weight of f. The signed edge domination number $\gamma_s{}'(G)$ of G is the minimum weight among all signed edge dominating functions of G. In this paper, we initiate the study of this parameter for G a complete multipartite graph. We provide the lower and upper bounds of $\gamma_s{}'(G)$ for G a complete r-partite graph with r even and all parts equal.

Keywords signed edge domination; signed edge domination number; complete multipartite graph

MR(2020) Subject Classification 05C69

1. Introduction

In this paper we in general follow [1] for notation and graph theory terminology. Specifically, let $G=(V,E)$ be a simple graph with vertex set V and edge set E, and let v be a vertex in V. The open neighborhood $N(v)$ of v is defined as the set of vertices adjacent to v. The closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. Let $E(v)$ be the set of all edges incident with v. Similarly, let e be an edge in E. The open neighborhood $N(e)$ of e is defined as the set of edges adjacent to e. The closed neighborhood of e is $N[e]=N(e)\cup\{e\}$. For a natural number $r\geq 2$, a graph is r-partite if its vertex set can be partitioned into r subsets, or parts, $V_1,V_2,\ldots,V_r$, in such a way that no edge has both ends in the same part. We denote an r-partite graph G with r-partitions $(V_1,V_2,\ldots,V_r)$ by $G[V_1,V_2,\ldots,V_r]$. If $G[V_1,V_2,\ldots,V_r]$ is simple and every vertex in each part is joined to every vertex in the other $r-1$ parts, then G is called a complete r-partite graph or complete multipartite graph. If $|V_1|=n_1,|V_2|=n_2,\ldots,|V_r|=n_r$, then the complete r-partite graph is denoted by $K_{n_1,n_2,\ldots,n_r}$. In particular, if $r=2$, we call a complete r-partite graph a complete bipartite graph; if $r=3$, we call a complete r-partite graph a complete tripartite graph.

Let $G=(V(G),E(G))$ be a non-empty graph. A function $f: V(G)\to\{-1,1\}$ is called a signed dominating function of G if $f[v]=f(N[v])=\sum_{u\in N[v]} f(u)\geq 1$, for every $v\in V(G)$. A survey on signed dominating functions can be seen in [2].

Received April 30, 2022; Accepted August 22, 2022
Supported by the National Natural Science Foundation of China (Grant No. 71774078).
E-mail address: zhaoyc69@126.com

A function $f : E(G) \to \{-1, 1\}$ is called a signed edge dominating function (SEDF for short) of G if $f[e] = f(N[e]) = \sum_{e' \in N[e]} f(e') \geq 1$, for every $e \in E(G)$. The weight $w(f)$ of f is the sum of the function values of all edges in G, that is, $w(f) = \sum_{e \in E} f(e)$. The signed edge domination number $\gamma_s{}'(G)$ of G is the minimum weight among all signed edge dominating functions on G, that is, $\gamma'(G) = \min\{w(f) | f \text{ is an SEDF on } G\}$. When $\gamma_s{}'(G) = w(f)$, f is called a $\gamma_s{}'$-function of G. Xu [3] introduced this concept and it has been studied in, for example [4–11].

The exact values of signed edge domination number of a complete bipartite graph have been given in [3]. The exact values of signed edge domination number of some class of complete tripartite graph have been provided in [11]. In this paper, we initiate the study of this parameter for G a complete multipartite graph. In Section 2, we obtain lower and upper bounds on $\gamma_s{}'(G)$ for a complete r-partite graph G with r even and all parts equal by directly constructing the minimum signed edge dominating functions of G.

2. Bounds

Given an SEDF f of $K_{n_1,n_2,\ldots,n_r}$, for every $i, j \in \{1, 2, \ldots, r\}$, we use the following notations: Denote the r parts of $K_{n_1,n_2,\ldots,n_r}$ by $V_1, V_2, \ldots, V_r$ with $|V_i| = n_i$ and $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$; Let $E(v) = \{e | e \text{ is incident with } v, v \in V\}$, $s_v = \sum_{e \in E(v)} f(e)$, $E(V_i) = \{e | e \text{ is incident with } v, v \in V_i\}$ and $f(E(V_i)) = \sum_{e \in E(V_i)} f(e)$, $E(V_i, V_j) = \{e | e = uv, u \in V_i, v \in V_j, i, j \in \{1, 2, \ldots, r\}, i \neq j\}$, $f(V_i, V_j) = \sum_{e \in E(V_i, V_j)} f(e)$ and $s_i = \sum_{v \in V_i} s_v$; For an integer k and some $i \in \{1, 2, \ldots, r\}$, define $V_i(k) = \{v_{ij} | \ s_{v_{ij}} = k, j = 1, 2, \ldots, n_i\}$; Also, denote $\bigcup_{j \in \{1,2,\ldots,r\} \setminus \{i\}} V_j$ by $\bigcup_{j \neq i} V_j$ and $\sum_{j \in \{1,2,\ldots,r\} \setminus \{i\}} f(V_j)$ by $\sum_{j \neq i} f(V_j)$ for some fixed $i \in \{1, 2, \ldots, r\}$.

By the definition of SEDF, we have the following immediate observation.

Observation 2.1 A function $f : E(K_{n_1,n_2,\ldots,n_r}) \longrightarrow \{-1, 1\}$ is an SEDF on $K_{n_1,n_2,\ldots,n_r}$ if and only if for any $u \in V_i$, $v \in V_j$, $i, j \in \{1, 2, \ldots, r\}, i \neq j$, the following formulae holds:

$$f[uv] = s_u + s_v - f(uv) \geq 1.$$

Now, we provide the lower and upper bounds of $\gamma_s{}'(K_{n,n,\ldots,n})$.

Theorem 2.2 *For a complete r-partite graph $K_{n,n,\ldots,n}$ with $r \geq 4$ even,*

$$\frac{(r-1)n}{4} \leq \gamma_s{}'(K_{n,n,\ldots,n}) \leq \frac{rn}{2}.$$

Proof First we give the following useful claim.

Claim 2.3 *If f is an SEDF on a complete r-partite graph $K_{n,n,\ldots,n}$, then $w(f)$ has the same parity with $\frac{r(r-1)}{2}n$.*

Proof of Claim 2.3 Since $w(f) = \sum_{e \in E} f(e)$, $|E| = \frac{r(r-1)}{2}n^2$ and $f(e) = 1$ or -1, $w(f)$ has the same parity with $\frac{r(r-1)}{2}n^2$, and thus has the same parity with $\frac{r(r-1)}{2}n$. □

In order to prove Theorem 2.2, we first prove $\gamma_s{}'(K_{n,n,\ldots,n}) \geq \frac{(r-1)n}{4}$. Let f be an SEDF f on $K_{n,n,\ldots,n}$.

Assume there exists some vertex $u \in V$, without loss of generality, assume $u \in V_1$ such that $s_u = \min\{s_v, v \in V\} = -k$ for some $k \geq 1$. Then by Observation 2.1, $s_v \geq k+1+f(uv)$ for every $v \in V \setminus V_1$, and $f(uw) = -1$ for every $w \in V_i(k), i \in \{2,3,\ldots,r\}$. So $\sum_{i=2}^{r} |V_i(k)| \leq \frac{r-1}{2}n + \frac{k}{2}$, for otherwise, $s_u < (r-1)n - 2(\frac{r-1}{2}n + \frac{k}{2}) = -k$, contradicting the assumption of $s_u = -k$. Therefore,

$$\sum_{v \in V \setminus V_1} s_v \geq (\frac{r-1}{2}n + \frac{k}{2}) \cdot k + (\frac{r-1}{2}n - \frac{k}{2}) \cdot (k+2) = (k+1)(r-1)n - k,$$

and thus,

$$w(f) = \frac{1}{2}\sum_{v \in V} s_v = \frac{1}{2}\Big(\sum_{v \in V_1} s_v + \sum_{v \in V \setminus V_1} s_v\Big) \geq \frac{1}{2}((k+1)(r-1)n - k - kn).$$

Consider $w(k) = \frac{1}{2}((k+1)(r-1)n-k-kn)$ as a function of k. Since $w'(k) = \frac{1}{2}((r-1)n-1-n) \geq 0$, $w(k)$ is an increasing function. So

$$w(f) = \frac{1}{2}((k+1)(r-1)n - k - kn) \geq \frac{1}{2}(2(r-1)n - 1 - n) > \frac{(r-1)n}{4}.$$

So, we always assume that $s_v \geq 0$ for all $v \in V$ in the rest argumentation of the lower bound.

Let t denote the cardinality of the set $\{i|$ there exists a vertex $v \in V_i$ such that $s_v = 0, i = 1,2,\ldots,r\}$. We consider the following three cases.

Case 1. $t \leq \frac{r}{2}$. Then $w(f) = \frac{1}{2}\sum_{v \in V} s_v \geq \frac{(r-t)}{2}n \geq \frac{rn}{4} \geq \frac{(r-1)n}{4}$.

Case 2. $\frac{r}{2}+1 \leq t \leq r$. Without loss of generality, suppose $|V_i(0)| > 0$ for every $i \in \{1,\ldots,t\}$. We claim that $\sum_{j=1,j\neq i}^{t} |V_j(0)| \leq \frac{r-1}{2}n$ for every $i \in \{1,\ldots,t\}$. For otherwise, suppose to the contrary that

$$\sum_{j=1,j\neq i}^{t} |V_j(0)| \geq \frac{r-1}{2}n + 1 \text{ for some } i \in \{1,\ldots,t\}.$$

Let $u \in V_i$ with $s_u = 0$. On the other hand, $f(uv) = -1$ for all $v \in \bigcup_{j\neq i,j=1}^{t} V_j(0)$ by Observation 2.1. But this results in $s_u < 0$, contradicting the assumption that $s_u \geq 0$. Now that $\sum_{j=1,j\neq i}^{t} |V_j(0)| \leq \frac{r-1}{2}n$ for every $i \in \{1,\ldots,t\}$, we add these t inequalities and have

$$(t-1)\sum_{j=1}^{t} |V_j(0)| \leq \frac{r-1}{2}nt,$$

$$\sum_{j=1}^{t} |V_j(0)| \leq \frac{r-1}{2(t-1)}nt = \frac{r-1}{2}n(1+\frac{1}{t-1}).$$

Therefore,

$$\begin{aligned} w(f) &= \frac{1}{2}\sum_{v \in V} s_v \geq \frac{1}{2}((tn - \frac{r-1}{2(t-1)}nt) \cdot 1 + (r-t)n \cdot 1) \\ &= \frac{1}{2}((1 - \frac{r-1}{2(t-1)})nt + (r-t)n) = \frac{1}{2}(rn - \frac{r-1}{2}n(1+\frac{1}{t-1})) \\ &\geq \frac{1}{2}(rn - \frac{r-1}{2}n(1+\frac{1}{\frac{r+2}{2}-1})) = \frac{1}{2}(rn - \frac{r-1}{2}n \cdot \frac{r+2}{r}) \end{aligned}$$

$$= \frac{n}{4}(r - 1 + \frac{2}{r}) \geq \frac{(r-1)}{4}n.$$

Case 3. $t = r$. Suppose there exists a vertex $v_i \in V_i$ such that $s_{v_i} = 0$ for every $i \in \{1, \ldots, r\}$. Then by Observation 2.1, $f(v_i v) = -1$ for every $v \in \bigcup_{j\neq i, j=1}^{r} V_j(0)$. Thus, $\sum_{j\neq i, j=1}^{r} |V_j(0)| \leq \frac{r-1}{2}n$, for otherwise, $s_{v_i} < 0$, contradicting the assumption that $s_{v_i} = 0$. By adding up these r inequalities, we have

$$(r-1)\sum_{j=1}^{r} |V_j(0)| \leq \frac{r(r-1)}{2}n,$$

and thus

$$\sum_{j=1}^{r} |V_j(0)| \leq \frac{rn}{2}.$$

As a result,

$$w(f) = \frac{1}{2}\sum_{v\in V} s_v \geq \frac{1}{2}\cdot(rn - \frac{rn}{2})\cdot 2 = \frac{rn}{2} \geq \frac{(r-1)}{4}n.$$

Now we have proved that ${\gamma_s}'(K_{n,n,\ldots,n}) \geq \frac{(r-1)}{4}n$.

Next we prove the upper bound by constructing an SEDF f on $K_{n,n,\ldots,n}$ such that $w(f) = \frac{rn}{2}$. We consider two cases in the following.

Case 1. n is even. Let f be defined as follows: For $e \in E_1 = E(V_1, V_2) \cup E(V_3, V_4) \cup \cdots \cup E(V_{r-1}, V_r)$, for example, $e \in E(V_1, V_2)$, let

$$f(v_{1i}v_{2j}) = \begin{cases} 1, & \text{if } i > \frac{n}{2},\ j > \frac{n}{2},\ i = j \text{ is even,} \\ (-1)^{i+j+1}, & \text{otherwise;} \end{cases}$$

For $e \in E_2 = E(V_2, V_3) \cup E(V_4, V_5) \cup \cdots \cup E(V_r, V_1)$, for example, $e \in E(V_2, V_3)$, let

$$f(v_{2i}v_{3j}) = \begin{cases} 1, & \text{if } i \leq \frac{n}{2},\ j \leq \frac{n}{2},\ i = j \text{ is even,} \\ (-1)^{i+j+1}, & \text{otherwise;} \end{cases}$$

For $e \in E \setminus (E_1 \cup E_2)$, suppose $e \in E(V_p, V_q)$ for some $p, q \in \{1, 2, \ldots, r\}$, without loss of generality, let $f(v_{pi}v_{qj}) = (-1)^{i+j+1}$.

By our construction, we can easily find that for every $p \in \{1, 2, \ldots, r\}$ and $i \in \{1, 2, \ldots, n\}$, $s_{v_{pi}} = 0$ when i is odd, $s_{v_{pi}} = 2$ when i is even. Also, noting the definitions of $f(e)$, it is easy to check that f is an SEDF on $K_{n,n,\ldots,n}$ by Observation 2.1, and we have

$$w(f) = \frac{1}{2}\sum_{v\in V} s_v = \frac{1}{2}\cdot\frac{n}{2}\cdot 2 = \frac{rn}{2},$$

which means that ${\gamma_s}'(K_{n,n,\ldots,n}) \leq \frac{rn}{2}$.

Case 2. n is odd. Let f be defined as follows: For $e \in E_1 = E(V_1, V_2) \cup E(V_3, V_4) \cup \cdots \cup E(V_{r-1}, V_r)$, for example, $e \in E(V_1, V_2)$, let

$$f(v_{1i}v_{2j}) = \begin{cases} 1, & \text{if } i = j \text{ is odd,} \\ (-1)^{i+j+1}, & \text{otherwise;} \end{cases}$$

For $e \in E \setminus E_1$, suppose $e \in E(V_p, V_q)$ for some $p, q \in \{1, 2, \ldots, r\}$, without loss of generality, let

$$f(v_{pi}v_{qj}) = \begin{cases} (-1)^{i+j}, & \text{if } p+q \text{ is odd}, \\ (-1)^{i+j+1}, & \text{if } p+q \text{ is even}. \end{cases}$$

By our construction, we can easily find that for every $p \in \{1, 2, \ldots, r\}$ and $i \in \{1, 2, \ldots, n\}$, $s_{v_{pi}} = 1$. Also, noting the definitions of all $f(e)$, it is easy to check that f is an SEDF on $K_{n,n,\ldots,n}$ by Observation 2.1, and we have

$$w(f) = \frac{1}{2}\sum_{v \in V} s_v = \frac{1}{2} \cdot n \cdot 1 = \frac{rn}{2},$$

which means that $\gamma_s{}'(K_{n,n,\ldots,n}) \leq \frac{rn}{2}$. This completes the proof of Theorem 2.2. $\square$

Concluding Remarks For further researches, we are interested in the exact value of $\gamma_s{}'(K_{n,n,\ldots,n})$ with the number r of the parts even, and interested in the case for r odd, which seems more difficult and complicated than the case for r even.

References

[1] T. W. HAYNES, S. T. HEDETNIEMI, P. J. SLATER. *Fundamentals of Domination in Graphs*. Marcel Dekker, New York, 1998.

[2] Liying KANG, Erfang SHAN. *Signed and Minus Dominating Functions in Graphs*. Springer, Cham, 2020.

[3] Baogen XU. *On signed edge domination numbers of graphs*. Discrete Math., 2001, **239**(1-3): 179–189.

[4] S. AKBARI, S. BOLOUKI, P. HATAMI, et al. *On the signed edge domination number of graphs*. Discrete Math., 2010, **309**(3): 587–594.

[5] L. W. BEINEKE, M. A. HENNING. *Opinion functions on trees*. Discrete Math., 1997, **167/168**: 127–139.

[6] Weidong CHEN, Enmin SONG. *Lower bounds on several versions of signed domination number*. Discrete Math., 2008, **308**(10): 1837–1846.

[7] M. A. HENNING, H. R. HIND. *Strict majority functions on graphs*. J. Graph Theory, 1998, **28**(1): 49–56.

[8] M. A. HENNING, P. J. SLATER. *Inequalities relating domination cubic graphs*. Discrete Math., 1996, **158**(1-3): 87–98.

[9] Erfang SHAN, T. C. E. CHENG. *Remarks on the minus (signed) total domination in graphs*. Discrete Math., 2008, **308**(15): 3373–3380.

[10] Baogen XU. *Two classes of edge domination in graphs*. Discrete Appl. Math., 2006, **154**(10): 1541–1546.

[11] A. KHODKAR, A. N. GHAMESHLOU. *Signed edge domination numbers of complete tripartite graphs: Part One*. Util. Math., 2017, **105**: 237–258.